Note on the Classic Stability Problem

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It has come to the writer's attention (private communication) that a question has arisen concerning his comments (STARR 1950) in regard to the stability of zonal flow. In terms of the problem as outlined previously, and in terms of the same symbolism, the question relates to the statement (1) that a monotone distribution of the vorticity ζ with respect to y_0 implies stability of zonal motion. The point at issue is whether an incipient small disturbance of otherwise *arbitrary* nature imposed upon such zonal flow can so alter the circumstances of the problem as to render the previous conclusion unjustified. The purpose of this second note is to show that such small arbitrary perturbations cannot lead to instability and hence need not be considered explicitly in the case of a sensibly montone vorticity distribution.

A disturbed vorticity pattern having one local extreme of vorticity superposed on a monotone distribution is shown in the figure. The entire zonal belt A may now be divided as shown into three regions A_1, A_2, A_3, the second being simply connected while the other two are doubly connected. Obviously the vorticity lines in A_2 do not encircle

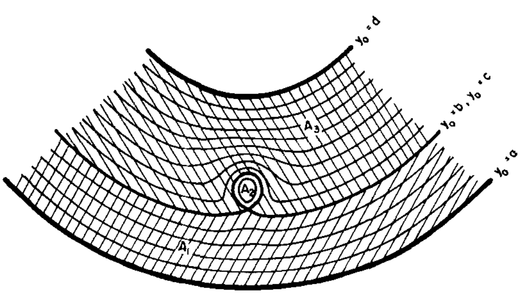

Figure 1. See text.

the zonal belt. It is to be observed that y_0 which i the mean value of y for the lines of constant ζ changes value abruptly from b to c as one passes from region A_1 to region A_3 along the common boundary. We may now form the integral $\triangle$ over the regions A_1 and A_3 where the ζ lines encircle the whole belt. This quantity is then related to the invariant I for the entire region A as follows:

$$\int_a^b \frac{d\zeta}{dy_0} \oint \varepsilon^2 d\lambda\, dy_0 + \int_c^d \frac{d\zeta}{dy_0} \oint \varepsilon^2 d\lambda\, dy_0 =$$

$$= \oint_{A_2} \zeta y^2 d\lambda + \frac{2}{r^2} \iint_{A_2} y \zeta\, dA - \frac{2}{r^2} I + \text{invariant}.$$

This equation may be verified by the methods previously used. The two integrals on the left could be written as a single one with proper interpretations as regards the discontinuity in y_0 between A_1 and A_3. The first term on the right is the line integral of ζy^2 around the region A_2 with respect to λ, while the second involves the area integral of $y\zeta$ over this same region. Here according to our assumptions $d\zeta/dy_0$ is of the same algebraic sign in the regions A_1 and A_2.

If we are to deal with an incipient small disturbance in the initial state, it is essential that each of of the four integrals mentioned above should at most be infinitesimals as compared with $2I/r^2$. Otherwise the vorticity distribution would be sensibly disturbed already at the initial moment, and the problem would not be one relating to the stability of sensibly zonal flow. Generally speaking this means that the area A_2 should be small initially (and therefore also later) as compared with A, and that ε in regions A_1 and A_3 should be infinitesimal as compared with $d—a$.

Since $d\zeta/dt = 0$ and the range of y is limited, the first two integrals on the right can never become finite in the subsequent motions of the fluid so that as a result, due to the invariance of I, the integrals on the left can never become finite. It thus follows that sensible departures from a zonal distribution of vorticity cannot develop later. Several further points should be noted.

(a) Any finite number of closed regions similar to A_2 may be included by generalizing the treatment. Likewise some of these regions may be contiguous with the latitude walls without producing a change in the result. In the end the treatment can include all possible topological configurations of the incipient disturbances, bringing them under the scope of the arguments set forth above.

(b) In all probability other and perhaps better arrangements of the argument given may be made. However in any case the crux of the problem lies in the fact that the development of finite anomalies in the vorticity distribution would require a finite change in the average value of y for all the vortices present. This is precluded by the invariance of I. On the other hand, if there were substantial fractions of the area A over which the vorticity distribution is not monotone, the constraint imposed by the invariance of I would no longer prevent the development of such anomalies, since contributions from different regions to such a change of the mean value of y for the vortices may then cancel.

(c) In most discussions of stability the question ultimately arises as to what is to be regarded as a suitable definition of this term. For the present problem such questions are to a large part due to the fact that we do not have sufficiently general explicit solutions of the vorticity equation at our disposal. Thus if a sufficiently complete and accurate discription of the physical behavior of hydrodynamic systems such as the one here treated were available, the question would be primarily one of terminology.

In may be noted, however, that the treatment given contains one particular instance where the meaning of stability is considerably simpler, namely the case of rotation as a solid ($d\zeta/dy_0$ = constant). This may serve as an illustration of the essential properties of a stable system, although from viewpoints *other than the present one* it possesses special characteristics differing from those of stable systems in general.

(d) It is interesting to speculate as to whether it is permissible to simplify vorticity distributions such as the one shown in the figure by the use of what may be termed VORTICITY CUTS. By this is meant the artificial introduction of lines along which the vorticity lines are extremely closely packed but along which the vorticity remains finite in intensity. By the appropriate use of such devices all possible vorticity distributions in the present problem may be reduced to the class of cases for which all the vorticity lines extend completely around the zonal belt, without altering the corresponding motions significantly. Conclusions concerning stability could then be reached more directly.

REFERENCE

STARR, V. P., 1950: Note on recent study of stability by R. Fjørtoft. *Tellus*, 2, pp. 321—322.