# On the Production of Mean Zonal Currents in the Atmosphere by Large Disturbances

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## *Abstract*

The problem of the production of mean zonal currents by large-scale horizontal disturbances in the atmosphere is attacked by computation of time-tendencies from the vorticity equation. It is shown that the earth's rotation, acting through the variable Coriolis parameter, generally produces a mean northward transport of westerly momentum, with the maximum transport occurring south of the middle of the zonal belt occupied by the disturbances, and therefore creating mean westerly and easterly currents in the central and outer parts respectively. Its effect on the disturbance may either be a damping or an amplification, depending on the asymmetry of the mean flow profile.

Six different mean flow profiles have been investigated to determine the nature of the interaction between a disturbance and the mean flow. For three profiles with inflection points, the interaction depends upon the wave length of the disturbance. If the wave length is longer than a critical value, an amplifying effect on the disturbance is produced accompanied by a divergence of westerly momentum in the region of maximum mean flow, and if shorter, a damping effect with a convergence of westerly momentum. On the other hand, the interaction produces only a damping effect for three mean flow profiles without inflection point.

Since the zonal currents in the atmosphere are seldom very strong and not far from symmetrical, the total effect of the earth's rotation and of the mean flow leads to horizontal damping, in which the contrast in the mean flow at different latitudes is increased and a mean northward transport of westerly momentum is produced. This is in qualitative agreement with observations, indicating that these factors are of real importance in the development and maintenance of the mean zonal flow.


## 1. Introduction

In previous studies, the author has discussed the mechanism of the mean meridional transport of westerly momentum by large-scale atmospheric disturbances (KUO, 1951 a, 1951 b), based on the properties of solutions of the linearized vorticity equation for horizontal, inviscid, barotropic, non-divergent flow. This will again be our subject. Although vertical motion and baroclinicity in the atmosphere are of paramount importance in some other aspects of the meteorological problem, it is believed that the omission of such factors does not significantly distort the average interactions between the purely horizontal components of the large-scale disturbances on the one hand and the mean flow and the earth's rotation on the other, as can be demonstrated by integrating the equations along the vertical with respect to pressure, or by comparing the simple vorticity equation with the general differential equation for quasi-geostrophic flow. In previous studies,

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the classical eigenvalue (perturbation) method was used, where the disturbances are assumed to be harmonic both in the coordinate $x$ and in time $t$, so that each fundamental harmonic is represented by a function of the type $f(y) \exp(ik(x-ct))$. The problem is then to find the permissible values of $c$ (the eigenvalues) and the function $f(y)$ (the eigenfunction) for different wave numbers $k$ and a given mean flow profile $U(y)$. Using this method it was possible to find some general properties of the disturbances related to the nature of the mean flow without actually solving the eigenvalue problem for each $k$ and $U$. It was shown that if the absolute vorticity of the mean zonal flow has extreme values in the zone considered, and increases with latitude where the mean zonal velocity $U$ is large and decreases with latitude where the mean zonal velocity $U$ is small, the linear vorticity equation permits solutions with non-vanishing real exponential time factor ($c$ is complex). In this case, disturbances of wave length shorter than a certain value will decrease exponentially with time and are therefore "damped"[1], and those longer than the critical wave length will increase exponentially, i.e., "amplify". The effect of the damped shorter disturbances is, in the mean, to transport zonal westerly momentum from regions of weaker to regions of stronger mean zonal flow, thereby accentuating the contrast in the existing zonal flow, leading to the development of a sharper jet type mean flow profile. The effect of the amplifying disturbances is in the reverse direction, flattening the existing mean flow profile.

If, on the other hand, the absolute vorticity of the mean flow does not have an extreme value, then no such eigenvalues exist for the inviscid vorticity equation. The failure of the eigenvalue method for this case is due to the assumption of a particular harmonic time dependence, which is a very restrictive assumption. If the problem is one relating merely to the stability or instability of steady mean flows which can be established in a real fluid, it is natural to take viscosity into consideration, whereupon the difficulty is removed and definite results can again be obtained by the eigenvalue method. Thus in the case mentioned above, most of the small disturbances will be damped in a flow of very high Reynolds number.

However, the inclusion of viscosity seems somewhat artificial when treating the large-scale motions of the atmosphere, since the Reynolds number is very large and the motion of a nonviscous fluid should also remain determinate. Furthermore, although a mean zonal current almost always exists, large disturbances are also invariably present; a purely zonal steady state flow has rarely been observed. Thus, in considering the production of zonal motions by large disturbances, we are confronted by an inherently nonlinear process, therefore, the question of the validity of the perturbation method arises in addition to the difficulties in its application mentioned above. Furthermore, in discussing this process, we are less concerned with the stability or instability of the existing mean flow, which are difficult to define when the disturbances are finite, than with the effects of the interactions between the disturbances and the mean flow or the earth's rotation. In view of these difficulties confronting the perturbation method, it becomes most desirable to approach this problem by another method.

The method to be used in the present investigation is to assume the existance of a mean flow and of certain disturbances and to compute the subsequent changes from the nonlinear vorticity equation. This method is similar to the one used by TAYLOR and GREEN (1937) in their study of the production of small eddies. The same method has also been used by PLATZMANN (1952) in a recent paper. In principle, the general solution corresponding to a given initial condition can be found by repeating the process and expanding the quantities in a Taylor's series in $t$. However, there is the question of convergence of the series, and it is found that the labor involved in obtaining the higher derivatives becomes prohibitive even for very simple initial disturbances. We must therefore be satisfied with only one or two terms in the expansion. But since we are not particularly interested in the

[1] The term "damped" and "amplified" are used, as in the previous investigations on horizontal non-divergent motions, merely to describe the interactions between the horizontal disturbances and the mean flow or the earth's rotation; they need not be identified with actual changes of any disturbance in the atmosphere, since these disturbances may be produced by the factors that have been neglected.

instantaneous rate of change, but in the end effect of the interaction, the question arises as to whether the single term represents only a transient effect.

Still another difficulty arises in the general interpretation of the results obtained from a specific model. Since the behavior of a disturbance is directly and continuously reflected in the form of the disturbance, the result obtained for any one disturbance is valid only for that disturbance. This difficulty is absent in some measure in the eigenvalue solutions, as both the time dependence of the solutions and the forms of the fundamental modes of the disturbances are determined by the eigenvalues and the eigenfunctions, so that all effects can be attributed to the character of the mean flow and to the scale of the disturbance. This is precisely what we would like to do in our present study. We must therefore attempt to eliminate those effects that are the result of the particular form of the initial disturbance.

One way of eliminating the effects that depend mainly on the initial form of the disturbance is to consider arbitrary or random disturbances, which may be expressed as a Fourier series in $x$ and $y$. So far as the effect of earth's rotation is concerned, we are able to extend the results to the general initial disturbance and demonstrate that whenever the disturbance is not represented by the exactly neutral solution of the vorticity equation, it always produces a mean northward transport of zonal momentum in almost all parts of the belt (northern hemisphere) except near the northern boundary, where a reverse transport may occur. On the other hand, no such generalization is possible regarding the interaction between the disturbance and the mean flow, which actually depends both on the form of the disturbance and on the type of mean flow, and therefore may change direction. We therefore choose only one simple initial wave-disturbance for the study of this effect, which initially does not produce a mean momentum transport. However, we shall take it to be different from the exactly neutral disturbances if such disturbances exist, either by taking a different amplitude function or by taking an arbitrary wave length so that some change will be produced later. Since all these initial changes are produced by the linear terms of the vorticity equation, the results obtained should be comparable with those obtained from the linear vorticity equation, at least qualitatively. It should be emphatically remarked that as no energy source is included in the simple vorticity equation, we merely assume the existence of an arbitrary disturbance rather than to identify it as given by the neutral solutions of this equation.

## 2. The production of mean zonal currents on a rotating earth.

Let us assume that in the atmosphere there exist disturbances at the initial moment, presumably produced through some external processes such as differential heating or the release of potential energy but which are now left to be controlled by the simple vorticity equation for nondivergent, barotropic, horizontal flow.

For motions over a spherical earth this controlling equation may be written

$$\frac{\partial \zeta}{\partial t} + \frac{u}{R \cos \varphi} \frac{\partial \zeta}{\partial \lambda} + \frac{v}{R} \frac{\partial}{\partial \varphi} (\zeta + f) = 0 \quad (2.1)$$

where $\lambda$ and $\varphi$ are the longitude and latitude, $f = 2\omega \sin \varphi$ is the Coriolis parameter, $R$ the radius of the earth, $u$ and $v$ are the linear west—east and south—north velocity components and $\zeta$ is the vertical component of the vorticity of the motion, respectively. For this motion, a stream function $\psi$ can be introduced, defined by

$$\left. \begin{aligned} u = R\dot{\lambda} \cos \varphi &= -R\frac{\partial \psi}{\partial \varphi}, \\ v \cos \varphi = R\dot{\varphi} \cos\varphi &= R\frac{\partial \psi}{\partial \lambda} \end{aligned} \right\} \quad (2.2)$$

where $\dot{\lambda}$ and $\dot{\Phi}$ are the velocity components in angular measure. In terms of this stream function and the new independent variable $\eta = \sin \varphi$, the vorticity equation (2.1) takes the symmetrical form

$$\frac{\partial \nabla^2 \psi}{\partial t} = \frac{\partial \psi}{\partial \eta} \frac{\partial \nabla^2 \psi}{\partial \lambda} - \frac{\partial \psi}{\partial \lambda} \frac{\partial (\nabla^2 \psi + f)}{\partial \eta} \quad (2.3)$$

where the symbol $\nabla^2$ denotes the two-dimensional Laplacian operator in spherical coordinates and $\nabla^2 \psi$ is proportional to the relative vorticity,

$$\nabla^2\psi \equiv \frac{1}{\cos^2\varphi}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right] \equiv R^2\zeta \tag{2.4}$$

To avoid complications, we at first study the simple initial disturbance with only one wave-component and north—south trough- and ridgelines, such that no mean momentum transfer is produced at the initial moment, and then extend the treatment to the more general disturbance with several wave-components and with tilted troughs and ridges, i.e., to the most general disturbance represented by

$$\psi = \sum_{m=0}^{\infty} H_m(\eta, t)e^{im\lambda} \tag{2.5}$$

where $H$ is generally complex.

Since the amplitude of the waves varies, we shall now consider the proper boundary conditions that should be applied to the motion.

From symmetry considerations, it seems natural to assume that there is no momentum flux across the equator, which can be satisfied by requiring either $u$ or $v$ to be zero. We shall assume $v$ is to vanish, which requires the vanishing of the amplitude functions identically along the equator. We note that the parts of the north—south velocity from the wave-components with $m$ larger than 1 must be zero at the pole, since the corresponding components of the stream function are symmetric about this point. For the component $m=1$, the corresponding stream function is not symmetric about the pole, and there is one stream line which cuts through this point, therefore $v$ is finite. This represents a single vortex centered off the pole. Therefore, we must require that the Fourier-component of $v$ with $m=1$ to be finite and all the other components with $m \geq 2$ to vanish at the pole, if this point is to be included in the region of consideration. This requires the functions $(1-\eta^2)^{-\frac{1}{2}} H_m(\eta, t)$ to be finite for $m=1$ and to be zero for all $m \geq 2$. In this section, we shall concentrate on the effect of the earth's rotation, and therefore assume that there is no basic current.

a. *A simple wave disturbance*. We first discuss the motion given by

$$\psi_0 = aF_m(\eta) \sin m\lambda \tag{2.6}$$

where $F_m(\eta)$ satisfies the boundary conditions. Substituting in (2.3), we get

$$\nabla^2\frac{\partial\psi}{\partial t} = -2\omega maF(\eta)\cos m\lambda + \frac{ma^2}{2}\,w(\eta)\sin 2m\lambda \tag{2.7}$$

where

$$w(\eta) \equiv \frac{dF}{\partial\eta}\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{dF}{d\eta}\right] - F\frac{d^2}{d\eta^2}\left[(1-\eta^2)\frac{dF}{d\eta}\right] + \frac{2m^2}{(1-\eta^2)^2}\eta F^2$$

Therefore, the solution $\frac{\partial\psi}{\partial t}$ must be of the following form

$$\frac{\partial\varphi}{\partial t} = 2\omega ma\gamma(\eta)\cos m\lambda + \frac{ma^2}{2}Z(\eta)\sin 2m\lambda \tag{2.8}$$

where $\gamma$ and $Z$ satisfy the differential equations

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d\gamma}{d\eta}\right] - \frac{m^2}{1-\eta^2}\gamma = -F \tag{2.9 a}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dZ}{d\eta}\right] - \frac{4m^2}{1-\eta^2}Z = W \tag{2.9 b}$$

and the boundary conditions

$$\left.\begin{aligned}(1-\eta^2)^{-\frac{1}{2}}\gamma = 0,\ (1-\eta^2)^{-\frac{1}{2}}Z = 0\\ \text{for } \eta=0 \text{ and } \eta=1\end{aligned}\right\} \tag{2.9 c}$$

Substituting the solution (2.8) in the expression for $\frac{\partial\overline{(uv)}}{\partial t}$ we find

$$\left.\begin{aligned}\frac{\partial\overline{(uv)}}{\partial t} &= -R^2\left(\overline{\frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial t\partial\lambda}} + \overline{\frac{\partial\psi}{\partial\lambda}\frac{\partial^2\psi}{\partial t\partial\eta}}\right) = \\ &= \frac{R^2m^2a^2}{2}\left[\gamma\frac{dF}{d\eta} - F\frac{d\gamma}{d\eta}\right]\end{aligned}\right\} \tag{2.10}$$

where the bars denote averages around the latitude circles. It is seen that $\frac{\partial\overline{(uv)}}{\partial t}$ depends only on $\gamma$ and $F$, that is, depends upon the interaction between the disturbance and earth's rotation, as represented by the term $v\frac{df}{Rd\varphi}$ in eq. (2.1). Since $Z$ does not contribute to $\frac{\partial\overline{(uv)}}{dt}$, we shall find $\gamma$ only. In order to deal with an arbitrary $F$, it is convenient to use Green's method, writing $\gamma$ in the form

$$\gamma = \int_0^1 G(\eta, \xi) F(\xi) d\xi \qquad (2.11)$$

where $G$ is the Green's function, satisfying the homogeneous part of (2.9 a) and the boundary conditions and has a jump of the first derivative for $\eta = \xi$. It is given by

$$G(\eta, \xi) = \frac{1}{2m}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{m}{2}}\left[\left(\frac{1+\eta}{1-\eta}\right)^{\frac{m}{2}} - \left(\frac{1-\eta}{1+\eta}\right)^{\frac{m}{2}}\right] \quad \text{for } \eta \leq \xi$$

$$= \frac{1}{2m}\left(\frac{1-\eta}{1+\eta}\right)^{\frac{m}{2}}\left[\left(\frac{1+\xi}{1-\xi}\right)^{\frac{m}{2}} - \left(\frac{1-\xi}{1+\xi}\right)^{\frac{m}{2}}\right] \quad \text{for } \eta \geq \xi \qquad (2.12)$$

Because of the particular form of the Laplacian operator (2.4), it is sometimes convenient to express $F_m(\eta)$ in terms of spherical harmonics. It is evident that any sum or product of the functions

$$F(\eta) = a p_n^{n-1}(\eta) = a_1 \eta (1-\eta^2)^{\frac{n-1}{2}} \qquad (2.13)$$

satisfies our boundary conditions. For simplicity, we shall simply take (2.13) as the amplitude function of the initial disturbance (2.6), and leave the more general case for treatment in section (c). Substituting in (2.11) and then in (2.10), we find

$$\frac{\partial(\overline{uv})}{\partial t} = \frac{R^2 a^2 m^2 \omega}{2}(1-\eta^2)^{\frac{n-3}{2}} \cdot \left\{ (1 + m\eta - n\eta^2) \cdot \left(\frac{1-\eta}{1+\eta}\right)^{\frac{m}{2}}\left[\int_0^\eta \left(\frac{1+\xi}{1-\xi}\right)^{\frac{m}{2}} p_n^{n-1}(\xi) d\xi - \int_0^1 \left(\frac{1-\xi}{1+\xi}\right)^{\frac{m}{2}} p_n^{n-1}(\xi) d\xi\right] + (1 - m\eta - n\eta^2) \cdot \left(\frac{1+\eta}{1-\eta}\right)^{\frac{m}{2}} \int_\eta^1 \left(\frac{1-\xi}{1+\xi}\right)^{\frac{m}{2}} p_n^{n-1}(\xi) d\xi \right\} \qquad (2.14)$$

It may be mentioned that disturbances with their relative vorticity proportional to the stream function, i.e., $\triangle^2 \psi = -K\psi$, are exactly neutral. For these disturbances, $w$ and therefore also $Z$ are identically zero and $\gamma$ is a constant multiple of the initial amplitude function which is given by $p_n^m(\eta)$. These disturbances merely rotate around the pole without change of shape, and no mean transport of momentum is produced by them at any time.[1] Thus the case with $m = n - 1$ belongs to this group of exactly neutral disturbances. We note that if we take a sum or product of the functions of eq. (2.13) as the amplitude function, no disturbance could be exactly neutral.

For all values of $m \neq n - 1$, (2.14) gives a positive $\frac{\partial \overline{uv}}{\partial t}$ at every point. It increases from zero at the equator to a positive maximum somewhere south of $\varphi = 45°$ and decreases to zero toward the pole. The value of this function for $m = 4$ and $n = 3$, when divided by the constant factor $4\bar{v}^2 U/D$ to make it non-dimensional, is represented by the dotted curve in Fig. 1.

Similar results may be obtained for disturbances with amplitude functions different from that given by (2.13), and it is found that if the maximum is stronger, then a slight southward transport in the region near the pole will occur, which also appears to be verified observationally.

*b. A single vortex.* As has been mentioned before, a requirement of $v = 0$ at the pole is not justified for the motion of a single vortex centered off the pole, or a wave motion of $m = 1$. Since this motion is also of some interest, we shall analyze its effect on the mean momentum transfer. As the motion represented by $p_2^1(\eta) \sin \lambda$ is exactly neutral, we shall take a different amplitude function. The simplest initial disturbance which satisfies the conditions that $v = 0$ at the equator and gives a finite $v$ at the pole and which is not exactly neutral is

$$\psi_0 = a\eta^2(1-\eta^2)^{\frac{1}{2}} \sin \lambda \qquad (2.15)$$

Substituting in (2.10) we find

$$\frac{\partial(\overline{uv})}{\partial t} = \frac{R^2 a^2}{60}\frac{\omega\eta}{1+\eta}\left\{\left(2 + \frac{1}{2}\eta - 3\eta^2\right) \cdot \right.$$

[1] For discussions of these exactly neutral disturbances see Neamtan (1946) and other papers.

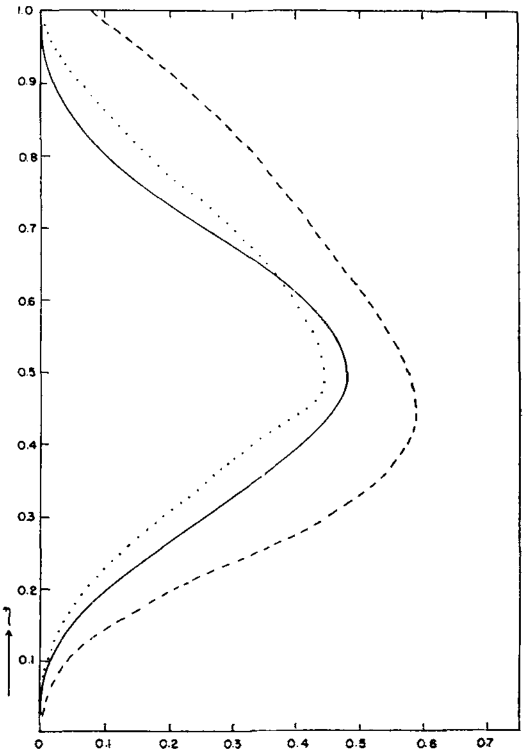


Fig. 1. Theoretical distribution of $\overline{uv}$, dashed curve corresponds to two sets of disturbances for profile (b), shifted a disturbance D/3. Full curve is that given by (3.3a) and the dotted curve is given by (2.14).

$$\left.\begin{aligned}&\cdot(10\eta^3+15\eta^4+6\eta^5-1)+\left(2-\frac{1}{2}\eta-3\eta^2\right)\cdot\\&\cdot(1+\eta)(1+\eta+\eta^2-9\eta^3-9\eta^4)\end{aligned}\right\} \quad (2.16)$$

which is also positive for all values of $\eta$, and has its maximum around 45°.

c. *Disturbances limited between two parallels of latitude.* Although the preceding results are obtained under the assumption that $v$ vanishes identically at the equator and at the pole but not at any other latitudes, similar results will be obtained whenever the amplitude of the disturbance varies with latitude, which may be considered as determined by the supply of energy to the disturbances in the atmosphere. Thus we shall investigate the disturbances that are limited to some other belt, and demand the vanishing of $v$ along two latitude circles $\varphi_1$ and $\varphi_2$. To satisfy these boundary conditions for unspecified positions of these parallels, it is more convenient to use the new coordinate system

$$x=\lambda,\ y=\log(\tan\varphi+\sec\varphi) \quad (2.17)$$

Then, after multiplying by $\cos^2\varphi$, the vorticity equation (2.3) becomes

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=-2\omega\,\mathrm{sech}^2 y\frac{\partial\psi}{\partial x}+\\&+\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}\end{aligned}\right\} \quad (2.18)$$

It is seen that the Laplacian operator on the left side has the same form as in Cartesian coordinates. The symmetry of the last two terms on the right has been preserved by using the operator $\nabla^2$ given by (2.4). We note that the earth rotation term takes the form

$$2\omega\cos^2\Phi\frac{\partial\psi}{\partial x}=2\omega\sec h^2 y\frac{\partial\psi}{\partial x}$$

in this equation. The origin of $y$ can be changed to latitude $\varphi_1$, by changing $\mathrm{sech}^2 y$ into $\mathrm{sech}^2(y+y_1)$. Then the boundary conditions will be $v=0$ for $y=0$ and $y=y_2$, corresponding to $\varphi_1$ and $\varphi_2$. We also assume the initial disturbance to be a single wave of the type (2.6). Then $\frac{\partial\psi}{\partial t}$ takes the same form (2.8), while now $\gamma$ satisfies the differential equation

$$\frac{d^2\gamma}{\partial y^2}-m^2\gamma=-F(y)\sec h^2(y+y_1)=-R(y) \quad (2.19)$$

Both $\gamma$ and $F$ should vanish at $y=0$ and $y=y_2$. Here we assume $y_2$ to be finite, i.e., $\varphi_2<90°$. Since $Z$ does not contribute to the change of the mean momentum transfer, we shall not discuss it.

We shall now specify the amplitude variations by assuming that $F$ has its maximum near the latitude $y=y_2/2$. Thus we put

$$F(y)=A(y)\sum_{n=1}^{N}b_n\Theta^n$$

where $\Theta=\sin\frac{\pi y}{y_2}$ and $A(y)$ is a monotone function of $y$. Each term of $F$ has its maximum near $y=y_2/2$ and satisfies the boundary conditions. As the term in the vorticity equation is linear in $\psi$, the effect of the disturbance with this amplitude function is similar to that given by a single term of it. Therefore we shall simply put $F$ to be proportional to $\Theta^n$. To simplify the integration, we take $A(y) = cs^2(y + y_1)$, so that $R(y) = \Theta^n$. Then $\gamma$ is given by

$$\gamma = \sum_{r=1}^{n} c_r \Theta^r + C_0 \left( 1 - \frac{csm(y - y_{2/2})}{cs\frac{\mu\pi}{2}} \right) \quad (2.20)$$

where $\mu = \frac{my_2}{\pi}$ and $cs$ stands for the hyperbolic cosine, and the coefficients are given by

$$c_{n-2l} = \left(\frac{y_2}{\pi}\right)^2 \cdot \frac{1}{n^2 + \mu^2} \cdot \frac{1}{(n-2)^2 + \mu^2} \cdots$$

$$\frac{1}{(n-2l) + \mu^2} \frac{n!}{(n-2l)!}$$

$$c_{n-2l-1} = 0$$

When $n$ is even, $c_0$ is given by the upper expression by putting $2l = n$; when $n$ is odd, $c_0 = 0$.

Substituting $\gamma$ in the expression for $\frac{\partial \overline{uv}}{\partial t}$ we find

$$\cos^4 \varphi \frac{\partial \overline{uv}}{\partial t} = R^2 m^2 a^2 \omega (M_1 + M_2) \quad (2.21)$$

where

$$M_1 \equiv \Theta^{n-1} \left\{ \frac{\pi}{y_2} \cos \frac{\pi y}{y_2} \sum_{r=1}^{n} (n-r) c_r \Theta^r + \frac{co}{cs\frac{\mu\pi}{2}} \cdot \right.$$

$$\cdot \left[ \frac{n\pi}{y_2} \cos \frac{\pi y}{y_2} \left( cs\frac{\mu\pi}{2} - csm\left(y - \frac{1}{2}y_2\right) \right) + \right.$$

$$\left. \left. + m\Theta Snm\left(y - \frac{1}{2}y_2\right) \right] \right\}$$

$$M_2 \equiv 2 \sin \varphi \Theta^n \gamma \quad (2.22)$$

Similar results are obtained if we choose $A = 1$; then $M_2$ will be larger. It can be shown that $M_1$ is positive for $y < y_2/2$ and negative for $y > y_2/2$, therefore, has the effect of producing a westerly current in the central part of the belt and easterly currents in the outer parts. On the other hand $M_2$ is positive for all $y$ and has its maximum near $y = y_2/2$, therefore produces a westerly current north of this latitude and an easterly current to the south. It can also be seen that for smaller values of $n$, $M_2$ is much larger than $M_1$. For example, $M_1$ disappears when $n = 1$. As $n$ increases, or as the maximum of the amplitude becomes more pronounced, $M_1$ becomes larger than $M_2$. The sum of these two terms gives a northward transport of momentum for almost all $y$ except near the northern boundary, where a reverse transport may result if $M_1$ is larger than $M_2$. The value of the transfer corresponding to $n = 1$ ($M_1 = 0$), with the factors sin $\varphi$ and cos $\varphi$ omitted, is plotted in Fig. 1, as the fully drawn curve. The case for $n > 1$ is something like those given by the curves $b$, $c$, or $d$ in Fig. 9.

These examples show that whenever the disturbance is not exactly neutral, the interaction between it and the earth's rotation produces a northward transport of momentum, except at very high latitudes where a reverse transport may occur. The maximum transport always occurs somewhere south of the mid-point of the belt in northern hemisphere. Figuratively, it may be said that this interaction tends to tilt the trough- and ridge-lines in a SE—NE direction in almost the entire region except near the northern boundary where they may take a SE—NW direction.

d. Disturbances with more wave-components. We now discuss the disturbance represented by

$$\psi_0 = a_l F_l(\eta) \sin l\lambda + a_m F_m(\eta) \sin m\lambda \quad (2.23)$$

where $l$ and $m$ are different integers. Substituting in (2.3) it can be seen that $\frac{\partial \psi}{\partial t}$ must take the form

$$\frac{\partial \psi}{\partial t} = 2\omega l a_l \gamma_l(\eta) \cos l\lambda + 2\omega m a_m \gamma_m(\eta) \cos m\lambda +$$

$$+ Z_l(\eta) \sin 2l\lambda + Z_m(\eta) \sin 2m\lambda + Z_3(\eta) \sin$$

$$(m + l)\lambda + Z_4(\eta) \sin (m - l)\lambda \quad (2.24)$$

The $Z$-terms are new wave-components while $\gamma$'s satisfy eq. (2.9 a) or (2.19) with the proper $F$-function on the right. Substituting in the expression for $\frac{\partial \overline{uv}}{dt}$, we get

$$\left. \begin{aligned} \frac{\partial (\overline{uv})}{\partial t} &= R^2 l^2 a^2{}_l \omega \left( \gamma_l \frac{dF_l}{d\eta} - F_l \frac{d\gamma_l}{dy} \right) + \\ &+ R^2 m^2 a^2{}_m \omega \left( \gamma_m \frac{dF_m}{d\eta} - F_m \frac{d\gamma_m}{d\eta} \right) \end{aligned} \right\} \quad (2.25)$$

It is seen that each wave-component produces its own momentum transfer because of the earth's rotation, if it is not exactly neutral. Thus the results obtained before can be extended to disturbances with any number of wave components.

e. Initial disturbances with tilted troughs. If this effect of the earth's rotation is always in the same direction regardless of the initial tilt of the disturbances, then we may say that it is not a temporary effect. Thus we shall consider the initial disturbance

$$\psi_0 = a_1 F_1(\eta) \sin m\lambda + a_2 F_2(\eta) \cos m\lambda \quad (2.26)$$

with $F_1 \neq k F_2$. For this disturbance, $\frac{\partial \psi}{\partial t}$ takes the form

$$\frac{\partial \psi}{\partial t} = 2m\lambda[a_1\gamma_1(\eta) \cos m\lambda - a_2\gamma_2(\omega) \sin m\lambda] + \\ + Z_1 \sin 2m\lambda + Z_2 \cos 2m\lambda + Z_3$$

in which $\gamma_1$ and $\gamma_2$ satisfy eq. (2.19), with the proper $F$. Substituting in (2.10) we find

$$\frac{1}{R^2 m^2 \omega} \frac{\partial \overline{uv}}{\partial t} = a_1^2\left(\gamma_1 \frac{dF_1}{d\eta} - F_1 \frac{d\gamma_1}{\partial \eta}\right) + \\ + a_2^2\left(\gamma_2 \frac{dF_2}{d\eta} - F_2 \frac{d\gamma_2}{d\eta}\right)$$

Since $\gamma_2$ is related to $F_2$ in the same way as $\gamma_1$ is related to $F_1$, we see that both the sine and the cosine components produce an increase of the transport irrespective of the initial transport. Thus our result also applies to the most general disturbance given by (2.5), and the effect will persist.

### 3. Modification of the mean zonal current

Let us now assume that there exists a mean zonal current $U = u(y)$ and study its interaction with the disturbance. Writing the streamfunction as $\overline{\psi}(y) + \psi(x, y, t)$ and $u$ for $-\frac{1}{R \cos^2 \varphi} \frac{\partial \overline{\psi}}{\partial y}$, it can be seen that the terms in (2.18) depending on the mean flow are given by $-u\frac{\partial \nabla^2 \psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - 2 \sin \varphi u\right)$, where $\nabla^2$ now stands for the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In the following, we shall neglect the term $2 \sin \varphi \cdot u$ and the variation of $\cos \varphi$ in the nonlinear terms, and consider the coordinates $x$, $y$ as has been multiplied by $R$. Equation (2.18) then beeomes

$$\left.\begin{aligned} \frac{\partial \nabla^2 \psi}{dt} &= -u\frac{\partial \nabla^2 \psi}{\partial x} - (f' - u'')\frac{\partial \psi}{\partial x} + \\ &+ \frac{1}{\cos^2 \varphi}\left(\frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y}\right)\end{aligned}\right\} \quad (3.1)$$

where the primes denote differentiation with respect to $y$. This equation has the same form as the equation for plane motion, except for $f'$. Since it is only an approximation to (2.18), we may still take $f'$ as given by $\frac{2\omega}{R}\cos^2 \varphi$. In what follows we shall investigate the interactions between different mean currents and the initial disturbance

$$\psi_0 = a \sin ly \sin kx \quad (3.2)$$

where $l = \pi/D$ and $D$ is the width of the belt. As the general effect of earth's rotation is independent of the presence of the mean zonal current and is already known, we shall find only its expression for this disturbance, by putting $f' = \frac{\omega}{R}(1 + \cos 2\varphi) = \frac{\omega}{R}(1 + \alpha \cos ly)$, with $\alpha < 1$. The increase of the mean momentum transfer produced by this effect is then equal to

$$\frac{\partial \overline{uv}}{dt} = \frac{\alpha\omega}{2R}\frac{k^2 a^2}{l}\frac{1}{\mu^2 + 4}\sin^3 ly \quad (3.3\text{ a})$$

which is positive everywhere and is very similar to that given by (2.14). It may be noted that if the amplitude of the disturbance is the $n$th power of $\sin ly$, then an antisymmetric part will also be present in this $\frac{\partial \overline{uv}}{\partial t}$, which originates from the constant part of $f'$ and is given by $M_1$ of (2.22). From (3.3 a) we find

$$\frac{\partial^2 u}{\partial t^2} = -\frac{3\alpha\omega}{R}\frac{k^2 a^2}{4(\mu^2 + 4)}\sin ly \sin 2ly \quad (3.3\text{ b})$$

These quantities may be added to the corresponding changes due to the mean current.

For the particular disturbance (3.2), the nonlinear terms on the right side of (3.1) disappear and the solution $\frac{\partial \psi}{\partial t}$ takes on the form

$$\frac{\partial \psi}{\partial t} = ak\gamma(y) \cos kx$$

where $\gamma$ is to satisfy

$$\gamma'' - k^2\gamma = [(k^2 + l^2)u + u'' - f'] \sin ly \quad (3.4)$$

and the conditions $\gamma(0) = \gamma(D) = 0$.

The rate of increase of the mean momentum transfer is given by

$$\frac{\partial(\overline{uv})}{dt} = \frac{a^2k^2}{2} (l\gamma \cos ly - \gamma' \sin ly) \quad (3.5)$$

From this we get the second derivative of the mean zonal current

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2(\overline{uv})}{\partial y \partial t} = \frac{a^2k^2}{2} \sin ly(\gamma'' + l^2\gamma) \quad (3.6)$$

Since the present problem is related to the stability of the mean flow, we may also write down the energy equation for the mean flow. Since $u_t = 0$ initially, $E_t = 0$ and

$$E_{tt} = \int_0^D uu_{tt}dy \quad (3.7)$$

Thus the disturbance might be said to be "amplified" when $E_{tt}$ is negative, "damped" when $E_{tt}$ is positive and neutral when $E_{tt}$ is zero. All these quantities are related to each other and their determination depends upon the solution of the vorticity equation. However, it can be shown that $u_{tt}$ not only occur in $E_{tt}$, but also in the higher derivatives. This can be seen from the increment of the kinetic energy of the mean flow relative to its initial value,

$$\left.\begin{aligned} \triangle E &= \frac{1}{2}\int_0^D (U^2 - U_0^2)dy = \\ &= \int_D^0 U_0 \triangle U dy + \frac{1}{2}\int_D^0 (\triangle U)^2 dy \end{aligned}\right\} \quad (3.8)$$

If $\triangle U$ is identified with $\frac{1}{2}t^2\overline{U}_{tt}$, it can be shown that the first term corresponds to $\frac{1}{2}t^2\overline{E}_{tt}$ in the Taylor expension of $\overline{E}$ and the second term corresponds to $\frac{1}{4!}t^4\frac{\partial^4 E}{\partial t^4}$. Thus, even when all the higher derivatives of $U$ are zero, $\frac{\partial^4 E}{\partial t^4}$ is still different from zero. Since this term is positive if $\triangle u$ is not identically equal to zero, it represents a damping effect upon the disturbance. Thus, $E_{tt}$ as a measure of the initial rate of amplification or damping of the disturbance generally underestimates the damping effect and overestimates the amplifying effect as compared to the information given by $U_{tt}$.

## 4. The mean zonal flow $U = U_0 + (U_m - U_0) \sin^2 ly$

Let us discuss this symmetrical mean zonal flow. In the expression $U_m$ is the maximum value of $U$ at the axis of the flow. The profile has two inflection points (where $U'' = 0$) midway between the axis of the flow and the walls, and also critical points (where $u'' = f'$) if $U_m$ is higher than a certain value. According to the previous investigation (Kuo, 1949), the inviscid linear vorticity equation will yield solutions with positive and negative real exponential time factors, so that there exist amplified and damped disturbances in the strict sense. We shall discuss this case in some detail to find whether the present method will also give comparable results.

When this expression for the mean flow is substituted into (3.4), the function $Y$ can be found by simple integration. Thus we find with the part depending on $f'$ omitted

$$\left.\begin{aligned} 4\gamma &= \frac{k^2 - 3l^2}{k^2 + 9l^2} A \sin 3ly - \\ &- \frac{(3k^2 - l^2)A + 4(k^2 + l^2)u_0}{k^2 + l^2} \sin ly \end{aligned}\right\} \quad (4.1)$$

The second time derivative of the mean flow, expressed in non-dimensional form, is then given by

$$U_{tt} = -\frac{k^2a^2\pi^2 A(\mu^2 - 3)}{(\mu^2 + 9)D^2} \sin ly \sin 3ly \quad (4.2)$$

where $\mu = k/l$ and $A$ is used as an abbreviation for $U_m - U_0$.

From (4.2) it may be seen that the interaction between the disturbance and the mean flow increases the mean flow in the central part of the zonal band from $ly = 60°$ to $ly = 120°$ and decreases the mean flow in the rest of the band if the wavelength $L = 2\pi/k$ of the disturbance is shorter than $2D/\sqrt{3}$. The change in the mean flow is in the reverse direction if the wavelength is longer than this critical value.

When $U_{tt}$ and the initial mean flow $U$ are substituted into (3.6) we find

$$\frac{4D}{a^2(k^2+l^2)A^2}E_{tt} = \frac{\pi^2}{2}\frac{(\mu^2-3)\mu^2}{(\mu^2+1)(\mu^2+9)} \quad (4.3)$$

Since $a^2(k^2+l^2)/4$ is the average disturbance kinetic energy per unit area, this equation with a minus sign expresses the normalized second time rate of change of kinetic energy of the perturbation. From (4.3) it will be seen that the wave length $L = 2D/\sqrt{3}$ is the neutral wave length which separates the longer amplifying disturbances from the damping shorter disturbances. It may be noted that this rate of change has a maximum negative value $(= -0.496\, A^2/D^2)$ for the wave length $L = 2.1\, D$ so that this disturbance may be termed the most unstable disturbance. In a flow of $U_m = 20$ m sec$^{-1}$, of width $D = 5 \times 10^6$ m, the most unstable disturbance will increase in energy by but 2.96 % during the first day. However, if the mean flow is stronger and contained in a narrower belt, the change may be very rapid. Thus in a flow of $U_m = 50$ m sec$^{-1}$ of width $D = 2 \times 10^6$ m, this most unstable disturbance will double its energy during the first day. It will also be noted that the earth's rotation does not contribute to $E_{tt}$ with this symmetric mean flow profile.

## 5. The higher time derivatives of the mean flow

As the kinetic energy of both the mean flow and the disturbance is finite for a finite disturbance, there must be an upper limit to the energy attainable by the disturbance when it is amplified or by the mean flow when the disturbance is damped. Whether this limit is actually attained and the rate of approach to the limit depends not only upon $\overline{U}_{tt}$ or $E_{tt}$, but upon higher derivatives as well. Thus, a discussion of the higher derivatives in one case is of interest.

Generally, in order to find $\frac{\partial^n u}{\partial t^n}$, it is necessary to compute all the time derivatives of $\psi$ up to the order $(n-1)$ which while possible in principle, would require an undue amount of labor if $n$ is large. Since $\frac{\partial^3 u}{\partial t^3}$ is zero initially let us find only $\frac{\partial^4 u}{\partial t^4}$ for the mean flow discussed in the preceding section. For simplicity, we shall take $f'$ as a constant $(=\beta)$ and put $U_0$ equal to zero. Then the fourth derivative is given by

$$\frac{4(\mu^2+9)^2D^2}{k^4a^2(\mu^2-3)A^3\pi^2}\frac{\partial^4 u}{\partial t^4} =$$

$$= \Big\{\frac{(\mu^2+9-2\beta')(3\mu^2-1-4\beta')-(\mu^2-3)(\mu^2+5)}{2(\mu^2+1)} -$$

$$- \frac{(\mu^2+9-2\beta')^2}{2(\mu^2+9)} - \frac{(\mu^2+9)(3\mu^2-1-4\beta')^2}{8(\mu^2+1)^2} -$$

$$- \frac{(\mu^2+5)(\mu^2+21)}{8(\mu^2+25)} + \frac{a^2l^2(4\mu^4-35\mu^2-207)}{(\mu^2+4)A^2}\Big\} \times$$

$$\times (\cos 4ly - \cos 2ly) + \frac{3(\mu^2+5)(\mu^2+9)}{4(\mu^2+25)} \cdot$$

$$\cdot \Big\{\frac{\mu^2+9-2\beta'}{\mu^2+9} + \frac{\mu^2+25-2\beta'}{\mu^2+25} - \frac{3\mu-1-4\beta'}{\mu^2+1} -$$

$$- \frac{4a^2l^2(2\mu^4-7\mu^2-69)}{(\mu^2+5)(\mu^2+4)A^2} \times (\cos 6ly - \cos 4ly) +$$

$$+ \frac{3(\mu^2-3)(\mu^2+5)}{4(\mu^2+25)} \times (\cos 8ly - \cos 2ly) \quad (5.1)$$

in which $\mu = k/l$ and $\beta' = \beta/l^2A$. The expression (5.1) is too complex to discuss generally. However, there are a few points worthy of mention. It may be noted that for the disturbance with $\mu^2 = 3$, these derivatives are zero so that the disturbance is neutral at least up to $\frac{\partial^4 u}{\partial t^4}$ and $\frac{\partial^4 E}{\partial t^4}$. The factor $A$ comes into this equation raised to a higher power than in the expression for $U_{tt}$, so the influence of the higher derivative is much higher with a stronger mean flow than with a weaker one. Computation shows that this fourth derivative generally has a sign opposite to that of $U_{tt}$, indicating that the time required by the system to reach the limiting condition is longer than that estimated from $U_{tt}$, or the change may

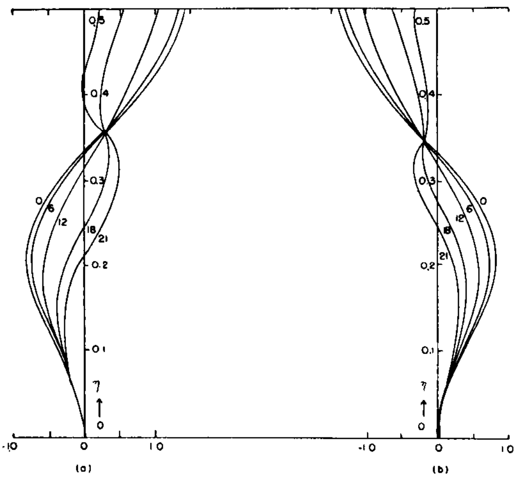


Fig. 2. The variations of $U_{tt}$ for the profile $U = A \sin^2 ly$, with $A = 20$ m sec$^{-1}$. (a) is for the damping disturbance $\mu^2 = 5$, (b) for the amplifying disturbance $\mu^2 = 1.5$.

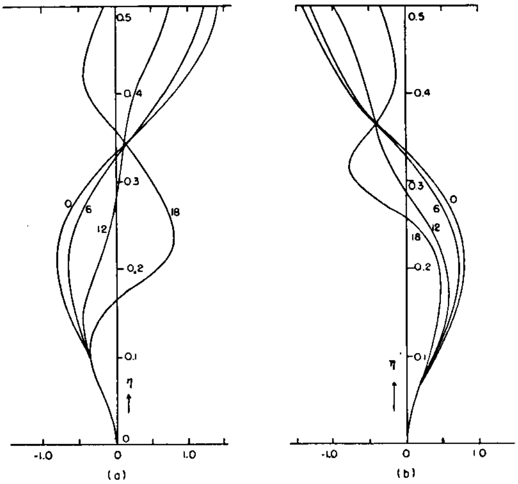


Fig. 3. The same as Fig. 2, with $A = 50$ m sec$^{-1}$.

be reversed. Since $\frac{\partial^4 u}{\partial t^4}$ is proportional to $k^4$, its effect must be small for disturbances of long wavelength.

To illustrate the effect of this fourth derivative, a few computed rates of change of the mean flow $U_{tt}$ are shown in Figures 2 and 3, expressed in units of $a^2k^2AD^{-2}$. Since the term containing $\alpha$ in (4.2) is neglected, the change is a symmetrical function of $y$ so that only half of the flow is plotted. Figure 2 shows the effects of a relatively weak mean zonal flow ($A = 20$ m sec$^{-1}$) and Figure 3 shows the effect of a stronger mean flow ($A = 50$ m sec$^{-1}$). These examples show that the parabolic extrapolation $\bar{u} - \bar{u}_0 = \frac{t^2}{2} u_{tt}$ cannot be used for time intervals longer than 24 hours.

To estimate the order of magnitude of the interaction between the disturbance and the mean flow, we may take the example plotted in Figure 2 a, which is for the damped disturbance with $\mu^2 = 5$, $al = 10$ m sec$^{-1}$, $A = 20$ m sec$^{-1}$, and $D = 5 \times 10^6$ m. Taking the mean value of $U_t$ for a time interval of one day, we find an increase of the mean zonal flow of about $2m$ sec$^{-1}$ in the middle of the stream and a decrease in the outer parts. These changes are of the same order of magnitude as those produced by the earth's rotation.

As mentioned above, the disturbance with $\mu^2 = 3$ is neutral at least up to $\frac{\partial^4 u}{\partial t^4}$ or $\frac{\partial^4 E}{\partial t^4}$. Since it is very difficult to find the higher derivatives, it appears desirable to examine the exactly neutral disturbance from another point of view, if one can be found. For the present case, let us consider the total stream function

$$\left.\begin{aligned} \varphi = -\left(U_0 + \frac{A}{2}\right) y + \frac{A}{2l} \sin 2ly + \\ + a \sin k(x - ct) \sin ly \end{aligned}\right| \quad (5.2)$$

which represents a system of alternating cyclonic and anticyclonic vortices embedded in a sine-curve mean zonal flow, propagated with a phase velocity $c$ without changing shape, and therefore is exactly neutral. At $t = 0$, this reduces to the initial motion we have been discussing. In order that this stream function satisfy the vorticity equation (3.1) with $f' = \beta$, the wave number $k$ and the phase velocity $c$ must satisfy the following relations:

$$k^2 = 3 l^2 \quad (5.3\text{ a})$$

$$c = U_0 + \frac{A}{2} - \frac{\beta}{k^2 + l^2} \quad (5.3\text{b})$$

It is seen that the wavelength and phase velocity of this neutral disturbance are determined by the mean zonal current and the width of the belt, so that it cannot be considered as representing a continuous spectrum of neutral disturbances unless $A$ is zero, or unless the mean flow is uniform. For the latter case, the condition (5.3 a) is unnecessary.

Since the stream function (5.2) reduces to the initial flow we have discussed, it may be concluded that the disturbance of the type (3.2) with $\mu^2 = 3$ is exactly neutral for this mean zonal flow if the variable part in $f'$ is neglected.

## 6. The mean currents $U = U_m \sin ly$ and $U = U_m \sin^4 ly$

The simple sine profile $U = U_m \sin ly$ has no inflection point within the walls, therefore is stable for small disturbances according to the linear theory. With this mean flow, the solution $Y$ of (3.4) is

$$Y = \frac{2U_m}{\mu^2+4}\left[\frac{Snky + Snk(D-y)}{SnkD} - 1 - \frac{\mu^2}{2}\sin^2 ly\right] \tag{6.1}$$

Substituting into (3.6), we obtain

$$\frac{D^2 U_{tt}}{k^2 a^2 U_m} = \frac{\pi^2(\mu^2+1)}{\mu^2+4} \cdot \sin ly\left[\frac{Snky + Snk(D-y)}{SnkD} - 1 + \frac{3\mu^2 \sin^2 ly}{2(\mu^2+1)}\right] \tag{6.2}$$

It may be seen that for any wavelength, this expression is positive in the central part of the zonal belt and negative in the outer part, indicating that the interaction between the disturbance and the mean flow is to accentuate the mean flow while the disturbance is damped. In this case, the energy integral $E_{tt}$ is given by

$$\frac{4D}{a^2(k^2+l^2)U_m^2} E_{tt} = \frac{4\pi^2\mu^2}{\mu^2+4}\left[\frac{4}{\mu(\mu^2+4)\pi}\tanh\frac{\mu\pi}{2} + \frac{9\mu^2}{16(\mu^2+1)} - \frac{1}{2}\right] \tag{6.3}$$

which is positive for all values of $k$, indicating that all the disturbances of type (3.2) are damped.

To find whether this is really the case, we might try to find an exactly neutral disturbance in this mean zonal flow. The only disturbance that can propagate with a constant phase velocity without change of shape is the one of $k = 0$, that is, only an infinitely long disturbance can remain exactly neutral and perfectly periodic. Since it has been shown that disturbances of finite wavelengths are damped, it must be concluded that all disturbances of the type (3.2) must be damped in this mean flow. This is in agrement with the result of the linear theory.

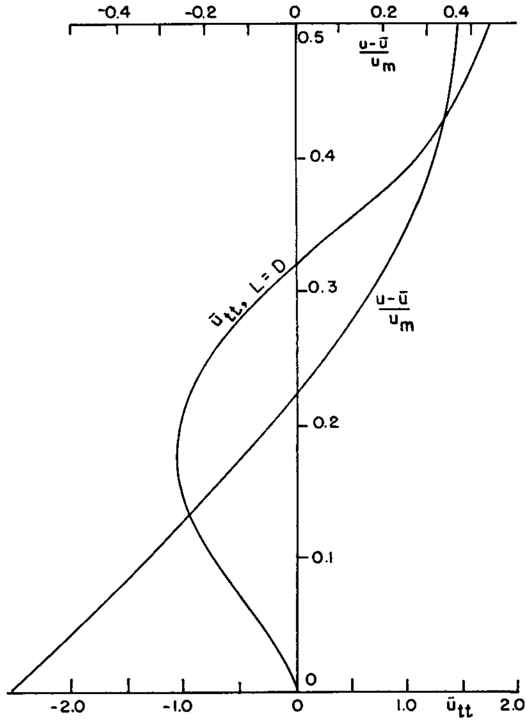


Fig. 4. The distribution of $U_{tt}$ for the mean flow $U = U_m \sin ly$.

The change of the mean zonal flow produced by the interaction between the disturbance and the mean flow, with $L = D$, is given by the curve $U_{tt}$ in Figure 4, while the other curve represents the initial mean flow expressed as the ratio $(U - \bar{U})/U_m$, where $U_m$ is the maximum and $\bar{U}$ the average value. From these two curves, it may be seen that the kinetic energy of the mean flow is increasing and the disturbance is being damped. For a disturbance with $L = D$, the relative second derivative of the kinetic energy given by (6.3) is equal to $0.5776\ U_m^2/D^2$, so that for $U_m = 20$ m sec$^{-1}$ and $L = D = 5 \times 10^6$ m (about 45 degrees of latitude) the kinetic energy changes by 3.45 % during the first day.

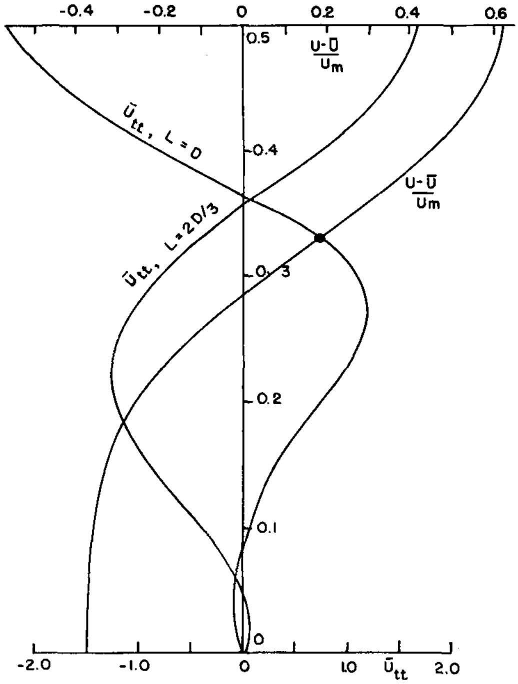


Fig. 5. The distribution of $U_{tt}$ for the mean flow $U = U_m \sin^4 ly$.

As a further example of a sine-curve profile, we consider the mean flow $U = U_m \sin^4 ly$, which has two inflection points at $y = D/3$ and $y = 2D/3$, and therefore may yield both amplifying and damping disturbances according to the linear theory. It can be found that for this profile $U_{tt}$ and $E_{tt}$ are given respectively by

$$\frac{D^2 U_{tt}}{k^2 a^2 U_m} = \frac{\pi^2}{4}\sin ly \cdot \left[\frac{3(\mu^2-15)}{\mu^2+25}\sin 5ly - \frac{5\mu^2-27}{\mu^2+9}\sin 3ly\right]\cdot$$

$$\frac{4D}{a^2(k^2+l^2)U_m^2}E_{tt} = \frac{\pi^2}{32}\mu^2\frac{7\mu^4+118\mu^2-945}{(\mu^2+1)(\mu^2+9)(\mu^2+25)} \quad (6.5)$$

According to $E_{tt}$, the disturbance with wavelengths longer than 0.8322 $D$ are amplified and those shorter than this value are damped. The most unstable disturbance is the one with $L = 1.7\,D$, for which the energy integral has the value $-0.5$. The changes of the mean flow produced by the interaction with the disturbances with $L = D$ (amplified) and $L = 2\,D/3$ (damped) are shown in Figure 5.

## 7. The polynomial profiles

If the mean zonal wind exhibits a profile of the third degree

$$U = a_0 + a_1 y + A_2 y^2 + a_3 y^3 \quad (7.1)$$

and a disturbance has the form (3.2), the solution of $Y$ in (3.4) is

$$Y = -\left[U + \frac{2(\mu^2-1)}{(\mu^2+1)^2}\frac{D^2}{\pi^2}U''\right]\sin ly - \frac{2l}{k^2+l^2}G(y)\cos ly - \frac{2l}{k^2+l^2}\frac{1}{SnkD}\cdot[G(D)Snky - G(0)Snk(D-y)] \quad (7.2)$$

where $G(y) = U'(y) + \frac{3k^2-l^2}{(k^2+l^2)^2}U'''(y)$. We shall apply this solution to three particular cases:

a. The constant shearing flow $U = U'y$ (Plane Couette flow), where $U'$ is a constant. Since the profile has no inflection point, according to the linear theory, it is stable for small disturbances. In our development,

$$\frac{D^2 U_{tt}}{k^2 a^2 U_m} = -\frac{\pi}{2}\sin 2ly + \frac{\pi}{Sn\mu\pi}\cdot[Sn\pi\mu(1-\eta) - Sn\mu\pi\eta]\sin\pi\eta \quad (7.3)$$

where $U_m = U'D$, the maximum value of $U$, $\eta = y/D$ and $\mu = k/l$. Then the energy integral $E_{tt}$ is given by

$$\frac{4DE_{tt}}{a^2(k^2+l^2)U^2{}_m} = \frac{\mu^2}{(\mu^2+1)^3}\cdot\left[(\mu^2+1)(\mu^2-3) + \frac{16}{\pi}\mu\coth\frac{\mu\pi}{2}\right] + \frac{4\alpha D^2\mu^2}{3(\mu^2+1)(\mu^2+4)U_m} \quad (7.4)$$

which is positive for all values of $\mu$, indicating that all disturbances are damped. In (7.3) it will be seen that if $U'$ is positive, the effect is to increase the zonal wind in the northern half of the belt and to decrease it in the southern half, the same direction of change as that produced by the earth's rotation. It will be noted that with this profile $\alpha$ also enters into the energy integral $E_{tt}$, increasing the kinetic energy of the mean flow. Since this profile is of small interest, no numerical example will be given.

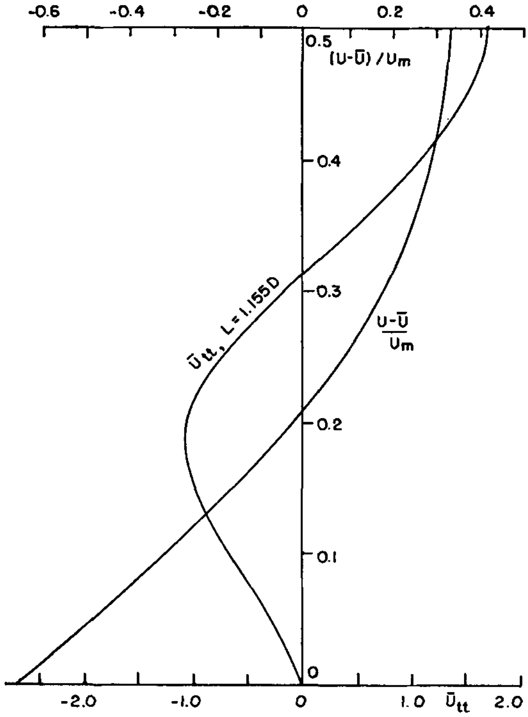


Fig. 6. The distribution of $U_{tt}$ for the mean flow $U = 4U_m\eta$ (1—$\eta$).

b. The parabolic profile $U=4U_m\eta(1-\eta)$ (Poiseuille flow). According to the linear theory, this form of profile is also stable for small disturbance, since $U''$ is equal to $-8\ U_m/D^2$ and there is no inflection point in the profile. From (7.2) we find

$$\frac{DU_{tt}}{k^2a^2U_m} = -2\pi(1-2\eta)\sin 2\pi\eta + \frac{4(\mu^2-3)}{\mu^2+1}\cdot \sin^2\pi\eta + \frac{4\pi\sin\pi\eta}{Sn\mu\pi}\left[Sn\mu\pi(1-\eta)+Sn\mu\pi\eta\right] \tag{7.5}$$

The energy integral $E_{tt}$ is given by

$$\frac{4DE_{tt}}{a^2(k^2+l^2)U_m^2} = \frac{16}{3}\left(1-\frac{6}{\pi^2}\right)\frac{\mu^2}{\mu^2+1} - \frac{64\mu^2}{3(\mu^2+1)^2}\cdot \left(1+\frac{3}{\pi^2}\right) - \frac{256}{\pi^2}\frac{\mu^2(3\mu^2-1)}{(\mu^2+1)^4} + \frac{256\mu^3}{\pi(\mu^2+1)^3}\tanh\frac{\mu\pi}{2} \tag{7.6}$$

which is positive for all values of $\mu$, so that all disturbances of the type (3.2) are damped.

The distribution of $U_{tt}$ produced by the interaction between the disturbance and the mean flow, given by (7.5), is represented in Figure 6 for $L=1.155\ D$. The relative second derivative of the energy of the disturbance given by (7.6) is 0.4875 $U_m^2/D^2$ for a disturbance of this wave length. It may be noted that this case is quite similar to that of the simple sine curve profile discussed in section 6.

c. The profile $u=27\,u_m\eta^2(1-\eta)/4$. This is an unsymmetric mean flow profile which has only one point of inflection, at $y=D/3$, as is shown in Figure 7a by the curve marked $(U-\bar{U})/U_m$. The perturbation theory shows that if the effect of the earth's rotation is neglected, or if the mean flow is strong, both damped and amplified disturbances exist, separated by a neutral disturbance. Because of the presence of a singularity in the linear vorticity equation, even the neutral disturbance is difficult to obtain, making it most desirable to study this case from the present point of view.

With this profile, we find

$$\frac{D^2U_{tt}}{k^2a^2U_m} = -\frac{27\pi}{8}\left[\eta(2-3\eta) - \frac{6}{\pi^2}\frac{3\mu^2-1}{(\mu^2+1)^2}\right]\cdot \sin 2\pi\eta - \frac{27}{4}\frac{\mu^2-3}{\mu^2+1}(1-3\eta)\sin^2\pi\eta + \frac{27\pi\sin\pi\eta}{4\ Sn\mu\pi}\left\{Sn\mu\pi\eta + \frac{6}{\pi^2}\frac{3\mu^2-1}{(\mu^2+1)^2}\cdot \left[Sn\mu\pi\eta - Sn\mu\pi(1-\eta)\right]\right\} \tag{7.7}$$

The energy integral is given by

$$\frac{4D}{k^2a^2U_m^2}\frac{\pi^2}{3^7}E_{tt} = -\frac{1}{16}\left(1-\frac{15}{2\pi^2}\right) + \frac{(\mu^2-1)\pi^2}{360(\mu^2+1)}\cdot \left[1+\frac{15}{2\pi^2}\left(1-\frac{9}{2\pi^2}\right)\right] - \frac{3\mu^2-1}{3(\mu^2+1)^3}\cdot \left[1+\frac{9(\mu^2+1)}{16\pi^2}\right] - \frac{3}{\pi^2}\frac{(3\mu^2-1)^2}{(\mu^2+1)^5} + \frac{\mu\pi}{6(\mu^2+1)^2}\cdot \left[1+\frac{6(3\mu^2-1)}{\pi^2(\mu^2+1)^2}\right]\coth\mu\pi + \frac{\mu(3\mu^2-1)}{\pi(\mu^2+1)^4}\cdot \operatorname{csch}\mu\pi + \frac{2}{\pi}\frac{\mu^2-1}{(\mu^2+1)^4}\left[1+\frac{12(3\mu^2-1)}{\pi^2(\mu^2+1)^2}\right]\cdot \coth\frac{\mu\pi}{2} + \frac{\alpha\omega D^2}{3^6R(\mu^2+4)}\left(\frac{40}{\pi}-3\right) \tag{7.8}$$

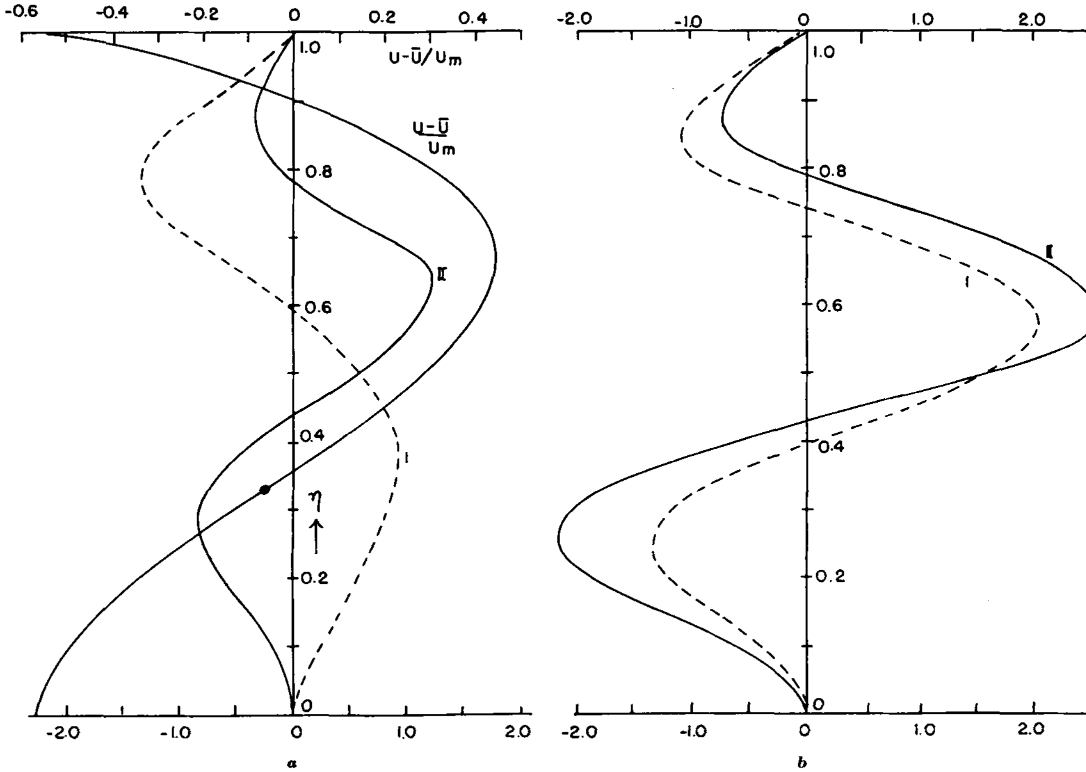


Fig. 7. The distribution of $U_{tt}$ for the mean flow $U = 27U_m\eta^2(1-\eta)/4$. (a) is for the longer disturbance $L = 3.464D$, (b) for $L = D$. The curves 1 represents the results when the earth rotation factor $\omega$ is neglected, curves 2, when the effect of $\omega$ is included.

It will be noted that the parameter $\omega$ of the earth's rotation enters into the last term of this energy integral because of the lack of symmetry of the mean flow. This term is always positive and therefore represents a damping effect.

Two examples worked out for this mean flow are plotted in Figures 7 a and 7 b. Figure 7 a is for a longer disturbance of $L = 3.464\ D$ and Figure 7 b is for a shorter disturbance of $L = D$. To demonstrate the relative importance of the mean flow and of the earth's rotation, the quantity $D^2U_{tt}/k^2a^2U_m$ has been computed both without and with the part given by (3.3 b). The results are represented by the curves 1 and 2 respectively. Curve 1 in Figure 7 a shows that when this part is neglected, the interaction between a long disturbance and the mean flow diminishes the mean flow in the northern part of the flow where $U$ is relatively high, and increases the mean flow in the southern part where $U$ is relatively low, and so abstracts kinetic energy from the mean flow and flattens it. The relative second derivative of the kinetic energy of the disturbance is $-0.1062\ U_m^2/D^2$. When the effect of the earth's rotation is included, this amplifying effect may be reversed. Thus, with $D = 5 \times 10^6 m$, $\frac{\alpha\omega}{R} = 1.0 \times 10^{-11}\ \mathrm{m}^{-1}\ \mathrm{sec}^{-1}$, and $U_m = 20$ m sec$^{-1}$, the last term on the right side of (7.8) is equal to 0.2308, so that the total relative second derivative of the kinetic energy is now equal to $0.1246\ U_m^2/D^2$. The corresponding modification of the mean flow is shown by curve 2 of Figure 7 a, which shows a general accentuation of the contrast of the mean flow for different zones.

For the shorter disturbance with $L = D$, both the mean flow and that of the earth's rotation have a damping effect so that the inclusion of the term with $\omega$ only magnifies the damping

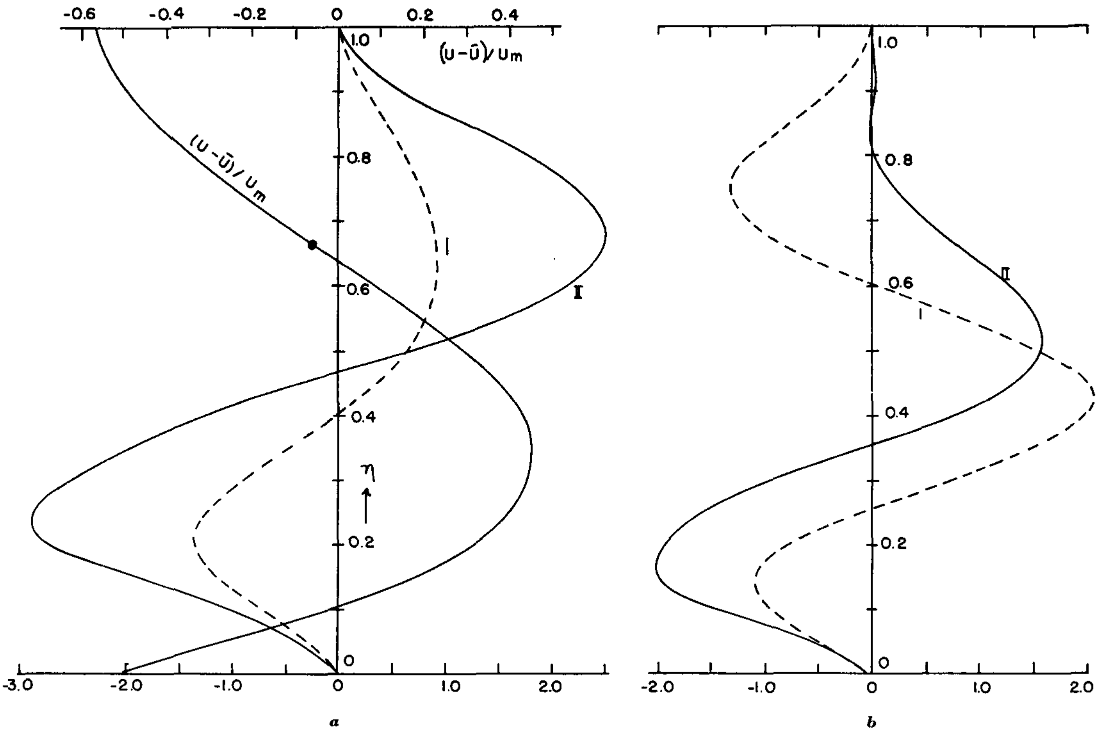


Fig. 8. Same as Fig. 7 for the mean flow $U = 27U_m\eta\ (1-\eta)^2/4$.

effect, as indicated by the curves 1 and 2 of Figure 7 b. The relative second derivative of the kinetic energy of the disturbance is 0.6955 $U_m^2/D^2$ without the parameter $\omega$ and 1.0955 $U_m^2/D^2$ with this parameter of the earth's rotation.

It should be pointed out that the earth's rotation may have an amplifying effect if the mean flow is unsymmetrical and is stronger in the southern part of the flow and weaker in the northern. To demonstrate this, let us take the mean flow $U = \frac{27U_m}{4}\eta(1-\eta)^2$ which is the image of the profile (c) about the line $\eta = \frac{1}{2}$. The change of mean flow is obtained by replacing $\eta$ in (7.7) by $1-\eta$. Figures 8 a and 8 b show the result of this substitution for $L = 3.464\ D$ and $L = D$ respectively. The energy integral is obtained from (7.8) by changing the sign of the last term containing the factor $\omega$. For these disturbances, the energy integral including the last term is $-0.337$ and 0.295 respectively.

## 8. The mean meridional transfer of zonal momentum

The mean transfer of zonal momentum produced by the large scale atmospheric disturbances has been investigated by a number of workers in recent years, and its meridional distribution is now roughly known. It will be most interesting, then to compare the theoretical results with that obtained from observational material.

In this section we shall discuss the mean momentum transfer corresponding to the disturbance (3.2) and the mean flow profiles (a) $U = U_m \sin^2 ly$, (b) $U = U_m \sin ly$, (c) $U = 27\ U_m\eta^2(1-\eta)/4$ and (d) $U = U_m \sin^4 ly$. For these four mean flow profiles, the rate of increase of the mean meridional transport of zonal momentum is given by the following equations:

$$\frac{D}{a^2k^2U_m}\frac{\partial \overline{uv}}{\partial t} = \frac{(\mu^2-3)\pi}{8(\mu^2+9)}(2\sin 2\pi\eta - \sin 4\pi\eta) \tag{8.1}$$

$$\frac{D}{a^2k^2U_m}\frac{\partial\overline{uv}}{\partial t} = \frac{\pi}{\mu^2+4}\cos\pi\eta\cdot\left\{\frac{\sinh\mu\pi\eta+\sinh\mu\pi(1-\eta)}{\sinh\mu\pi} - \frac{\mu\tan\pi\eta\,[\operatorname{csch}\mu\pi\eta - \operatorname{csch}\mu\pi(1-\eta)]}{\sinh\mu\pi} - 1 + \frac{\mu^2}{2}\sin^2\pi\eta\right\} \tag{8.2}$$

$$\begin{aligned}\frac{D}{a^2k^2U_m}\frac{\partial\overline{uv}}{\partial t} = {} & \frac{27}{4}\frac{1}{\mu^2+1}\frac{1}{\sinh\mu\pi}\cdot\Big\{\mu\sin\pi\eta\cosh\mu\pi\eta - \cos\pi\eta\sinh\mu\pi\eta + \\ & + \frac{6(3\mu^2-1)}{(\mu^2+1)^2\pi^2}[\mu\sin\pi\eta(\cosh\mu\pi\eta + \cosh\mu\pi(1-\eta)) - \cos\pi\eta\cdot \\ & \cdot(\sinh\mu\pi\eta - \sinh\mu\pi(1-\eta))]\Big\} + \frac{27\eta(2-3\eta)}{16}\cdot \\ & \cdot\left(\frac{\mu^2-3}{\mu^2+1} - \cos 2\pi\eta\right) - \frac{81}{4(\mu^2+1)^3\pi^2}\cdot \\ & \cdot[(\mu^4-1)\cos 2\pi\eta - \mu^4 + 6\mu^2 - 1] - \\ & - \frac{27(1-3\eta)}{4(\mu^2+1)\pi}\sin 2\pi\eta\end{aligned} \tag{8.3}$$

$$\begin{aligned}\frac{D}{a^2k^2U_m}\frac{\partial\overline{uv}}{\partial t} = {} & \frac{\pi}{16}\frac{\mu^2-15}{\mu^2+25}\sin 6\pi\eta - \\ & - \frac{\pi}{4}\frac{\mu^4+10\mu^2-135}{(\mu^2+9)(\mu^2+25)}\sin 4\pi\eta + \\ & + \frac{\pi}{16}\frac{5\mu^2-27}{\mu^2+9}\sin 2\pi\eta\end{aligned} \tag{8.4}$$

To each of these we should add the transport produced by earth's rotation, given by (3.3 a). For the mean flow profiles (a), (c), and (d), the nature of this interaction depends upon the wave length of the disturbance. However, since the mean zonal flow in the atmosphere is generally not very strong, this interaction represents in the mean a horizontally damping effect on the large atmospheric disturbances, therefore we shall only discuss the damped disturbances with $L = D$ for (a), (b), and (c) and $L = 2\,D/3$ for (d). It may also be pointed out that for these large-scale disturbances, this average meridional distribution of the transfer is not much effected by the occasional presence

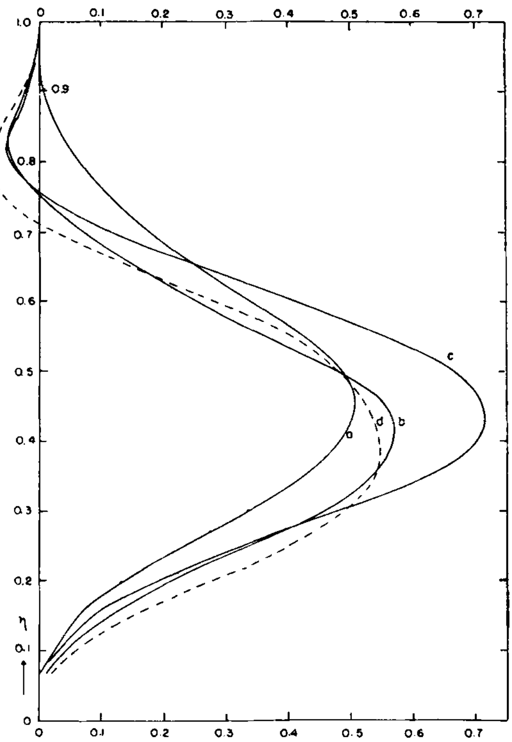


Fig. 9. Theoretical distributions of $\overline{uv}$ for the mean flow profiles (a), (b), (c), and (d).

of amplifying disturbances, as it is determined largely by the earth rotation effect. Taking $D = 6.6\times10^6$ m, $\frac{\alpha\omega}{R} = 1.1\times10^{-11}\,\mathrm{m}^{-1}\,\mathrm{sec}^{-1}$ and $U_m = 20$ m $\mathrm{sec}^{-1}$, the nondimensional rates of increase of the mean momentum transfer given by these equations are as represented by the curves (a), (b), (c), and (d) in Figure 9. Similar transports can be produced by earth rotation alone, if the disturbance is more concentrated as have been discussed before.

Since the mean transfer of zonal momentum is initially zero for the type of motion we are considering, we may take the rates of change as representative of the actual transfer and compare them with the results obtained from observations.

In Figure 10, the curve 1 is the six months average (January through June, 1950) total meridional transport of zonal momentum across unit length of the latitude circle, from earth's surface to the 100 mb level, based on

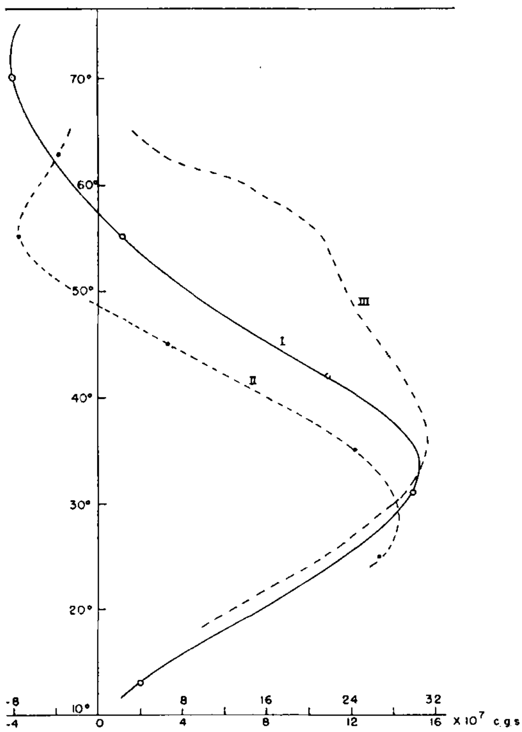


Fig. 10. Observed distribution of $\overline{uv}$: I, 6 months average (January through June, 1950). Hemispherical. II, February 1949, American Continent. II, January 1949, Hemispherical.

the results obtained by Starr and White from actual wind studies along the latitude circles 13° N, 31° N, 42° N, 55° N and 70° N (STARR and WHITE, 1951; 1952 a; 1952 b; and unpublished computations). Curve 2 is the average total transport over the American Continent in February, 1949 (STARR, 1951), and Curve 3 is obtained from Mintz's (1951) results of geostrophic transport for January, 1949. The units for the two curves 2 and 3 are twice as large as for the curve 1. If we put the maximum point of the curves in Figure 9 at the maximum points of the Curves 1 and 2 of Figure 10, it can be seen that they are in rough agreement. The magnitude of the transfer represented by the curves in Figure 9 is also correct if we take the mean north—south velocity to be about 5 to 7 m $sec^{-1}$, and let the increase take place for a period of one day. For the month of January, 1949, Mintz's mean geostrophic zonal wind profile indicates that two jet streams of westerly current were present, a stronger one around 30° N and a weaker one around 50° N. This suggests that the curve 3 in Figure 10 may be interpreted by superimposing two sets of disturbances with the one having a smaller intensity shifted about 20 latitudinal degrees to the north. Thus, if we take the curve b in Figure 9 and add another one with an intensity equal to two-thirds of that of b and shifted a distance equal to $D/3$ toward north, we get the dashed curve in Figure 1. It is seen that this curve fits the curve 3 in Figure 10 nicely. The momentum transfer produced by the interaction between the disturbance and earth's rotation alone without any mean flow, is also shown in Figure 1 by the full curve.

## 9. Summary and Conclusions

The problem of the production of mean zonal flow by atmopheric disturbances is attacked by the computation of tendencies. The investigation shows that the effect of the earth's rotation is always to produce a mean northward transport of westerly momentum and a tilt of the troughs and ridges in the SW—NE direction over almost all the regions except near the northern boundary, with the maximum transport and maximum tilting occurring south of the mid-point between the two latitudinal walls assumed. This interaction may either be a damping effect or an amplifying effect for the disturbance, depending on the asymmetry of the existing mean flow profile. For the symmetric mean flow profiles, this process always results in a slow damping of the disturbance, and appears in the fourth time derivative of the mean flow energy. From the nature and distribution of this effect, it would appear that it must have some bearing on the meridional distribution of the mean northward transfer of zonal momentum observed in the atmosphere.

In regard to the interaction between the disturbance and the mean flow, the six examples examined indicate that the presence or absence of an inflection point in the mean flow profile is an important and decisive factor, so far as the particular disturbance is concerned. Thus, in the three cases in which $U''$ is of one sign throughout, all disturbances of the type discussed are damped, while in the cases in which $U''$ changes its sign, both damped and amplified disturbances exist. The disturbances of wavelength longer than a critical wavelength abstract kinetic energy from the mean flow and flatten the profile, and are therefore amplified, while disturbances of shorter wavelength are damped, accentuating the mean flow. The three sinecurve profiles discussed above also indicate that the range of the amplifying wavelength increases as the inflection point of the profile moves away from the boundaries.

Since the mean zonal current in the atmosphere is generally not very strong, the interaction between the disturbance and the mean flow generally represents a horizontal damping process similar to the effect of earth's rotation, while the possible instability of the horizontal motion associated with a stronger mean zonal current happens only occasionally.

To compare our theoretical results with observational facts, the meridional distribution of the mean northward transfer of zonal momentum are computed by this method for four different mean velocity profiles. These examples show that the effects of earth's rotation and that of the mean flow are of the same order of magnitude for the large-scale atmospheric motions. When taken together, the combined effect is to produce a mean northward transport of zonal momentum corresponding to a general SW—NE orientation of the troughs and ridges in almost all parts of the belt except a possible reversal near the northern boundary. It is shown that if the mean north—south velocity is about 5 m $sec^{-1}$ and the increase of these transports takes place for a period of one day, then the computed transports agree roughly with the results obtained from observational material, indicating that the processes discussed are of real importance for the development of the mean zonal current.

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