A Power Law Representation of the Heat Transfer in the Lowest 100 metres of the Atmosphere

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Abstract

On the assumptions that the flux of heat in the air is governed by an interchange coefficient which can be represented by a simple power law of the height above the surface, that the flux of heat in the ground is by molecular processes and that during the daylight hours the variation of temperature can be represented by a sine function, the phase angle of the diurnal temperature wave is calculated and is shewn to be in good agreement with observations. The phase angle of the annual temperature wave is also discussed.

Calculations of K_H the coefficient of eddy conductivity of heat and of its mode of variation with height in unstable conditions suggest that it is approximately equal to K_M the coefficient of eddy viscosity.

1. Introduction

Despite the fact that the mean temperature distribution within the lowest 100 metres of the atmosphere has been described in considerable detail in a series of geophysical memoirs (JOHNSON, 1929; BEST, 1935; FLOWER, 1936; JOHNSON and HEYWOOD, 1938; BEST, KNIGHTING, PEDLOW and STORMONTH, 1950) the theoretical conclusions from these observations have been conflicting, partly owing to the complexity of the problem and lack of any clear cut picture of the physical processes involved in the mechanism of heat transfer, and partly to the effects of the different topography of the stations.

In most discussions of heat transfer in the atmosphere (cf. SUTTON, 1949), it has been assumed that molecular conduction is negligible, except within the lowest cm or so above the surface, that radiation may be neglected and that, at least in the mid period of the day, convection is mainly responsible for the transfer of heat in the vertical, although there is disagreement as to the roles played by 'free convection' due to buoyancy effects and 'forced convection' due to frictional turbulence. On the other hand, RIDER and ROBINSON (1951) have stated that in the lowest two metres above the ground the absorption of long wave radiation by the air is at least comparable in effect with convection.

In view, therefore, of the uncertainty as to the mechanism of heat transfer, Richardson's direct approach, which avoids the necessity of making any assumptions as to the nature of that mechanism, has been adopted in this paper.

RICHARDSON (1919) defined K the coefficient of eddy diffusivity for any transferable property χ of the atmosphere by

$$\frac{D\chi}{Dt}=\frac{\partial}{\partial z}\left\{K\frac{\partial \chi}{\partial z}\right\} \qquad 1.00$$

where the only limitation imposed upon χ is that χ must represent some property which is unchanged by transfer to a different level.

Whilst the coefficient K in this equation was originally called the coefficient of eddy diffusivity, this interpretation is not essential and if in heat transfer part of the temperature change at any level is due to factors other than eddy motion, equation 1.00 can still be used to represent the whole of the observed variation, although in this case K is a composite diffusion coefficient which takes into account all the causative factors.

It is evident that K as defined in this manner may be a function of χ and z and hence may vary from property to property if the transferable property under consideration influences the motion. Comparison of values of K for various transferable properties computed by means of this equation thus affords some evidence as to whether, as has been suggested by various writers on turbulent motion from Reynolds onwards, the transfer of heat momentum and water vapour is affected by the same mechanism.

In the case of controlled wind tunnel experiments PASQUILL (1942) found that the coefficients of eddy diffusivity and momentum were practically identical, but in a second paper (1949) dealing with the diffusion of water vapour in the lowest two metres above the surface he concluded that whilst under stable conditions there was equality between the coefficients of eddy diffusivity for water vapour and heat, in unstable conditions the latter was substantially greater.

RIDER and ROBINSON (1951) however questioned Pasquill's conclusions, and by a further series of measurements within the lowest two metres above the surface shewed that the wind and temperature profiles in nearly all cases had the same form and established a strong probability that the coefficients of eddy viscosity, eddy diffusivity and eddy conductivity were approximately equal under all circumstances.

2. The present position of K_H

It has been noted by many writers that near the ground the diurnal variation of temperature can be represented approximately by a sine wave of period 24 hours, whilst at great heights it is negligible. If therefore, horizontal advection can be neglected the problem of heat transfer in the lowest atmosphere reduces to the problem of obtaining a solution of

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z}\left\{K_H \frac{\partial \Theta}{\partial z}\right\} \qquad 2.00$$

where Θ is the potential temperature of the air at a height z above the surface at a time t, subject to the boundary conditions

$$\Theta = \Theta_0 + a \sin qt \text{ when } z \to 0 \qquad 2.01$$

$$\Theta = \Theta_0 \qquad \text{when } z \to \infty \qquad 2.02$$

where $q = \dfrac{2\pi}{24 \times 60 \times 60} = 7.3 \times 10^{-5}$,

and a is a constant.

The solution of 2.00 which fits these conditions with K_H = a constant was given by G. I. TAYLOR (1915), and is

$$\Theta = \Theta_0 + a \sin (qt - bz) \exp - bz \qquad 2.03$$

where $b^2 = \dfrac{q}{2K_H}$

Thus on the assumption that K_H = a constant, the lag in the time of maximum temperature with height should be a linear function of the height above the surface.

BEST (1935) however, from a very careful set of measurements between 2.5 cm and 17 m above the surface, found that the lag varied very closely as $z^{0.2}$ whilst calculations, which he made on the assumption that K_H could be treated as a constant over certain shallow layers of the atmosphere, indicated that K_H increased with height according to the power law

$$K_H = \text{const. } z^{1.8} \qquad 2.04$$

so that the assumption that K_H is a constant is clearly unacceptable. Subsequently however COWLING and WHITE (1941) showed that methods which assumed that K_H could be regarded as independent of height through even a shallow layer led to erroneous results and could give values of K_H which were not even correct as regards order of magnitude. Values of K_H, which they computed by a variety of methods, indicated that K_H increased very rapidly with height above the surface, the rate of increase being greater by day than by night, and it was shewn by the present writer (1948) that their values of K_H under night inversion conditions could be fitted very closely by a simple power law $K_H =$ const. $z^{2/3}$.

In the case of momentum transfer it has been shewn by many writers, including the present writer (1948), that K_M can be represented by a power law of the form

$$K_M = cz^{1-m} \qquad 2.05$$

where the index m always lies between 0 and 1 and varies from a value of about $\frac{1}{7}$ under adiabatic conditions in the daytime to a value of between $\frac{1}{3}$ and $\frac{1}{2}$ under night inversion conditions, so that under inversion conditions the modes of variation with height of K_M and K_H are approximately the same. Under unstable conditions however the evidence from Leafield (JOHNSON and HEYWOOD, 1938) and Rye (BEST and OTHERS, 1950) is conflicting.

SUTTON (1948) from an examination of the Leafield data found that in the hours around noon on a clear summer day $\frac{\partial \Theta}{\partial t}$ became approximately constant with height and time above 1.2 m, and by making certain assumptions as to the amount of heat convected from the surface and assuming that the horizontal flux of heat in the mid hours of the day was negligible (cf. however section 4) concluded that the variation with height of the vertical flux of heat could be ignored below about 100 m. By plotting log $\frac{\partial \Theta}{\partial z}$ against log z, Sutton found that over the limited range from 7 m to 45 m $\frac{\partial \Theta}{\partial z}$ could be closely represented by const. $z^{-1.75}$, so that on the assumption that the vertical flux of heat was constant over this height interval it followed that over the same range

$$K_H = \text{const. } z^{1.75} \qquad 2.06$$

Sutton therefore argued that as the direct application of the solution of 2.00 with $K_H =$ const. z^{1-m}, where m was a small positive fraction < 1, to the case of heat transfer in the hottest hours of the day led to results which were not in agreement with observations, then the mechanism responsible for heat transfer must differ from that for momentum transfer and suggested that this was due to buoyancy forces.

On the other hand KNIGHTING and BEST (1950), in a discussion of the Rye data, whilst making the assumption that $\frac{\partial \Theta}{\partial t}$ was approximately constant and $= A$ in the mid hours of a hot summer day, made no assumptions as to the constancy with height of the vertical flux of heat, and by integrating 2.00 when $K_H = cz^{1-m}$ obtained

$$\Theta = \frac{Az^{1+m}}{c(1+m)} + \frac{Bz^m}{cm} + C \qquad 2.07$$

where A, B and C are constants. By fitting this equation to a series of observations of potential temperature at four heights between 1.1 m and 106.7 m and eliminating A, B and C, they derived the following equation for m, i.e.

$$\begin{vmatrix} \Theta_4 - \Theta_1 & z_4^{1+m} - z_1^{1+m} & z_4^m - z_1^m \\ \Theta_3 - \Theta_1 & z_3^{1+m} - z_1^{1+m} & z_3^m - z_1^m \\ \Theta_2 - \Theta_1 & z_2^{1+m} - z_1^{1+m} & z_2^m - z_1^m \end{vmatrix} = 0 \qquad 2.08$$

and by solving this equation graphically, found a value of $m = 0.45$. It will be noted that the value of m obtained in this manner is independent of $A = \frac{\partial \Theta}{\partial t}$, providing that A is a constant $\neq 0$, but that the values of c and B depend upon the values of m and A.

Unlike the negative value of m derived by SUTTON (1948) from the Leafield data on the assumption that the first term on the right hand side of 2.07 could be neglected in comparison with the second, this value of m lies within the possible range of values derived from considerations of momentum transfer, although differing considerably from that appropriate to an unstable lapse rate. An examination of the Rye temperature data on clear summer days however shews that $\frac{\partial \Theta}{\partial t}$ falls from about 4.5×10^{-4}° F. sec^{-1} at 1030 to about 1.0×10^{-4} ° F. sec^{-1} at 1230, so that calculations of m from small differences of potential temperature at Rye based upon the assumption of constancy of $\frac{\partial \Theta}{\partial t}$ must accordingly be suspect.

When the value of m was recomputed from the Leafield summer data using the observations at the four heights 12.4 m, 30.5 m, 57.4 m and 87.7 m, using the method suggested by KNIGHTING and BEST (1950), it was found that the potential temperatures over the period 1000 G.M.T. to 1300 G.M.T. could be fitted very closely with a value of $m = \frac{1}{7}$.

With this value of m together with the value of $A = 3.3 \times 10^{-4}$ °F sec^{-1} found by SUTTON (1948), the values of c and B calculated from 2.07 are 22 and -3.1 respectively, so that the Leafield observations yield

$$K_H = 22\, z^{6/7} \qquad 2.09$$

For comparison the observed values of the differences of potential temperature over various height intervals on a clear summer day and values calculated from

$$\left.\begin{aligned} \Theta - \Theta^{12.4} &= \frac{3.3 \times 10^{-4}}{22 \times 8/7}\{z^{8/7} - 1240^{8/7}\} - \\ &- \frac{3.1}{22 \times 1/7}\{z^{1/7} - 1240^{1/7}\} \end{aligned}\right\} 2.10$$

are shewn in Table I.

The potential temperatures used are means from 1000 G.M.T. to 1300 G.M.T. so that each entry in the second row of Table I represents the mean of about 30 observations.

Table I. Observed and computed differences of potential temperature.

Height interval in m	30.5—12.4	57.4—12.4	87.7—12.4
Observed differences in °F.	—0.31	—0.46	—0.52
Calculated differences in °F	—0.30	—0.46	—0.52

The expression for K_M derived by the present writer (1948) was

$$K_M = m\, z_0^{2m} V_h h^{-m} z^{1-m} \qquad 2.11$$

where V_h was the velocity measured at a standard height h above the surface, z_0 was a parameter characteristic of the degree of roughness of the surface and m was a thermal parameter.

The mean value of V_h during the period 1000 G.M.T. to 1300 G.M.T. on clear summer days at Leafield at a height of 12.7 m was 4.2 m sec^{-1} and whilst the value of z_0 over the grass covered surface is not known, it is unlikely to be widely different from the value of $z_0 = 2.6$ cm. found by a variety of methods for the rolling grassland of Salisbury Plain. With this value of z_0 and a value of $m = \frac{1}{7}$, which was derived from velocity profiles measured near the surface under unstable conditions,

$$K_M = 28.4\, z^{6/7} \qquad 2.12$$

Thus whilst with small differences of potential temperature, such as those in Table I, the possibility of observational error cannot be excluded, it would seem that, contrary to Sutton's conclusions, the Leafield observations in the hottest hours of a clear summer day and at heights greater than 1.2 m are not inconsistent with predictions made from the heat transfer equation when $K_H = cz^{1-m}$ where m has a value of approximately $\frac{1}{7}$, and where c is of the same order of magnitude as $mz_0^{2m} V_h h^{-m}$.

KNIGHTING (1950), in a mathematical discussion of the heat transfer equation, shewed that if any experimental results were analysed by means of solutions obtained from 2.00 and this led to values of m which were negative, then the solutions were not applicable, or alternatively, if the experiments resulted in values of m which were negative, no use having been made of any solutions, then equation 2.00 was invalid and could not be used to represent the equation.

In this connection, it is perhaps not without interest to note that if Sutton's negative value of m is used to calculate c from 2.07 using the Leafield observations at 12.4 m, 30.5 m, and 87.7 m, then $c = -5 \times 10^{-2}$ so that K_H is negative which is physically impossible.

KÖHLER (1932), on the assumptions that near the ground the diurnal variation of temperature could be represented by a sine wave of period 24 hours and that K_H could be represented by cz^{1-m} where c and m were constants, obtained a solution of 2.00 in terms of Bessel Functions of an imaginary argument and from this solution derived formulae which could be used for comparison with observations near the ground. With the aid of these formula Köhler computed the variation of the amplitude with height for various values of c and m over the height range from the surface to about 30 metres and by comparing JOHNSON's (1929) measurements at Porton of the mean amplitudes at heights of 1.2 m, 7.1 m and 17.1 m with curves derived from these computations for $m = \frac{1}{7}$, found that a close fit was obtained with

$$K_H = 13.8 z^{6/7}$$

BEST (1935) who employed formula 12 of Köhler's paper in order to discuss the variations of the lag with height of the time of maximum temperature found that for large values of z the lag tended to a law

$$\text{lag} = \text{const.}\ z^{\frac{1+m}{2}} \qquad 2.13$$

and by comparing this with his empirical result obtained a negative value of $m = -0.6$ which is incompatible with values derived from the momentum theory.

Later however, Best (Meteorological Magazine, November 1935) concluded that the assumption that formula 12 could be used for heights as low as those considered in his paper (1935) was not justified and that the deduction that $m = -0.6$ was invalid.

If therefore we exclude Sutton's conclusions, which appear to be open to some criticism, it appears that the Leafield observations both at night-time and in the daytime are not inconsistent with predictions made from solutions of the heat transfer equation on the assumption that K_H = const. K_M.

In the following sections, where the consequences of this assumption are examined in more detail, it is shewn that the predicted values are also in very close agreement with observations at Ismailia and at Kew.

3. The diurnal temperature wave at the earth's surface

It was shewn by the present writer (1948) that K_M = a const. z^{1-m}, where m is a function of $\frac{\partial \Theta}{\partial z}$ varying from a value of about $\frac{1}{2}$ to $\frac{1}{3}$ in the early hours of the morning to a value of about $\frac{1}{7}$ or less at midday and the Leafield observations suggest that a similar expression holds for K_H. Such a consideration in the case of heat transfer however renders equation 2.00 quite unmanageable, but since the variation of m during the daylight hours is slight the solution of 2.00 with m = a const. can be used to discuss the temperature changes during this period. We thus require a solution of

$$\frac{\partial \Theta}{\partial t} = c \frac{\partial}{\partial z}\left\{ z^{1-m} \frac{\partial \Theta}{\partial z} \right\} \qquad 3.00$$

where c is independent of t and z subject to

$$\left.\begin{array}{ll} \Theta = \Theta_0 + a \sin qt & \text{when } z \to 0 \\ \Theta = \Theta_0 & \text{when } z \to \infty \end{array}\right\} \qquad 3.01$$

where a is the amplitude of the diurnal temperature wave at the surface and

$$q = \frac{2\pi}{24 \times 60 \times 60} = 7.3 \times 10^{-5}$$

Writing

$$\xi = \frac{z^{1+m} q}{(1+m)c} \qquad 3.02$$

and

$$\Theta = \Theta_0 + (\exp\, iqt) \cdot f(\xi) \qquad 3.03$$

in 3.00 gives

$$(1+m)\xi \frac{\partial^2 f}{\partial \xi^2} + \frac{\partial f}{\partial \xi} - if = 0 \qquad 3.04$$

The solution of this equation was given by the present writer (1948), and substituting this in 3.03 and selecting the real part, the solution of 3.00 which fits the required boundary conditions is readily found to be

$$\Theta = \Theta_0 + a\varphi(\xi) \sin qt - a\psi(\xi) \cos qt \qquad 3.05$$

where

$$\varphi(\xi) = \left| \begin{array}{l} = \left\{ -\frac{\xi^2}{\underline{|2}(2+m)} + \frac{\xi^4}{\underline{|4}(2+m)(3+2m)(4+3m)} - \ldots \right\} - \\ \quad - C\xi^{\frac{m}{1+m}} \left\{ 1 - \frac{\xi^2}{\underline{|2}(1+2m)(2+3m)} + \ldots \right\} + \\ \quad + D\xi^{\frac{m}{1+m}} \left\{ \frac{\xi}{1+2m} - \ldots \right\} \end{array} \right. \qquad 3.06$$

and

$$\psi(\xi) = -\left\{\xi - \frac{\xi^3}{\lfloor 3(2+m)(3+2m)} + \ldots\right\} + D\xi^{\frac{m}{m+1}}\left\{1 - \frac{\xi^2}{\lfloor 2(1+2m)(2+3m)} + \ldots\right\} + C\xi^{\frac{m}{1+m}} \cdot \left\{\frac{\xi}{1+2m} - \frac{\xi^3}{\lfloor 3(1+2m)(2+3m)(3+4m)} + \ldots\right\} \quad 3.07$$

and C and D are functions of m which were tabulated loc. cit. for various values of m and whose values for $m = \frac{1}{7}$ are 1.1161 and 0.2221 respectively. Theoretical expressions for C and D derived by A. F. Crossley are given in an appendix to this paper, and it can be seen from equations 6.24 and 6.25 of the appendix that

$$\left.\begin{aligned} C &= \frac{\Gamma(2-n)}{\Gamma(1+n)}(1-n)^{-(1-n)} \cos\frac{n\pi}{2}, \ \& \\ D &= \frac{\Gamma(2-n)}{\Gamma(1+n)}(1-n)^{-(1-n)} \sin\frac{n\pi}{2} \end{aligned}\right\} \quad 3.08$$

where

$$n = \frac{m}{1+m} \quad 3.09$$

so that

$$\sqrt{C^2+D^2} = \frac{\Gamma(2-n)}{\Gamma(1+n)}(1-n)^{-(1-n)} \quad 3.10$$

and

$$\frac{D}{C} = \tan\frac{n\pi}{2} = \tan\frac{m}{m+1}\cdot\frac{\pi}{2} \quad 3.11$$

Equation 3.05 may be written

$$\Theta = \Theta_0 + a\chi(\xi)\sin(qt-\lambda) \quad 3.12$$

where

$$\tan\lambda = \frac{\psi(\xi)}{\varphi(\xi)}, \ \& \ \chi(\xi) = \sqrt{\psi^2(\xi)+\varphi^2(\xi)} \quad 3.13$$

and these functions are tabulated and utilized in the following section. It can be seen from 3.12, 3.13, 3.06 and 3.07 that very near the surface the lag with height of the time of maximum temperature is given by

$$\lambda = \tan^{-1}\frac{\psi(\xi)}{\varphi(\xi)} = \tan^{-1}\frac{D\xi^{\frac{m}{m+1}}}{1 - C\xi^{\frac{m}{m+1}}} = \text{const. } \xi^{\frac{m}{m+1}}$$

approximately = const. z^m 3.14 whilst at great heights, from the asymptotic expansion cf. 2.13 the lag is given by

$$\text{lag} = \text{const. } z^{\frac{1+m}{2}} \quad 3.15$$

Thus with $m = \frac{1}{7}$ the lag varies as $z^{\frac{1}{7}}$ near the surface and as $z^{\frac{4}{7}}$ at great heights so that it is not altogether surprising that BEST's (1935) comparison of Köhler's asymptotic solution for great heights with observations near the surface gave misleading results.

The time of maximum temperature at the surface can be obtained from a consideration of the fluxes of heat in the air and ground as follows. The convective flux of heat $C(0)$ from the surface

$$= Lt_{z\to 0} - c_p\varrho c z^{1-m}\frac{\partial\Theta}{\partial z} \quad 3.16$$

and hence from 3.02, 3.05, 3.06, 3.07 and 3.08

$$\left.\begin{aligned} C(0) = mc_p\varrho ca\left[\frac{q}{c(1+m)}\right]^{\frac{m}{m+1}} \\ \{C\sin qt + D\cos qt\} \end{aligned}\right\} 3.17$$

$$= mc_p\varrho ca\left[\frac{q}{c(1+m)}\right]^{\frac{m}{m+1}}\sqrt{C^2+D^2}\sin(qt+\alpha) \quad 3.18$$

where

$$\alpha = \tan^{-1}\frac{D}{C} \quad 3.19$$

but from 3.11

$$\frac{D}{C} = \tan\frac{m}{m+1}\cdot\frac{\pi}{2}$$

and hence

$$\alpha = \frac{m}{m+1}\cdot\frac{\pi}{2} \quad 3.20$$

Thus the convective flux of heat from the surface can be represented by a sine wave and is a maximum at a time t before the time of maximum temperature given by

$$t = \frac{m}{m+1} \cdot \frac{\pi}{2} \cdot \frac{24}{2\pi} = \frac{6m}{m+1} \text{ hours} \qquad 3.21$$

and it will be noted that this phase difference is a function of m only and is independent of the wind speed, amplitude of the diurnal wave of temperature or nature of the surface.

The flux of heat in the ground may be derived by means of a similar analysis, with k_1, the coefficient of thermal diffusivity of the soil, replacing $K_H = cz^{1-m}$, the coefficient of eddy diffusivity of the air. The corresponding solutions may be derived very simply by writing $m = 1$ in 3.16 and replacing c_p (the specific heat of the air) by c_1 (the specific heat of the soil), ϱ (the density of the air) by ϱ_1 (the density of the soil) and c by k_1 (the coefficient of thermal diffusivity) whenever they occur in equations 3.17 et sequitur.

Hence the ground flux of heat $G(0)$ is given by

$$G(0) = c_1 \varrho_1 k_1 a \left(\frac{q}{2k_1}\right)^{1/2} \sqrt{C_1^2 + D_1^2} \sin (qt + \alpha_1) \qquad 3.22$$

where from 3.10 and 3.20

$$\sqrt{C_1^2 + D_1^2} = \sqrt{2} \text{ and } \alpha_1 = \frac{\pi}{4} \qquad 3.23$$

and thus

$$G(0) = c_1 \varrho_1 k_1^{1/2} a q^{1/2} \sin \left(qt + \frac{\pi}{4}\right) \qquad 3.24$$

Thus the ground flux of heat can be represented by a sine wave and is a maximum at a time t before the time of surface maximum temperature given by

$$t = \frac{\pi}{4} \cdot \frac{24}{2\pi} = 3 \text{ hours} \qquad 3.25$$

and again it can be seen that this phase difference is independent of the amplitude of the diurnal temperature wave, nature of soil etc.

This result was first obtained by SCHMIDT (1925) who also, on the assumption of a constant coefficient of eddy diffusivity in the air, found that the ground flux of heat was in phase with the convective flux of heat.

Now the heat balance equation on the assumption that there are no horizontal gradients of temperature and water vapour is

$$C(0) + E(0) + R(0) + G(0) = 0 \qquad 3.26$$

where $C(0)$ is the convective flux of sensible heat at the surface, $E(0)$ is the convective flux of latent heat, $R(0)$ is the net radiative flux of heat and $G(0)$ is the flux of heat into the ground. Fluxes directed away from the surface are considered positive.

Over a desert surface it may be assumed that $E(0)$ is zero, and hence

$$R(0) = -C(0) - G(0) \qquad 3.27$$

and $\therefore$

$$\left.\begin{aligned} R(0) = & - m c_p \varrho c a \left[\frac{q}{c(1+m)}\right]^{\frac{m}{m+1}} \sqrt{C^2 + D^2} \cdot \\ & \cdot \sin\left(qt + \frac{m}{m+1} \cdot \frac{\pi}{2}\right) - c_1 \varrho_1 k_1^{1/2} a q^{1/2} \sin\left(qt + \frac{\pi}{4}\right) \end{aligned}\right\} \qquad 3.28$$

$$= A \sin (qt + \gamma) \qquad 3.29$$

where

$$\left.\begin{aligned} A = a \sqrt{\Big\{} & m^2 c_p^2 \varrho^2 c^2 \left[\frac{q}{c(1+m)}\right]^{\frac{2m}{m+1}} (C^2 + D^2) + \\ & + c_1^2 \varrho_1^2 k_1 q + 2 m c_p \varrho c \left[\frac{q}{c(1+m)}\right]^{\frac{m}{m+1}} \cdot \\ & \cdot \sqrt{C^2 + D^2} \varrho_1 c_1 k_1^{1/2} q^{1/2} \cos \frac{\pi}{4}\left(\frac{1-m}{1+m}\right)\Big\} \end{aligned}\right\} \qquad 3.30$$

and

$$\left.\tan \gamma = \frac{m c_p \varrho c \left[\frac{q}{c(1+m)}\right]^{\frac{m}{m+1}} \sqrt{C^2 + D^2} \cdot \sin \frac{m}{m+1} \cdot \frac{\pi}{2} + c_1 \varrho_1 k_1^{1/2} q^{1/2} \sin \frac{\pi}{4}}{m c_p \varrho c \left[\frac{q}{c(1+m)}\right]^{\frac{m}{m+1}} \sqrt{C^2 + D^2} \cdot \cos \frac{m}{m+1} \cdot \frac{\pi}{2} + c_1 \varrho_1 k_1^{1/2} q^{1/2} \cos \frac{\pi}{4}}\right\} \qquad 3.31$$

Thus the net radiative flux of heat at the surface $R(0)$ is also represented by a sine wave which is out of phase with the temperature wave.

The net radiative flux of heat on a clear day is however a maximum at local noon, and hence the time of maximum temperature at the surface occurs at a time $12+t$ hours where

$$t=\gamma \times \frac{2\pi}{360} \times \frac{24}{2\pi} = \frac{\gamma}{15} \text{ hours} \qquad 3.32$$

where γ is given by 3.31.
It can be seen from 3.31 that γ must lie between $\frac{m}{m+1} \cdot \frac{\pi}{2}$ and $\frac{\pi}{4}$ and hence over the desert it follows that the time of maximum must occur not earlier than $\frac{6m}{m+1}$ hours and not later than 3 hours after local noon, the actual time depending upon the relative magnitudes of the convective and ground fluxes of heat, but being independent of the amplitude of the diurnal wave of temperature. With K_H const. with height, $m=1$ so that the time of maximum would always occur 3 hours after local noon. If $K_H \propto K_M$, so that $c \propto m z_0{}^{2m} V_h h^{-m}$, then an increase in wind speed, or to a lesser extent an increase of the surface roughness, should result in an earlier time of maximum.

Over a grass covered surface from which evaporation is taking place the effect of the flux of water into the air is difficult to assess directly. According to BOWEN (1926) however, the ratio between the flux of latent heat and sensible heat is constant for any given surface and may be obtained from simultaneous measurements of humidity and temperature at two heights, and although it is not possible to prove this theoretically, RIDER and ROBINSON (1951), who calculated values of Bowen's ratio μ near the surface on clear summer days at Kew, found that μ was constant with height and shewed little variation from day to day. If therefore the fluxes of sensible and latent heat are in phase, equations 3.27 to 3.31 can still be used with $(1+\mu)$ $C(0)$ replacing $C(0)$ over a grass surface. Hence if the other variables remain unaltered, the time of maximum will be earlier over a grass surface from which evaporation is taking place than over a dry surface, but it will still occur between the limits of $\frac{6m}{m+1}$ and 3 hours after local noon.

4. Comparison with observations

As this theory depends upon the fundamental assumption that there is no horizontal flux of heat, it does not appear possible to use the results from either Rye or Leafield other than qualitatively.

The station at Rye for example, is only 5 km from the sea and is affected by sea breezes especially on clear summer days, and the irregularities in the mean temperature curve for such days make it impossible to determine the time of maximum with any degree of confidence. Furthermore an examination of the individual daily records, from which the mean curve is constructed, shews that on many occasions in the hours around midday the temperature instead of rising is falling at all heights, so that any theory which depends upon the simple propagation of a sine wave upwards from the surface is clearly incapable of explaining the observed vertical thermal structure at Rye on such occasions.

Leafield on the other hand, although free of complications such as a sea breeze, is situated on a spur of the Cotswold Hills with a rapid fall of 100 m in 3 km to the north and a fall of the same amount in about 4 km to the south, and the meteorological observations at Leafield have been found to be unrepresentative of conditions over more level sites. JOHNSON and HEYWOOD (1938) for example, list several anomalies in the meteorological elements at Leafield on clear summer days and ascribe them to the fact that during the hottest hours of the day, the surface air at Leafield mainly consists of air which has come from the relatively great height of about 100 m above the lower ground to the north and south of the spur.

The desert site at Ismailia however is singularly free from topographical defects except in the summer months when it is sometimes affected by a slight 'sea breeze' from Lake Timsah 3 km to the south east, and hence the observations from here, and mainly those on clear December days (FLOWER, 1936) have been utilised to test out the theory.

Now from 3.18 the maximum value of $C(0)$ with $m=\frac{1}{7}$ is

$$\left.\begin{aligned} C(0) &= \frac{1}{7} \times \frac{1}{4} \times \frac{1}{800} \times c^{7/8} \times a \times \\ &\quad \times \left(\frac{7.3 \times 10^{-5}}{8/7}\right)^{1/8} \times 1.138 \end{aligned}\right\}$$

$$= ac^{7/8} \times 1.520 \times 10^{-5} \text{ g cal. cm}^{-2} \text{ sec}^{-1} \quad 4.00$$

where a is the amplitude of the diurnal temperature wave at the surface, and from 3.21 $C(0)$ is a maximum 45 minutes before the time of maximum at the surface.

Similarly, using the values of c_1, ϱ_1 and k_1 for Ismailia given by FLOWER (1936) i.e., $c_1 = 0.19$, $\varrho_1 = 2.63$ and $k_1 = 10^{-2}$ the maximum value of $G(0)$ is from 3.22

$$\begin{aligned} G(0) &= 0.19 \times 2.63 \times 0.1 \times a \times (7.3 \times 10^{-5})^{1/2} \\ &= a \times 4.05 \times 10^{-4} \text{ g cal. cm}^{-2} \text{ sec.}^{-1} \quad 4.01 \end{aligned}$$

and hence from 3.30 the maximum value of $R(0)$ is

$$R(0) = a\left[(4.05)^2 + c^{7/4}(0.1520)^2 + 2 \times 4.05 \times \times 0.1520 \times c^{7/8} \cos\frac{3\pi}{16}\right] \quad 4.02$$

whilst from 3.31

$$\tan\gamma = \frac{c^{7/8} \times 0.1520 \sin\frac{\pi}{16} + 4.05 \sin\frac{\pi}{4}}{c^{7/8} \times 0.1520 \cos\frac{\pi}{16} + 4.05 \cos\frac{\pi}{4}} \quad 4.03$$

Whilst at Ismailia there appear to be no direct observations of $R(0)$, $C(0)$ or $G(0)$ which can be used to calculate c and a, $R(0)$ and $G(0)$ can be estimated with a reasonable degree of confidence.

Thus on a clear December day the maximum downward flux of short wave radiation at Ismailia may be taken as approximately 140×10^{-4} gram cal. cm^{-2} sec^{-1} and the albedo of the desert surface may be taken as approximately 20 %. The former value is based on American observations in California and is supported by observations at Kew Observatory made when the solar elevation is the same as that at Ismailia, whilst the latter is based upon mesasurements by Ångström and observations from American aircraft at high levels over Death Valley, California (1951). The incoming long wave radiation of atmospheric origin, using Flower's data of temperature and vapour pressure, is about 75 gram cal.cm^{-2} sec^{-1} whilst the outward long wave radiation from a desert surface at a temperature of 300° A is about 112×10^{-4} gramcal.cm^{-2}sec^{-1}. Hence the net radiative flux of heat, which Dr G. D. Robinson of Kew Observatory, to whom I am indebted for this estimate, considers may possibly be in error by as much as 10%, is 75×10^{-4} gram cal.cm^{-2}sec^{-1}.

Dr Robinson has also drawn my attention to unpublished measurements of the ground flux of heat in arid soil in Palestine by Ashbel which give $G(0) = 50 \times 10^{-4}$ gramcal.cm^{-2}sec^{-1} and by W. Swinbank in semi-desert soil near Melbourne which also give values of $G(0)$ of about 50×10^{-4} gramcal.cm^{-2}sec.$^{-1}$.

With a value of $G(0) = 50 \times 10^{-4}$ gram cal.cm^{-2}sec^{-1}, the amplitude a of the diurnal wave at the surface is, from 4.01, 12.4° C. With this value of $a = 12.4°$ C and a value of $R(0) = 75 \times 10^{-4}$ gram cal.cm^{-2}sec^{-1} the value of c from 4.02 = 21.9, whilst from 4.00 with $c = 21.9$ and $a = 12.4°$ C, $C(0) = 28.4 \times 10^{-4}$ gramcal.cm^{-2}sec^{-1}.

Whilst there is some uncertainty as to the value of $C(0)$ at midday, the above value is in accordance with the usual concept of its value, and it is of interest to note that the value of c is very nearly the same as that derived from the Leafield observations in the hottest hours around noon on a clear summer day with similar wind speeds.

With this value of $c = 21.9$ from 4.03,

$$\tan\gamma = \frac{2.27 \sin\frac{\pi}{16} + 4.05 \sin\frac{\pi}{4}}{2.27 \cos\frac{\pi}{16} + 4.05 \cos\frac{\pi}{4}} = \tan 33°17' \quad 4.04$$

and hence the time of maximum temperature at the surface at Ismailia on clear December days should occur $33\frac{17}{60} \times \frac{24 \times 60}{360} = 133$ minutes after the time of local noon, i.e. at 1413 local time.

The values of $\chi(\xi)$ and λ from 3.06, 3.07, and 3.13 with $m = \frac{1}{7}$ for selected values of ξ are shewn in Table II.

Table II. Values of $\chi(\xi)$ and λ when $m = \frac{1}{7}$.

ξ	10^{-5}	10^{-4}	10^{-3}	10^{-2}	5×10^{-2}	10^{-1}
$\chi(\xi)$	.737	.650	.537	.391	.272	.224
λ	3°55′	5°55′	9°32′	17°42′	29°01′	35°55′

The lag in the time of maximum temperature in minutes from equation 3.12 is $\frac{\lambda \times 24 \times 60}{360} = 4\lambda$, and hence from 3.02 using the value of $c = 21.9$, Table III shewing the amplitude and lag at various heights can be constructed.

Table III. Percentage amplitude and lag in minutes at various heights.

Height	3.07 cm	22.04 cm	1.65 m	12.40 m	50.78 m	92.94 m
Percentage Amplitude	73.7	65.0	53.7	39.1	27.2	22.4
Lag in minutes......	16	24	38	71	116	144

Table IV shews the time of maximum temperature at four heights above the surface as determined by Flower from the December clear day observations and those derived from Table III together with the time of maximum temperature at the surface i.e. 1413 hours, and it can be seen that the agreement between the two sets of times is remarkably close.

Table IV. Times of maximum temperature at various heights.

Height in m	1.1	16.2	46.4	61.0
Observed times...	1447	1524	1605	1623
Calculated times..	1447	1530	1605	1616

It is of interest to note that in view of the doubtful accuracy of determining times of maximum temperature at various heights from smooth curves drawn through mean hourly values, Flower (1936) carried out a harmonic analysis of the variation of temperature at each height and determined the time of maximum from the phase angle of the diurnal harmonic.

Flower also calculated the times of occurrence of the maximum temperature at each of the four heights for each month of the year and, noting that the time of maximum especially at a height of 1.1 m varied very closely with the equation of time, suggested that the variation of temperature at the surface was equal to the equation of time. It will be seen that provided the variations in the mean monthly fluxes of convective heat are small, this follows from the fact that the radiative flux of heat is a maximum at local noon.

If, as might be expected by analogy with momentum transfer, c is proportional to the mean wind speed measured at some standard height, then the effect of the wind speed upon the time of maximum temperature can readily be determined. The value of c in 4.03 was determined at a time when the mean wind speed at a height of 15.2 m was 3.8 m sec^{-1} and hence if the other factors remain unchanged the values of γ over a desert surface are, from 4.04, given by

$$\tan \gamma = \frac{2.27\left(\frac{V}{3.8}\right)^{7/8} \sin\frac{\pi}{16} + 4.05 \sin\frac{\pi}{4}}{2.27\left(\frac{V}{3.8}\right)^{7/8} \cos\frac{\pi}{16} + 4.05 \cos\frac{\pi}{4}} \qquad 4.05$$

so that an increase in the wind speed results in an earlier time of maximum.

The values of γ and the corresponding times of maximum temperature for various values of V_h are given in Table V.

Table V. Variation with wind speed of time of maximum temperature over the desert.

Wind speed at 15.2 m in m. sec.$^{-1}$	0	2½	5	7½	10
γ..........	45°	36°19′	31°21′	28°34′	26°10′
Time of maximum at surface........	1500	1425	1406	1354	1345
Time of maximum at 1.1 m.........	—	1504	1439	1425	1414
Time of maximum at 16.2 m.........	—	1556	1519	1459	1445
Time of maximum at 46.4 m.........	—	1635	1548	1523	1507
Time of maximum at 61.0 m.........	—	1646	1558	1532	1515

There are unfortunately no observations in the Ismailia report (1936) which can be used to check this dependence of the time of maximum upon the wind speed, but that at least the order of magnitude of the effect is correct can be seen from tables given by JOHNSON and HEYWOOD (1938) shewing the times of maximum at various heights at Leafield in summer for various wind speeds.

Table VI.

Wind speed at 12.7 m in m. sec.$^{-1}$	2.5	4.2	7.5
Time of maximum at 1.2 m	1505	1455	1430
,, ,, ,, ,, 12.4 m	1546	1535	1500
,, ,, ,, ,, 30.5 m	1600	1550	1512
,, ,, ,, ,, 57.4 m	1616	1606	1522

A curious feature of the Leafield observations, which has no counterpart in either the Rye or Ismailia observations, is the fact that on clear summer days at Leafield the times of maximum temperature at all heights occur very nearly an hour later than the corresponding times of maximum on a mean June day, and in fact occur outside the theoretical times of maximum based on the assumption of very light winds and a very high value of the thermal conductivity. This is consistent with other anomalous features of the Leafield observations on clear summer days noted by JOHNSON and HEYWOOD (1938) and lends support to their suggestion that these were due to advection and lateral mixing.

As already stated, the value of V_h at Ismailia at a height of 15.2 m during the midday hours was 3.8 m, and assuming that the parameter z_0 for Ismailia is of the same order of magnitude as that for the rolling downland of Salisbury Plain, then K_M with $m=\frac{1}{7}$ is given by

$$\left.\begin{aligned} K_M &= \frac{1}{7}(2.6)^{2/7}\frac{380}{(1520)^{1/7}}z^{6/7} \\ &= 25.6z^{6/7} \end{aligned}\right\} \qquad 4.06$$

whereas with $m=\frac{1}{7}$

$$K_H = 21.9z^{6/7} \qquad 4.07$$

which also lends some support to the suggestion that $K_H=\text{const.}\ K_M$ where the constant is approximately unity.

From 3.05, 3.06, 3.07 and 3.08, it can be seen that around midday and within a few metres of the ground Θ is given to a good degree of approximation by

$$\Theta = \Theta_0 + a\sin qt - a\xi^{\frac{m}{m+1}}(C\sin qt + D\cos qt) \qquad 4.08$$

so that from 3.02 and 3.11

$$\left.\begin{aligned} \Theta = \Theta_0 + a\sin qt - az^m\left[\frac{q}{(m+1)c}\right]^{\frac{m}{m+1}} \cdot \\ \cdot\sqrt{C^2+D^2}\sin\left(qt+\frac{m}{m+1}\cdot\frac{\pi}{2}\right) \end{aligned}\right\} \qquad 4.09$$

and hence near the surface

$$\Theta = \Theta_g - \beta z^m \qquad 4.10$$

where Θ_g is the value of Θ at the surface and

$$\beta = a\left[\frac{q}{(m+1)c}\right]^{\frac{m}{m+1}}\sqrt{C^2+D^2}\sin\left(qt+\frac{m}{m+1}\cdot\frac{\pi}{2}\right) \qquad 4.11$$

It follows therefore that if K_H and K_M are represented by similar power laws then the temperature and velocity profiles near the ground should be the same, and conversely that if the temperature and velocity profiles near the ground are the same then K_M and K_H are represented by similar laws.

Now RIDER and ROBINSON (1951), from observations at Kew Observatory, shewed that when the winds and temperatures meassured at several heights between 10 cm and 200 cm were superposed with adjustment for scale, so that the profiles as drawn by hand coincided at 25 cm and 100 cm, the wind and velocity profiles in nearly all cases had the same form.

Table VII shews the velocities from 25 cm to 200 cm for all unstable cases which they listed for clear summer days at Kew 1949 at times between 1100 and 1 400 G.M.T. reduced to a standard velocity of 200 cmsec^{-1} at a height of 2.02 m. The wind velocities are mean values over a 30 minute period which the authors considered was sufficiently long to give smooth mean profiles and suffi-

Table VII. Velocities near the surface at Kew on clear sunny days.

Case No.	1	2	3	4	5	6	7	8	9	10	1/9th law
Date	18/6	20/6	20/6	21/6	21/6	23/6	23/6	24/6	1/7	1/7	$200\left(\frac{z}{202}\right)^{1/9}$
Height \ Time in cm.	1121	1121	1410	1110	1400	1103	1432	1419	1122	1428	—
25	159	161	158	152	150	155	153	163	158	156	159
37	162	165	166	163	166	165	163	173	148	161	166
58	173	177	171	174	174	173	175	179	168	173	170
76	176	180	179	178	177	179	180	176	171	175	179
102	187	187	189	188	185	188	186	189	184	185	185
202	200	200	200	200	200	200	200	200	200	200	200

ciently short to exclude the effects of the diurnal variation. For comparison the velocities computed from the simple power law $V = 200\left(\frac{z}{202}\right)^{1/9}$ which gives a closer fit than a $\frac{1}{7}$th law are also tabulated.

It can be seen that over this height range the observed velocities can be represented reasonably closely by a simple power law where the index $m = \frac{1}{9}$, and hence since the ve ocity and temperature profiles when adjusted for scale coincide, it follows that over this height range the temperature data can be represented by

$$\Theta = \Theta g - \beta z^{1/9} \qquad 4.12$$

where Θ is the value of Θ at the surface, and where

$$\beta = \frac{V_{202}}{(202)^{1/9}} \cdot \frac{\Theta_{25} - \Theta_{202}}{V_{202} - V_{25}} \qquad 4.13$$

The values of β in °F. cm$^{-1/9}$ for the above occasions have been computed and are listed in Table VIII.

Table VIII. Values of β on clear summer days.

Case No.	1	2	3	4	5	6	7	8	9	10
β.......	6.3	5.8	5.3	5.6	5.0	6.1	4.2	5.8	5.9	4.9

The mean value of β from Table VIII is 5.5, and hence near the surface where Θ is given by 4.12

$$K_H = cz^{8/9} \qquad 4.14$$

The convective flux of heat at the surface is thus

$$C(0) = Lt_{z\to 0} - K_H \varrho c_p \frac{\partial \Theta}{\partial z}$$

$$= c \times \frac{1}{4} \times \frac{1}{800} \times 5.5 \times \frac{1}{9} \times \frac{5}{9}$$

$$= 1.06 \times 10^{-4} \times c \text{ gram cal. cm}^{-2}\text{ sec}^{-1} \qquad 4.15$$

The mean value of $C(0)$ for the occasions listed in Table VII and computed by Rider and Robinson from the heat balance equation is $C(0) = 37.3 \times 10^{-4}$ gram cal. cm^{-2} sec^{-1} 4.16

and hence from 4.15 and 4.16

$$c = 35$$

This value of c is subject to some uncertainty, as the computations of $C(0)$ from the heat balance equation depend upon the fundamental assumption that there is no horizontal flux of heat, and in view of the topography at Kew it is possible that this does not hold. Thus RIDER and ROBINSON (1951) noted that the temperature fluctuations which they observed near the ground appeared to be inconsistent with this assumption, whilst values of $C(0)$ computed by LANDER and ROBINSON (1952) from simultaneous observations of wind and temperature fluctuations were consistently only about $\frac{1}{2}$ to $\frac{1}{3}$ the values computed from the heat balance equation, so that the value of c lies somewhere between about 12 and 35.

Now the mean value of V_h at 2.02 m for these observations was about 3 m. sec^{-1} and with a value of $m = \frac{1}{9}$ and a value of $z_0 =$ 2.6 cm, the value of $K_M = 22.9z^{8/9}$. Whether however this value of z_0 is approximately true for Kew is uncertain, for whilst the actual observations were made over a surface of level grass which was kept mown to an average length of 1 to 2 cm, this grass surface was surrounded by a solid fence about 1.5 m high which was about 50 m away at the nearest point, whilst in some directions there were very large trees and buildings, including the observatory itself which was on rising ground to the north west of the site. In view of the uncertainty of both c and z_0, all that can be said with any degree of confidence is that under unstable conditions the modes of variation of K_H and K_M at Kew are the same, with $m = \frac{1}{9}$ approximately, whilst the observed values of the convective heat flux are not inconsistent with those predicted on the assumption that $K_H = \text{const.}\ K_M$, where the constant of proportionality is approximately unity.

The general agreement between the observed and calculated values at Leafield, Ismailia and Kew suggest that, except perhaps in the lowest few cm, the mechanism for heat transfer is the same as that for momentum transfer and that the coefficients of eddy viscosity and eddy conductivity are approximately the same.

5. The annual temperature wave

The results of the previous sections can be used with slight modification to discuss the annual variation of temperature. For example during the summer months at Ismailia the annual variation of temperature can be closely represented by a single sine wave of period one year. Similarly during the period May to September the average monthly lapse rate of potential temperature between 1.1 m and 61 m is positive, i.e. the potential temperature decreases with height so that during this period the index m can be assumed to be approximately constant and equal to about $^1/_7$.

The solution of 3.00 with $m = {^1/_7}$ will therefore be utilized to discuss the variation in the date of occurrence of the maximum air temperature at Ismailia.

The phase lag of the annual temperature wave against the annual convective flux wave is from 3.20 given by

$$\alpha = \frac{m}{m+1} \cdot \frac{\pi}{2} \qquad 5.00$$

and since for annual values $q_a = \frac{q}{365}$, the convective flux of heat from the surface is a maximum at a time t_1 before the time of maximum temperature at the surface given by

$$\left.\begin{aligned} t_1 &= \frac{m}{m+1} \cdot \frac{\pi}{2} \cdot \frac{12}{2\pi} \\ &= \frac{3}{8} \text{ months} \\ &= 11^1/_4 \text{ days approximately} \end{aligned}\right\} \quad 5.01$$

The phase lag of the annual temperature wave against the annual ground flux wave is from 3.23 given by

$$\alpha_1 = \frac{\pi}{4}$$

and hence

$$\left.\begin{aligned} t_2 &= \frac{\pi}{4} \times \frac{12}{2\pi} \\ &= 1^1/_2 \text{ months} \end{aligned}\right\} \quad 5.02$$

and it will be noted that, as in the case of the diurnal wave, these times are independent of the amplitude of the annual temperature wave and of the physical properties of the air and soil, etc.

The phase lag of the annual temperature wave against the net radiative flux wave is given by 3.31 with q_a replacing q. Thus using the same values of the soil constants at Ismailia as in 4.01 and the same value of c, the time interval from 4.03 is

$$\left.\begin{aligned} t_3 &= 33^\circ 17' \times \frac{2\pi}{360} \times \frac{12}{2\pi} \\ &= 33 \text{ days} \end{aligned}\right\} \quad 5.03$$

Now from 3.12, the lag in days at various heights is $\frac{\lambda \times 365}{360} = \lambda$ approximately, whilst from 3.02, assuming that c does not increase with the increased time over which the means are now taken, the heights corresponding to the specified values of ξ in Table III have to be multiplied by $(365)^{7/8}$.

The dates of maximum air temperature

Table IX. Dates of maximum air temperature at Ismailia at various heights.

Height in m	1.1	16.2	46.4	61.0
Observed date of maximum................	July 10th	July 29th	July 30th	July 31st
Calculated date of maximum...............	July 26th	July 28/29	July 30th	July 30/31

from Flower's (1936) harmonic analysis of the monthly mean temperatures at 4 heights, during the period October, 1931 to October, 1932, together with those derived by the present theory for the same heights, are given in Table IX.

The agreement between the observed and predicted dates is very good except at 1.1 m where Flower's date of maximum, in the light of his values at greater heights, appears to be somewhat early, and suggests that c is independent of the period over which means are taken.

It is possible that this discrepancy is due to the limited period over which means are taken, or it may be that the present theory has oversimplified the problem of the annual temperature wave.

Acknowledgment: I am indebted to the Director of the Meteorological Office for permission to publish this paper.

6. Appendix by A. F. Crossley, M. A.

Equation 13 (FROST, 1948) c.f. also 3.04 of the present paper, is

$$(1+m)\xi\frac{\partial^2 W}{\partial \xi^2}+\frac{\partial W}{\partial \xi}-iW=0 \qquad 6.00$$

Substituting

$$W=y\xi^{\frac{m}{2}} \text{ and } x=2(1-n)^{1/2}e^{-\frac{i\pi}{4}}\xi^{1/2} \qquad 6.01$$

where

$$n=\frac{m}{1+m} \qquad 6.02$$

gives

$$\frac{\partial^2 y}{\partial x^2}+\frac{1}{x}\frac{\partial y}{\partial x}+\left(1-\frac{n^2}{x^2}\right)y=0 \qquad 6.03$$

which is Bessel's equation of order n

Since $\quad 0 \leqslant m \leqslant 1 \qquad 6.04$

therefore $\quad 0 \leqslant n \leqslant {}^1/_2 \qquad 6.05$

so that (except for $n=0$) $y=J_n(x)$ and $y=J_{-n}(x)$ are independent solutions of 6.03 and hence

$$W_1=\xi^{\frac{n}{2}}J_{-n}(x), \text{ and } W_2=\xi^{\frac{n}{2}}J_n(x) \qquad 6.06$$

are independent solutions of 6.00.

Now

$$\left.\begin{aligned}J_n(x)=\frac{x^n}{2^n\Gamma(1+n)}\Bigg\{1-\frac{x^2}{2^2(1+n)}+\\ +\frac{x^4}{2^4\underline{|2}(1+n)(2+n)}-\ldots\Bigg\}\end{aligned}\right\} \qquad 6.07$$

and hence the two solutions of 6.00 are

$$\left.\begin{aligned}W_1=\xi^{n/2}J_{-n}(x)=\frac{(1-n)^{-n/2}e^{\frac{in\pi}{4}}}{\Gamma(1-n)}\cdot\\ \cdot\left\{1+i\xi+\frac{i^2\xi^2}{\underline{|2}(2+m)}+\ldots\right\}\end{aligned}\right\} \qquad 6.08$$

and

$$\left.\begin{aligned}W_2=\xi^{n/2}J_n(x)=\frac{(1-n)^{n/2}e^{-\frac{in\pi}{4}}}{\Gamma(1+n)}\cdot\\ \cdot\left\{1+\frac{i\xi}{1+2m}+\frac{i^2\xi^2}{\underline{|2}(1+2m)(2+3m)}+\ldots\right\}\end{aligned}\right\} \qquad 6.09$$

Except for the constant factors these solutions coincide with equations 17 and 18 (FROST 1948). We require asymptotic expressions for the two series which (following Frost) we separate into their real and imaginary parts. Thus

$$f_1(\xi)+if_2(\xi)=\Gamma(1-n)e^{-\frac{in\pi}{4}}(1-n)^{n/2}\xi^{n/2}J_{-n}(x) \qquad 6.10$$

and

$$f_3(\xi)+if_4(\xi)=\Gamma(1+n)e^{\frac{in\pi}{4}}(1-n)^{-n/2}\xi^{n/2}J_n(x) \qquad 6.11$$

Now from 6.01 amp $x=-\frac{\pi}{4}$, and the corresponding asymptotic expression for $J_n(x)$ is (GRAY, MATTHEWS and MACROBERT, Bessel Functions, p. 61)

$$J_n(x) \to \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \quad 6.12$$

$$\left.\begin{aligned} &= \frac{e^{\frac{i\pi}{8}} \xi^{-\frac{1}{4}}}{\pi^{1/2}(1-n)^{1/4}} \cdot \\ &\cdot \cos\left\{2^{1/2}(1-n)^{1/2}(1-i)\xi^{1/2} - \frac{\pi}{4} - \frac{n\pi}{2}\right\} \end{aligned}\right\} \quad 6.13$$

or putting

$$\alpha = \frac{\pi}{4} + \frac{n\pi}{2} \text{ and } u = \{2(1-n)\xi\}^{1/2} \quad 6.14$$

$$\left.\begin{aligned} J_n(x) &\to \frac{e^{\frac{i\pi}{8}} \xi^{-\frac{1}{4}}}{\pi^{1/2}(1-n)^{1/4}} \cdot \\ &\cdot \{\cos(u-\alpha)\cos h\, u + i \sin(u-\alpha) \sin h\, u\} \end{aligned}\right\} \quad 6.15$$

$$\to \frac{\xi^{-\frac{1}{4}} e^u}{2\pi^{\frac{1}{2}}(1-n)^{\frac{1}{4}}} \cdot e^{i\left(u - \frac{\pi}{8} - \frac{n\pi}{2}\right)} \quad 6.16$$

Similarly

$$J_{-n}(x) \to \frac{\xi^{-\frac{1}{4}} e^u}{2\pi^{\frac{1}{2}}(1-n)^{\frac{1}{4}}} \cdot e^{i\left(u - \frac{\pi}{8} + \frac{n\pi}{2}\right)} \quad 6.17$$

Therefore

$$f_1(\xi) + i f_2(\xi) \to \cdot$$

$$\cdot \frac{\Gamma(1-n) \cdot (1-n)^{-\frac{(1-2n)}{4}} \xi^{\frac{2n-1}{4}} e^{i\left(u - \frac{\pi}{8} + \frac{n\pi}{4}\right)}}{2\pi^{\frac{1}{2}}} \quad 6.18$$

and

$$f_3(\xi) + i f_4(\xi) \to \cdot$$

$$\cdot \frac{\Gamma(1+n)(1-n)^{-\frac{(1+2n)}{4}} \xi^{\frac{(2n-1)}{4}} e^{i\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right)}}{2\pi^{\frac{1}{2}}} \quad 6.19$$

Equations 26 and 27 (Frost, 1948) therefore become

$$0 = -\varrho_0 G_0 \Gamma(1-n)(1-n)^{-\frac{(1-2n)}{4}} \cdot$$
$$\cdot \cos\left(u - \frac{\pi}{8} + \frac{n\pi}{4}\right) + \Gamma(1+n)(1-n)^{-\frac{(1+2n)}{4}} \cdot$$
$$\cdot \left\{C \cos\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right) - D \sin\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right)\right\} \quad 6.20$$

and

$$0 = -\varrho_0 G_0 \Gamma(1-n)(1-n)^{-\frac{(1-2n)}{4}} \cdot$$
$$\cdot \sin\left(u - \frac{\pi}{8} + \frac{n\pi}{4}\right) + \Gamma(1+n)(1-n)^{-\frac{(1+2n)}{4}} \cdot$$
$$\cdot \left\{D \cos\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right) + C \sin\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right)\right\} \quad 6.21$$

whence

$$C \cos\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right) - D \sin\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right) =$$
$$= \varrho_0 G_0 \frac{\Gamma(1-n)}{\Gamma(1+n)}(1-n)^n \cos\left(u - \frac{\pi}{8} + \frac{n\pi}{4}\right) \quad 6.22$$

$$C \sin\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right) + D \cos\left(u - \frac{\pi}{8} - \frac{n\pi}{4}\right) =$$
$$= \varrho_0 G_0 \frac{\Gamma(1-n)}{\Gamma(1+n)}(1-n)^n \sin\left(u - \frac{\pi}{8} + \frac{n\pi}{4}\right) \quad 6.23$$

whence

$$C = \varrho_0 G_0 \frac{\Gamma(1-n)}{\Gamma(1+n)}(1-n)^n \cos\frac{n\pi}{2} \quad 6.24$$

and

$$D = \varrho_0 G_0 \frac{\Gamma(1-n)}{\Gamma(1+n)}(1-n)^n \sin\frac{n\pi}{2} \quad 6.25$$

These forms are suitable for computation. The figures in Table II (Frost 1948) are confirmed (apart from some minor differences in the fourth place of decimals).

Also

$$\tan \psi_0 = \frac{D}{C} = \tan\frac{n\pi}{2} \quad 6.26$$

$$\therefore\ \psi_0 = \frac{n\pi}{2} = \frac{m}{m+1} \cdot \frac{\pi}{2} \quad 6.27$$

which confirms equation 30 (Frost 1948).

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