Multiple-Parameter Models of the Atmosphere for Numerical Forecasting Purposes[1]

By BERT BOLIN, University of Stockholm

(Manuscript received January 29, 1953)

Abstract

A general discussion of multiple parameter models of the atmosphere for forecasting purposes is given and various ways of arriving at such models are presented. For some two-parameter models it is possible to derive a simple graphical method of integration which may be used before high-speed electronic computors are available. A numerical example is given. Finally the accuracy of a two-parameter representation is investigated more closely and a comparison is made with the accuracy of the radiosonde data. The conclusion is reached that a two-parameter model is adequate for representing the field of pressure-heights with an accuracy which is only slightly less than the accuracy of the radiosonde observations themselves. Hence models of three or more parameters are of interest merely to the extent that they are capable of describing in a more satisfactory manner the *processes* of the atmosphere.

1. Introduction

The barotropic model of the atmosphere has proven to give a very useful first approximation to the actual motion of the atmosphere for computational purposes. The few complete numerical forecasts that have been published up to now show that the success of the barotropic model is comparable with the success of standard methods for forecasting at use in the weather services. However, the computations are extremely voluminous as is seen for example from the article by CHARNEY, FJØRTOFT and VON NEUMANN (1950). Therefore the method seems not applicable in prcatice until we have large-capacity high-speed electronic computors at our disposal.

One of the main reasons for the large amount of computations required is the necessity to fulfil the Courant-Friedrich-Lewy's criterion for computational stability. This defines the maximum time interval Δt over which extrapolation in time may be carried out, when the grid size Δs is given. In order to have a fairly good resolution the grid-size must not be chosen too large and Δt turns out to be 1—2 hours. This seems to be a surprisingly short time period to a practical meteorologist, who is accustomed to extrapolate over periods of 24 hours with considerable success. It is quite obvious that these time extrapolations are done in different ways. In the former case a strict Eularian method is adopted and no direct use is made of the fact that the absolute vorticity is conserved. In the practical extrapolation, on the other hand, certain quasi- conservative fields are moved over the map and the local effects are computed by integrating in time at various points on the map. The question arises if a similar method of integration can be advised for the barotropic model. This has lately been done by FJØRTOFT (1952). By his method one person can make a complete 24-hour forecast for an area of about 10^7 km^2 in three hours using nothing but graphical methods of addition and subtraction. The extrapolation in time is carried out in one time-step through the introduction of a certain mean stream field which varies slowly and in

[1] This work has been supported by grants from the Knut and Alice Wallenberg Foundation, Stockholm and the U. S. Weather Bureau, Washington D. C.

which the vorticity field may be advected. One might expect, however, that even if the vorticity field is advected by the actual wind this may be done over time periods of 6—12 hours before it is necessary to compute a new stream field. The forecast equations of the barotropic model may be written.

$$\frac{\partial \eta}{\partial t} = J(\eta, z) \tag{1}$$

$$\eta = \frac{g}{f} \nabla^2 z + f \tag{2}$$

$$\eta(t + \Delta t) = \eta(t - \Delta t) + \frac{\partial \eta}{\partial t} \cdot 2\Delta t \tag{3}$$

where z is the height of the pressure surface defining the motion, η is the absolute vorticity, g is the acceleration of gravity, f is the Coriolis parameter and $J(\alpha_1, \alpha_2) = \partial\alpha_1/\partial x \ \partial\alpha_2/\partial y - \partial\alpha_2/\partial x \ \partial\alpha_1/\partial y$. In order to obtain $\partial\eta/\partial t$ we must know the two fields of η and z. However, the relative changes with time of z are considerably slower than corresponding changes of η and it may therefore be possible to use (1) and (3) alternatively over a few time steps without computing new values z from (2) at every instant. This procedure in which the vorticity field is advected in the initial wind field over several elementary time steps may be utilized for computations with a 2-parameter model. In this case it does not seem be possible to derive a slowly varying mean field in which some invariant quantity is advected, as was done by FJØRTOFT (1952), but we may thus try to use the actual wind field itself and extrapolate over 6—12 hours before deriving the new stream field. Such a method will, of course, not be of immediate practical value in the way Fjørtoft's method for the barotropic model is, since we cannot make a 24 hour forecast in one step but have to use two or three, and each time step will mean 8—10 hours' work for one person. Still, it may be of some interest for preliminary computations before high-speed electronic computors are avarilable and a few examples will be given here.[1]

[1] The reader is referred to an article by CHARNEY and PHILLIPS (1953) for a detailed discussion of these problems.

Before we discuss this method in detail it is of some interest to approach the multiple-parameter models of the atmosphere for computational purposes in a somewhat more general way than has been done previously (PHILLIPS 1951, ELIASSEN 1952, EADY 1952, ARNASON 1952).

2. Multiple-parameter models

In all models of the atmosphere advanced so far for computational purposes, one makes use of the geostrophic approximation in order to express the wind field in terms of the pressure field in the vorticity equation. A set of dynamically consistent equations incorporating the geostrophic approximation was derived by CHARNEY (1948) which implies essentially an elimination of oscillations of the inertia-gravity type that develop in connection with the adjustment of the wind and pressure fields to each other. This is probably a good approximation when we deal with large-scale quasi-barotropic phenomena, but becomes more questionable when we try to incorporate baroclinic processes which are of a smaller scale. We shall, however, in this analysis adopt this approximation and leave the question of the goodness of the geostrophic approximation for further investigations We shall here also neglect effects of friction and non-adiabatic processes.

The multiple-parameter models now essentially represent attempts to simplify the complicated three-dimensional atmospheric structures and behavior patterns by introducing simple assumptions about the distribution of the dependent variables with respect to the vertical coordinate. The name of the model indicates the number of parameters used in order to describe the field of motion. This representation is used partly to simplify the computations and partly to get more accurate expressions for the derivatives along the vertical than finite differences will give, in particular if only a few gridpoints are used. As our vertical coordinate we may choose z (height), p (pressure) or Θ (potential temperature) or any function of these three. The dynamic and adiabatic equations become considerably simpler if p or Θ are chosen. In the former case $\partial z/\partial t$ and ω ($= dp/dt$) will be the unknown quantities, which are determined from the vorticity equation and the adiabatic equation, in the latter case $\partial\psi/\partial t$ (where $\psi = c_p T + gz$) is our only dependent variable and may be determined from the vorticity equation. The adiabatic equation is automatically fulfilled by our choice of coordinate system. It is well-known, however, that the boundary conditions at the ground become considerably more complicated in that case. We shall use p as the vertical coordinate in the analysis below.

The two basic equations that govern the motion are the vorticity equation and the adiabatic equation

$$V(x, y, p, t) = \frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta + \omega \frac{\partial \eta}{\partial p} - $$
$$- (f + \zeta) \frac{\partial \omega}{\partial p} + \mathbf{k} \cdot \nabla \omega \times \frac{\partial \mathbf{v}}{\partial p} = 0 \quad (4)$$

$$\frac{\partial \ln \Theta}{\partial t} + \mathbf{v} \cdot \nabla \ln \Theta + \omega \frac{\partial}{\partial p} (\ln \Theta) = 0 \quad (5)$$

$\mathbf{v}$ is the horizontal velocity, and $\mathbf{k}$ denotes the vertical unit vector. The continuity equation has already been used to eliminate the divergence term from the vorticity equation. The boundary conditions at the top and at the bottom of the atmosphere are (ϱ denotes density)

$$\omega = 0 \text{ at } p = 0 \quad (6)$$

$$\frac{\partial z}{\partial t} + \mathbf{v} \cdot \nabla z - \frac{\omega}{g \varrho} = 0 \text{ at } p = p_0 \quad (7)$$

Equation (5) may be transformed with the aid of the hydrostatic equation and we obtain

$$\frac{\partial}{\partial p} \left(\frac{\partial z}{\partial t}\right) + \mathbf{v} \cdot \nabla \frac{\partial z}{\partial p} - \frac{\omega}{g \varrho} \frac{\partial}{\partial p} (\ln \Theta) = 0 \quad (8)$$

where, of course, $\partial \ln \Theta / \partial p$ also is a unique function of z.

In order to solve the two equations (4) and (8) with the boundary conditions (6) and (7) we may now follow the procedure used by ELIASSEN (1952) and EADY (1952) and assume certain parametric representations of z and ω and thus also of $\partial z/\partial t$:

$$z(x, y, p, t) = H(p) + \sum_{n=1}^{N} z_n(x, y, t) \cdot A_n(p) \quad (9)$$

$$\omega(x, y, p, t) = \sum_{m=1}^{M} \omega_m(x, y, t) \cdot B_m(p) \quad (10)$$

z_n and ω_m are functions to be determined from the field of motion, while $A_n(p)$ and $B_m(p)$ are power expressions of p given once and for all. $H(p)$ is the height of the various pressure surfaces in the standard atmosphere. Both EADY and ELIASSEN use $N = 2$ and $M = 1$ (however EADY has z as the vertical coordinate instead of p). The boundary conditions are simplified to $\omega = 0$ at $p = p_0$ and $p = 0$ in ELIASSEN's model and EADY puts $w = 0$ at the top and the bottom of the atsmosphere. The simplest function $B_m(p)$ satisfying this boundary condition is a parabolic function of the vertical coordinate.

Another possibility is to solve for ω from equation (8) and introduce in (4) which then contains only one unknown function $\partial z/\partial t$. The boundary condition (6) must be transformed in a similar way in order to eliminate ω. It is now sufficient to introduce one assumption such as (9), which has to satisfy the vorticity equation and the boundary conditions. It is of course also possible to eliminate $\partial z/\partial t$, but in that case we have to use both assumptions (9) and (10). We shall proceed as EADY and ELIASSEN.

It is quite obvious that the approximate expressions for $\partial z/\partial t$ and ω given by (9) and (10) cannot satisfy the equations at all levels. At each level where we require them to be satisfied we obtain one relation containing $\partial z_n/\partial t$ and ω_m but $N + M$ represents the maximum number of conditions we may derive from the two equations (9) and (10) and the two boundary conditions without overdetermining the problem. It is also seen from equation (4) that the expressions we finally obtain for $\partial z/\partial t$ cannot give an exact solution. The nonlinear terms in the equations will contain products of the functions $A_n(p)$ and $B_m(p)$. Thus $\partial\eta/\partial t$ and also $\partial z/\partial t$ as well as ω cannot be represented by the functions $A_n(p)$ and $B_m(p)$ originally chosen. Higher powers of p will appear but are automatically excluded by the method of representation. However, this is true for any numerical solution of a non-linear equation of this type. The very fact that the function z according to the differential equations very soon will contain all possible powers represents one of the major reasons why this equation cannot be solved by known analytical methods at present.

From a physical point of view it is not unreasonable to disregard these higher powers of p. It means that we assume that the energy originally contained in the large perturbations dissipates via the small ones but that no energy transfer takes place in the opposite direction. Of course the same thing is true in the horizontal dimensions where we disregard the small perturbations which cannot be caught by the grid we use. Even if these harmonics were important for the development of atmospheric disturbances of the scale we observe in the atmosphere, it is very unlikely that for example the geostrophic approximation would be valid in this case. This assumption does not permit us to consider the detailed structure of the motion.

It was indicated above that one way to obtain a sufficient number of equations for the determination of $\partial z/\partial t$ and ω is to assume that the two equations (4) and (8) as well as the boundary conditions are satisfied at a proper number of levels. However, this is not the most rational way, in particular not when M and N are small. Instead we may proceed in the same way as EADY and devide the interval $p = p_0$ to $p = 0$ into a sufficient number of intervals and require that the two basic equations are satisfied if integrated over these intervals. Two conditions are again obtained by satisfying the two boundary conditions. Another method was employed by ELIASSEN who required that certain weighted pressure means of the vorticity equation and adiabatic equation should be satisfied if integrated over the whole interval $p = p_0$ to $p = 0$. The question of how to determine the appropriate equations is important particularly when the number of parameters is small.

In the case of the *barotropic or one-parametric model* it is quite obvious that the single equation needed has to be obtained from the dynamics, in other words from the vorticity equation. This is done by satisfying the equation at one level approximately in the middle of the atmosphere (the equivalent barotropic level) or by demanding that the vorticity equation be satisfied after integration over the vertical. This model does not permit any conclusions concerning conditions at the upper and lower boundaires.

The next model is one where *two parameters* are used to describe the field of motion, $N = 2$. M can be chosen arbitrarily to some extent, but little is gained if $M > 2$. (ELIASSEN, and similarly EADY, puts $M = 1$. The $2 + M$ equations needed to determine the time-derivatives of the parameters may again be derived in various ways. ELIASSEN (and also EADY) has chosen his $B_1(p)$ so that the two boundary conditions at the top and the bottom of the atmosphere are automatically fulfilled and all three equations needed ($N + M = 3$ in this case) are derived from integral requirements regarding the vorticity equation and the adiabatic equation as was indicated in a previous paragraph. In this way the model is fitted to the observed structure of the atmosphere. However, it can be shown (see sec 3) that these integral conditions are equivalent to the requirement that the equations be valid at certain levels. These levels may not be chosen arbitrarily, but are determined by the character of the functions $A_1(p)$, $A_2(p)$ and $B_1(p)$. The two levels, where the vorticity equation is satisfied, may then be called the *equivalent-baroclinic levels for a two-parameter model of the atmosphere.*

It is true that the two-parameter models described above permit unstable waves to develop, which have many characteristics in common with actually observed disturbances, but some essential features seem not to be accounted for. The assumption introduced both by EADY and ELIASSEN regarding the distribution of the vertical velocity means that the level of non-divergence is a horizontal or quasihorizontal surface which does not vary with time. It is easily seen from equation (8) that in order to remove this restriction we must use a more detailed field of z. If ξ denotes any of the variables x, y, t we obtain from (9)

$$\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial p}\right) = \sum_{n=1}^{N} \frac{\partial z_n}{\partial \xi}\frac{d A_n}{dp} \qquad (11)$$

If only one function $dA_n/dp \neq 0$ as Eliassen assumes, nothing is gained by introducing more than one function $B_m(p)$. If on the other hand both $A_1(p)$ and $A_2(p)$ are mutually independent functions which vary with p we may incorporate a more realistic field of vertical motion. Since at least one of these functions contains second powers of p the static stability may vary horizontally in this model, even though only two parameters are needed in order to describe the field of z. Now being able to introduce two parameters for describing the field of vertical motion we need a fourth equation to determine the problem. This fourth equation may be obtained in many different ways but it seems most natural to the author to try to allow for a more refined lower boundary condition by assuming that $B_1(p_0) \neq B_2(p_0) \neq 0$ and by using equation (7) to obtain a relation between ω_1 and ω_2. If we extend the model in this way it does not seem possible to derive a similar correspondance between the finite difference method and the parameter representation similar to the one indicated above. This is intimately related to the fact that the level of non-divergence now is not necessairly an isobaric surface. However, further discussion will have to be deferred until more experience has been gained through actual computations.

As yet very few attempts have been made to use more than two parameters for describing the field of motion. ARNASON (1952, 1953) has discussed such models and it is clear from the presentation above that such models can be derived in many different ways. However, the work involved increases very rapidly, mainly depending upon the fact that the non-linear terms of the type $\mathbf{v} \cdot \nabla\eta$ will be split up in N^2 terms when the expressions (9) and (10) are introduced. It is therefore questionable whether this is the most practical way to proceed if we want to incorporate additional details. It may be better to use finite difference methods also along the vertical axis, an approximation that becomes the better the more layers are used.

If using a parametric representation of the kind discussed here it is quite important to choose the parameters in such a way they are not correlated to each other since otherwise the full descriptive capacity of the multiple parameter representation is not utilized. This criticism applies to both ELIASSEN's and EADY's 2-parameter models as to ARNASON's multiple-parameter model. In principle *this is the reason why a more detailed field of motion may be described by means of only two parameters than is the case in the models mentioned above.*

3. A simplified graphical method for forecasting with a two-parametric model

The following analysis is based on the model suggested by ELIASSEN, but may also be applied to EADY's model. It seems not possible, however, to extend it to more detailed models. We shall neglect the vertical advection of vorticity as well as the term $\mathbf{k} \cdot \nabla\omega \times \partial\mathbf{v}/\partial p$ and finally assume that $\zeta \ll f$ in the divergence term of the vorticity equation. The definitions of h and $A(p)$ are arbitrary to a certain degree and will be stated at a later stage. The two forecast equations derived by ELIASSEN (1952) are

$$\frac{\partial\bar{\zeta}}{\partial t} + \bar{\mathbf{v}} \cdot \nabla(\bar{\zeta} + f) + a\mathbf{v}_T \cdot \nabla\zeta_T = 0 \quad (12)$$

$$\frac{\partial\zeta_T}{\partial t} + \mathbf{v}_T \cdot \nabla(\bar{\zeta} + f) + \bar{\mathbf{v}} \cdot \nabla\zeta_T + b\mathbf{v}_T \cdot \nabla\zeta_T - rf\left(\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h\right) = 0 \quad (13)$$

where

$$\left.\begin{aligned} z(x,y,p,t) &= \bar{z}(x,y,t) + H(p) + A(p)\,[h(x,y,t) - h_n] \\ \bar{\mathbf{v}} &= \frac{g}{f}\nabla\bar{z}\times\mathbf{k} \qquad \mathbf{v}_T = \frac{g}{f}\nabla h\times\mathbf{k} \\ \bar{\zeta} &= \frac{g}{f}\nabla^2 z \qquad \zeta_T = \frac{g}{f}\nabla^2 h \end{aligned}\right\} \quad (14)$$

a, b, and r are constants determined from $A_m(p)$ and $B_m(p)$ and h_n is a constant = normal value of h. These two equations were obtained from (4) by assuming

$$\int^{p_0} V(x,y,p,t)\,dp = 0; \quad \int_0^{p_0} p \cdot V(x,y,p,t)\,dp = 0 \quad (15)$$

Except for the constants these equations are identical with those derived by EADY using a method similar to the one outlined in the previous section. The differences in the constants are insignificant from a principal point of view.

In order to be able to use a graphical computing method similar to the one advanced by FJØRTOFT for the barotropic case we must transform (12) and (13) into two conservation theorems (cf CHARNEY, PHILLIPS, 1953). In fact it is possible to find two quantities each of which are invariant at one level. ELIASSEN pointed out that if two levels p_1 and p_2 are chosen so that $A\,(p_1)$ and $A\,(p_2)$ are the two roots of the equation

$$[A\,(p)]^2 - b\,A\,(p) - a = 0 \qquad (16)$$

we may write

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{A\,(p_2)\,J_1 - A\,(p_1)\,J_2}{A\,(p_2) - A\,(p_1)} = 0 \qquad (17)$$

$$\frac{\partial \zeta_T}{\partial t} + \frac{J_1 - J_2}{A\,(p_1) - A\,(p_2)} - r \cdot f \cdot \left(\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h\right) = 0 \qquad (18)$$

where

$$J_1 = [\bar{\mathbf{v}} + A\,(p_1)\,\mathbf{v}_T] \cdot \nabla[\bar{\zeta} + f + A\,(p_1)\,\zeta_T]$$
$$J_2 = [\bar{\mathbf{v}} + A\,(p_2)\,\mathbf{v}_T] \cdot \nabla[\bar{\zeta} + f + A\,(p_2)\,\zeta_T] \qquad (19)$$

Multiplying (18) with $A\,(p_1)$ and $A\,(p_2)$ and adding to (17) gives us respectively

$$\frac{d_1}{dt}(\zeta_1 + f) - A\,(p_1) \cdot r \cdot f \left(\frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h\right) = 0 \qquad (20)$$

$$\frac{d_2}{dt}(\zeta_2 + f) - A\,(p_2) \cdot r \cdot f \left(\frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h\right) = 0 \qquad (21)$$

where the individual derivatives d_1/dt and d_2/dt refer to the levels p_1 and p_2 respectively. According to the definitions $\mathbf{v}_T \cdot \nabla h = 0$ and thus (20) and (21) may be transformed into

$$\frac{d_i}{dt}(\zeta_i + f) - A\,(p_i) \cdot r \cdot f \cdot \frac{d_i h}{dt} = 0 \quad (i = 1, 2) \qquad (22)$$

Introduce the finite difference expression for ζ_i according to

$$\zeta_i = \frac{4\,g\,m^2}{f \cdot d^2}\,(\bar{z}_i - z_i) \quad (i = 1, 2) \qquad (23)$$

where m is the scale factor for the map projection used, d is the gridsize used in computing $\bar{z}_i$, which represents the mean value of z in the four neighbouring points. We now define h to be the difference in height between the two levels derived above. Thus also $A\,(p)$ is determined. Introducing a function $J\,(\varphi)$ as FJØRTOFT (1952) does we obtain

$$\frac{\partial}{\partial t}[\bar{z}_i - z_i - K_i\,(\varphi)\,(z_1 - z_2)] = -\mathbf{v}_i \cdot \nabla[\bar{z}_i - z_i + J\,(\varphi)] - K_i\,(\varphi)\,\mathbf{v}_i \cdot \nabla(z_1 - z_2) \qquad (24)$$

where

$$K_i\,(\varphi) = A\,(p_i) \cdot \frac{r \cdot f^2 \cdot d^2}{4\,g\,m^2} \qquad (25)$$

To bring $K_i\,(\varphi)$ inside the ∇-operator with a minimum of error we introduce a quantity G being a constant and defined as the mean value of $z_1 - z_2$ over the area. Thus we obtain

$$K_i\,(\varphi)\,\mathbf{v}_i \cdot \nabla(z_1 - z_2) = K_i\,(\varphi)\,\mathbf{v}_i \cdot \nabla(z_1 - z_2 - G) = \mathbf{v}_i \cdot \nabla(K_i\,(\varphi) \cdot \{z_1 - z_2 - G\}) - (z_1 - z_2 - G) \cdot \nabla K_i\,(\varphi) \qquad (26)$$

where the second term may be neglected. We finally obtain

$$\frac{\partial}{\partial t}[\bar{z}_i - z_i + J\,(\varphi) - K_i\,(\varphi)\,(z_1 - z_2 - G] = -\mathbf{v}_i \cdot \nabla[\bar{z}_i - z_i + J\,(\varphi) - K_i\,(\varphi) \cdot (z_1 - z_2 - G)] \quad (i = 1, 2) \qquad (27)$$

These two equations are well suited for graphical solution.

We shall now compare these equations with those obtained by using finite differences in order to evaluate the derivatives along the vertical (cf. CHARNEY, PHILLIPS, 1953). We then choose three levels at equal distances: level a: 250 mb; level b: 500 mb; level c: 750 mb. At level a and c we get by applying the vorticity equation

$$\frac{d_a}{dt}(\zeta_a + f) = \frac{f\omega_b}{2 \cdot \Delta p} \qquad (28)$$

$$\frac{d_c}{dt}(\zeta_c + f) = -\frac{f\omega_b}{2 \cdot \Delta p} \qquad (29)$$

($\Delta p = 250$ mb in this case). The finite difference form of the adiabatic equation at level b gives

$$\frac{\partial}{\partial t}(z_c - z_a) + \mathbf{v}_b \cdot \nabla (z_c - z_a) + \frac{\omega_b}{g \varrho_b} s_b = 0 \qquad (30)$$

s being the vertical stability. If we introduce certain mean values for s_b and ϱ_b which are independant of time and put $\mathbf{v}_b = {}^1/_2 (\mathbf{v}_a + \mathbf{v}_c)$ we can express ω_b in terms of quantities at the two levels a and c and introduce in (28) and (29). The two equations obtained in this way will be of exactly the same form as (22), except for the numerical constants. However, in deriving (22) to be valid at $p = p_1$ and $p = p_2$ we have considered the normal stratification of the atmosphere, while the three levels chosen in the finite difference procedure were chosen more or less arbitrarily. By defining $A(p)$ from actual soundings *we can derive the two levels where the vorticity equation preferably should be applied in finite difference form.*

The interpretation of these two levels may be expressed as follows: If the wind field is assumed to vary with height in the manner expressed by (14) there are two levels (but only two, depending upon the non-linear variation of the vorticity advection) with the aid of which the mean vorticity advection of the atmosphere and the differential vorticity advection within this atmosphere may be expressed.

4. Discussion of numerical values and accuracy of representation

a. Determination of the levels p_1 and p_2

ELIASSEN (1952) has given approximate expressions for the empirical constants that enter the equations and that determine the two levels p_1 and p_2. As far as the author knows one month's radiosonde observations were used and $A(p)$ was determined with the aid of equation (14) using the method of least squares. It should first be pointed out that this function may be determined in many different ways and ultimately it should be chosen to give the best possible forecasts. However, the following remarks will be given already here.

Equation (14) is used in various ways in the equations (12) and (13) namely in order to compute the density or temperature, the geostrophic wind and the geostrophic vorticity. In order to obtain these quantities one or several differentiations of z have to be performed. If now $A(p)$ is determined by fitting the observed z-values according to (14) we cannot be certain that this will be the best function $A(p)$ to represent the *wind field* or the *vorticity* field with the aid of two parameters. We have determined $A(p)$, first by fitting equation (14) directly to the observed values of z, secondly by computing a series of hodographs at different points on the map using in each case three radiosonde observations, then determining the variation of the thermal wind with height and finally determining the function $A(p)$ which will give the smallest error of

$$\nabla z_{obs} - \nabla \bar{z} = A(p) \cdot \nabla h \qquad (31)$$

In both cases z was first determined from

$$\bar{z} = \frac{1}{p_0} \int_0^{p_0} z\, dp, \qquad (32)$$

which integral was evaluated by finite differences using $\Delta p = 100$ mb. As h we have here chosen the observed difference of z between 200 mb and 1,000 mb to represent the thermal structure of the whole troposphere. Admittedly a fairly small amount of data has been used for these determinations: about 50 radiosonde observations in a winter situation and about the same number in a summer situation. In both cases the material covers the whole North American continent. The function $A(p)$ has been computed for winter and summer separately and is given in table 1 column 2 and 3 respectively and graphically in fig. 1. The method first indicated above has been used to obtain these

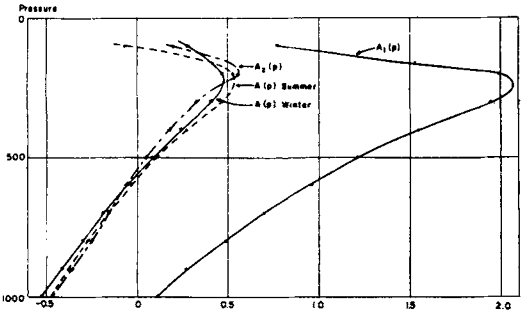

Fig. 1. $A(p)$ computed for a winter and a summer situation. $A_1(p)$ and $A_2(p)$ for the summer situation.

values. Practically the same results are obtained if equation (31) is used. Some differences between summer and winter are observed as is to be expected, however, mainly at high altitudes. The variations between tropical and polar regions probably are larger, but are difficult to take into consideration since the same value of $A(p)$ must be used over the whole area in the equations (12) and (13).

The values given here differ considerably from those given by ELIASSEN (1952). Because of this the values of a and b also become different. We obtain $a = 0.10$, which corresponds to $a = 0.30$—0.35 with Eliassen's definition of h. b seems not to be significantly different from zero. From equation (16) and (14) we then obtain

$$\begin{cases} p_1 \approx 350 \text{ mb} \\ p_2 \approx 825 \text{ mb} \end{cases}$$

which thus are the approximate pressures at the equivalent-baroclinic levels of this two-parameter model of the atmosphere.

b. Accuracy of a two-parameter representation

There are two different factors that have to be considered when going from simple to more complicated models of the atmosphere for forecasting purposes. Primarily, how would it be possible to treat the hydrodynamic equations more exactly without increasing the computational work too much, secondly, how much details may we incorporate in our models without going beyond the accuracy of the input data, the temperature soundings? It is therefore of particular interest to investigate the accuracy of the representation given by equation (14) or more generally by assuming $N = 2$ in equation (9). A few estimates have therefore been made using the summer case. Column 4 of table 1 gives the standard deviation (σ_1) of the value of $z_{obs}(x, y, p, t) - H(p)$ and in column 5 the standard deviation (σ_2) is given of the errors of $z(x, y, p, t)$ if using the representation proposed by Eliassen. The standard deviation of the corresponding temperature error (σ_{T_2}) at various levels is given in column 6. It is clearly seen that conditions near the ground are not very well described, while the temperature field in the rest of the troposphere is represented with an accuracy of about 2.5° C. Between 200 and 300 mb the accuracy is again slightly less, obviously depending upon the variation of the height of the tropopause. It should be remembered that the function $A(p)$ used here has been determined to fit this particular case, and thus these figures probably become slightly larger if the application is made to another weather situation. However, the

Table 1.

1	2	3	4	5	6	7	8	9	10	11	12
mb	$A(p)$ winter case	$A(p)$ summer case	σ_1 10 feet	σ_2 10 feet	σ_{T_2} °C	$A_1(p)$	$A_2(p)$	σ_3 10 feet	σ_{T_3} C°	σ_s °C/100	σ_4 10 feet
1000	—0.53	—0.47	18	10	7.5	0.11	—0.47	8	6.5		1
900	—0.41	—0.38	17	6	4.0	0.27	—0.36	4	3.5	0.50	2
800	—0.30	—0.27	23	4	3.0	0.49	—0.25	4	2.5	0.30	3
700	—0.19	—0.16	31	5	2.0	0.70	—0.16	5	2.0	0.25	4
600	—0.05	—0.03	42	6	1.5	0.96	—0.06	6	1.5	0.10	5
500	0.08	0.10	53	7	2.0	1.21	0.05	7	1.5	0.10	7
400	0.24	0.27	68	9	2.0	1.55	0.18	8	2.0	0.15	9
300	0.41	0.47	82	10	4.5	1.94	0.33	8	3.5	0.15	12
200	0.47	0.53	88	10	3.5	1.99	0.56	8	3.0	0.15	15
100	0.28	—0.07	39	21		0.77	0.20	19			—

values of $A(p)$ seem to vary only a small amount from one case to another as can been seen from a comparison between the winter and summer cases (col. 2 and 3). Normally we therefore may expect these values to be perhaps 10—20 % larger, at 200 and 100 mb maybe still somewhat more.

In section 2 it was very briefly indicated how a somewhat more adjustable 2-parameter model might be obtained by trying to determine *two* empirical functions $A_1(p)$ and $A_2(p)$ instead of just one as in Eliassen's model. Before developing the equations that would govern such a model it is of some interest to see if such an attempt would give a more accurate representation of reality. In the columns 7 and 8 of table 1 these two functions are given as obtained in this particular case (cf fig. 1). As our parameter z_1 we have chosen $\bar{z} - \overline{H(p)}$ and for the other parameter z_2

$$z_2(x, y, t) = [z_{obs}(x, y, 200, t) - H(200) - A_1(200) \cdot z_1(x, y, t)] - [z_{obs}(x, y, 1000, t) - H(1000) - A_1(1000) \cdot z_1(x, y, t)]$$

In the columns 9, 10 and 11 the standard deviations of the errors in this representation of $z(x, y, p, t)$,$[\sigma_3]$, and the corresponding standard deviations of the errors of the temperature (σ_{T3}) and static stability (σ_s) are given. A slight improvement is obtained if compared with the representation proposed by Eliassen, but it is surprisingly small. The error of the determination of stability is quite large in the layers close to the surface of the earth but varies between 0.10 and 0.15° C/100 m in the middle and upper part of the troposphere. However, the errors of this representation still show certain systematic characteristics that probably are real. This cannot be judged, however, until we have made a comparison with the accuracy of the various radiosondes in use at present.

A comparison between six different kinds of radiosondes has recently been made in Payerne, Switzerland (cf. e.g. NYBERG, 1952). From this investigation we have determined the standard deviation of the error in computing the height of various pressure levels and these values have been entered in column 12 of table 1. *Above 600 mb the errors in the observations are as large as or larger than the errors introduced by a two-parameter representation.*

In relatity the situation is somwehat better than these figures indicate. In USA for example the same kind of radiosonde is used over the whole continent which reduces the errors, in particular when computing gradients of z. However, these figures clearly indicate that *the exactness of the dynamic equations must not be carried too far but must also be determined from the accuracy of the observations.*

The fact that the errors in the representation of z remain comparatively small up to about 200 mb depends upon our choice of the second parameter as the difference between 200 mb and 1,000 mb. In order to increase the accuracy at lower levels, where the observations justify it, this second parameter should be chosen as the difference between for example 300 mb and 1000 mb or may be even 400 mb and 1000 mb.

5. An example of a baroclinic forecast

We shall finally give an example of forecasting with the two-parameter model using the graphical method indicated in section 3. We have chosen a case over USA in which a marked baroclinic development took place. The 300 mb and 700 mb maps have been used since the heights of the 350 and 825 mb levels were not available. The variable $\bar{z}$ has in this case been computed from $\bar{z} = 0.4\,[z(x, y, 300, t) + z(x, y, 700, t)]$, while h has been assumed to be $h = (z(x, y, 300, t) - z(z, y, 700, t))$. As a comparison a barotropic forecast with Fjørtoft's method has been made using the 500 mb map. The initial maps as well as the observed and forecasted 12-hour changes are shown in figs. 2—10. The effects of the mountains have been included in a way that was described by BOLIN, CHARNEY, (1951). In tables 2 and 3 some figures are given to express the success of the forecasts.

Table 2. Correlation coefficients between observed and computed 12-hour changes

Obs. 500-mb change — Comp. barotropic change	0.72
Obs. 300 + 700-mb change — Comp. baroclinic $z_1 + z_2$ change	0.79
Obs. 300 — 700-mb change — Comp. baroclinic $z_1 - z_2$ change	0.72

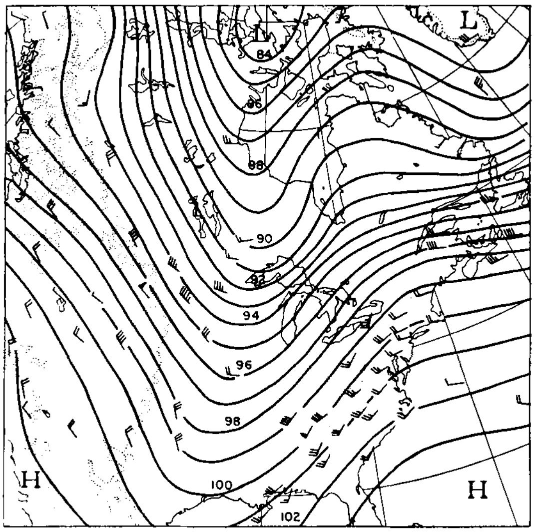

Fig. 2. Contour-field of the 700 mb surface Feb. 6, 1951, 15 GMT. Lines are drawn for every 100 feet.

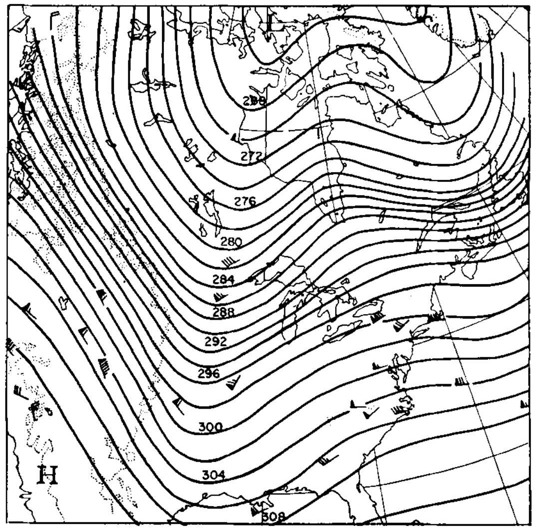

Fig. 3. Contour-field of the 300 mb surface Feb. 6, 1951, 15 GMT. Lines are drawn for every 200 feet.

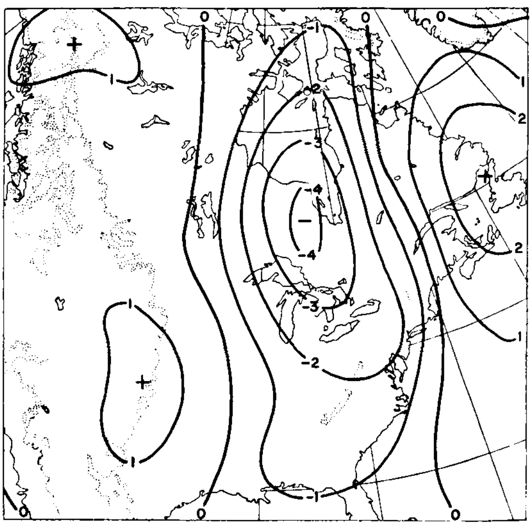

Fig. 4. Computed 12-hour change of the 700 mb surface from Feb. 6, 15 GMT—Feb. 7, 03 GMT 1951. Unit 100 feet.

Fig. 5. Observed 12-hour change of the 700 mb surface from Feb. 6, 15 GMT—Feb. 7, 03 GMT 1951. Unit 100 feet.

Table 3. Standard deviations of observed and computed 12-hour changes (100 feet).

Obs. 500 mb change	2.3
Barotropic forecast (using 500 mb)	1.9
Obs. change of 300 + 700 mb	4.6
Comp. change of $z_1 + z_2$	4.4
Obs change of 300 — 700 mb	2.0
Comp. change of $z_1 - z_2$	1.7

A slight improvement is obtained if compared with the barotropic forecast and in particular the intensification of the trough over the Middle West is better described by the 2-parameter model. When inspecting the maps more closely there are two facts that should be brought out.

1. The computed centre of maximum fall is systematically displaced to the north of the observed centre, an observation that has also

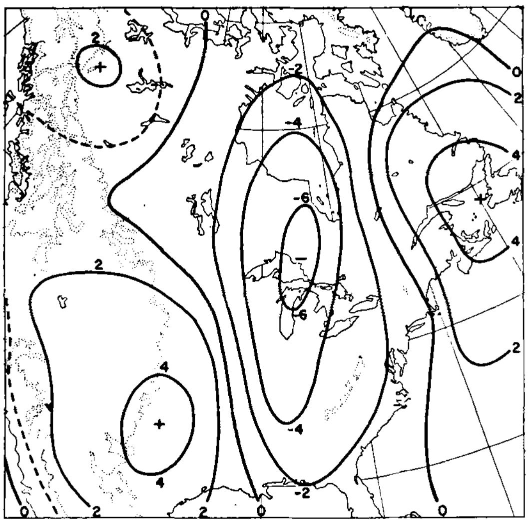

Fig. 6. Computed 12-hour change of the 300 mb surface from Feb. 6, 15 GMT—Feb. 7, 03 GMT 1951. Unit 100 feet.

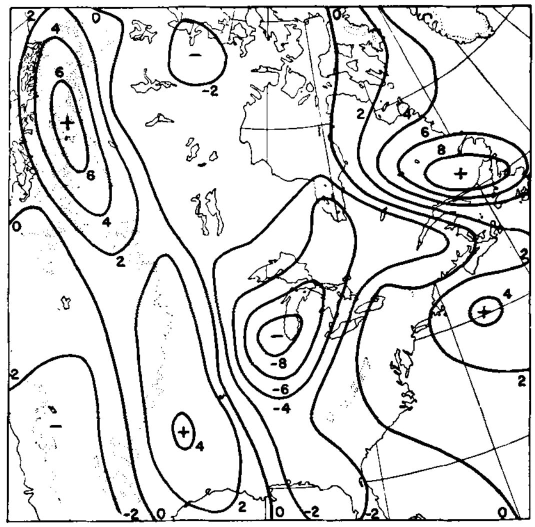

Fig. 7. Observed 12-hour change of the 300 mb surface from Feb. 6, 15 GMT—Feb. 7, 03 GMT 1951. Unit 100 feet.

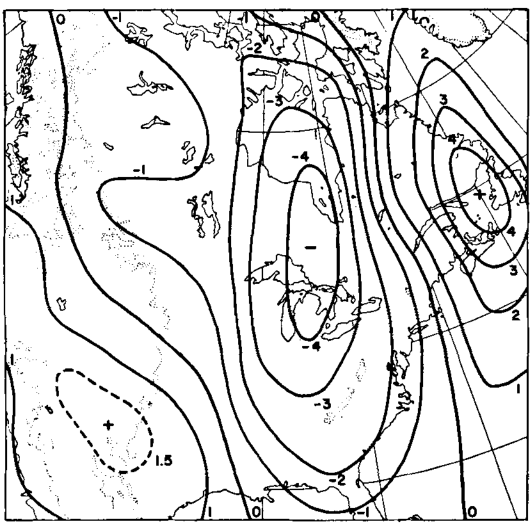

Fig. 8. Computed 12-hour change of the 500 mb surface from Feb. 6, 15 GMT—Feb. 7, 03 GMT 1951. Unit 100 feet.

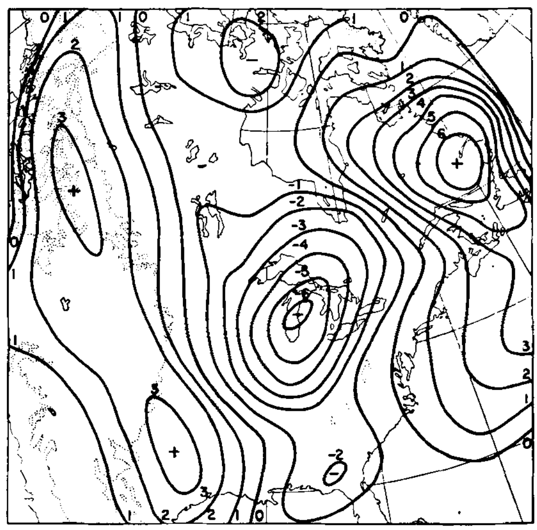

Fig. 9. Observed 12-hour change of the 500 mb surface from Feb. 6, 15 GMT—Feb. 7, 03 GMT 1951. Unit 100 feet.

been made in Princeton (Charney, Phillips, 1953).

2. The observed changes are considerably more concentrated into sharp maxima than is computed. This seems to be an indication that the influence between different points is too large in the way the forecast is made. It may also to some extent depend upon using too large a grid-size, which will smooth the field. Some smoothing is also introduced by the graphical method.

However, the final judgement cannot be made until a series of forecasts has been made, preferably by using a machine.

REFERENCES

ARNASON, G., 1952, 1953: A Baroclinic Model of the Atmosphere Applicable to the Problem of Numerical Forecasting in Three Dimensions. I *Tellus* **4**, pp 356—373, II *Tellus* **5**, No 3.

BOLIN, B., CHARNEY, J., 1951: Numerical Tendency Computations from the Barotropic Vorticity Equation. *Tellus* **3**, pp 248—257.

CHARNEY, J., FJØRTOFT, R., VON NEUMANN, J., 1950: Numerical Integration of the Barotropic Vorticity Equation. *Tellus* **2**, pp 237—254.

CHARNEY, J., PHILLIPS N., 1953: To be published in *J. Meteor*, **10**, 2.

EADY, E. T., 1952: Note on Weather Computing and the so-called $2\,^1/_2$-dimensional Model. *Tellus* **4**, pp 157—167.

ELIASSEN, A., 1952: Simplified Models of the Atmosphere, Designed for the Purpose of Numerical Weather Prediction, *Tellus* **4**, pp 145—156.

FJØRTOFT, R., 1952: On a Numerical Method of Integrating the Barotropic Vorticity Equation. *Tellus* **4**, pp 179—194.

NYBERG, A., 1952: On the Comparison of Radiosonde Data in Payerne May 1950. *Sveriges Meteorologiska och Hydrologiska Institut Medd.* Serie B, No. 9. 22 pp.

PHILLIPS, N., 1951: A simple Three-dimensional Model for the Study of Large-scale Extratropical Flow Patterns. *J. Meteor*, **8**, pp 381—394.