Energy Conversions in the Atmosphere on the Scale of the General Circulation

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Abstract

From the equations of balance established for the different forms of energy (potential, kinetic and internal energies), the energy fluxes, the rates of energy production and conversion are deduced. Provided weighted mean values are considered the zonally averaged value of the kinetic energy may be decomposed into kinetic energy of the mean motion and the mean kinetic energy of the large-scale eddies. The corresponding equations of balance are established. The axial symmetry with respect to the earth's axis allows the introduction of the kinetic energy of the zonal motion and of the motion in the meridional planes. Their equations of balance are given. The conversions between the different species of mechanical energies are envisaged and the influence of the inertial stability on the conversion of "meridional" eddy-kinetic energy into "zonal" eddy-kinetic energy is pointed out. The definition of the large-scale eddy diffusion of heat is deduced from the first law of thermodynamics. This definition contains the flux of "sensible heat" and of "latent heat" as an approximation. Neglecting viscosity and small-scale turbulence, the general formulas are applied to the large atmospheric disturbances. A tentative formulation of the second law of thermodynamics is given.

1. Introduction

For the investigation of the different possible energy conversions in the atmosphere, on the scale of the general circulation, it is necessary to introduce a state of mean motion such that the averaged quantities vary slowly from one point (x^1, x^2, x^3) to another and from one instant (t) to another. This condition is fulfilled when the averages are taken for a one-, two- or three-dimensional domain of space, or for a time interval, or for a space-time domain, which is large with respect to the volume and the local lifetime of the perturbations (large-scale eddies) imbedded in the mean flow.

In this article only mean values along latitude circles will be considered. Let $\overline{X}\,(x^2, x^3, t)$ be the zonal mean value of the function $X\,(x^1, x^2, x^3, t)$ where x^1 designates the longitude λ. The corresponding weighted mean value $\hat{X}$ is defined by $\overline{\varrho}\,\hat{X} = \overline{\varrho X}$, ϱ being the specific mass of the air. The fluctuations X' and X'' of X with respect to the mean values $\overline{X}$ and $\hat{X}$ are given by $X = \overline{X} + X' = \hat{X} + X''$, with the conditions $\overline{X'} \equiv 0$, $\overline{\varrho X''} \equiv 0$, $X'' = \overline{X''} + X'$ and $-\overline{\varrho}\overline{X''} = \overline{\varrho' X''} = \overline{\varrho' X'} = \overline{\varrho X'} = \overline{\varrho' X}$, (VAN MIEGHEM, 1949).

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Following a general method outlined elsewhere (VAN MIEGHEM, 1949) we establish successively the equations of balance of the kinetic, potential and internal energies of the zonally averaged motion and of the mean kinetic energy of the large-scale eddies. In the atmosphere of a rotating planet, the concepts of "zonal" and "meridional" kinetic energies are introduced and the equations of balance for these energies are also given. From the divergence form of the first member of an equation of balance, the flux of the envisaged property is deduced. The second member of the equation of balance defines the rate of production of that property. The rate of production of any of the above mentioned energy forms may be expressed as a sum of terms such that each of them appears also in the rate of production of only one other energy form, but with the opposite sign. Such a term may then be interpreted as a rate of conversion between the two energy forms (energy transformation function of J. E. MILLER, 1950).

Finally, evidence is given of the influence of the inertial stability on the conversion between the "zonal" and the "meridional" kinetic energy of the large-scale eddies.

2. Equations for non-averaged properties

The system of coordinates which is most appropriate for the study of the general circulation is one of non-local coordinates, symmetric with respect to the earth's axis. Let x^1 be the azimuthal coordinate (the longitude λ) and (x^2, x^3) arbitrary coordinates in the meridional plane, $\gamma_{ij} \equiv \gamma_{ji}$ the coefficients of the metric form in the variables x^i $(i, j = 1, 2, 3)$, γ^{ij} the algebraic minor of γ_{ij} divided by the determinant $\gamma \equiv \|\gamma_{ij}\|$, $\mathrm{v}^i \equiv \frac{dx^i}{dt}$ and $\mathrm{v}_i = \gamma_{ij}\,\mathrm{v}^j$ respectively the contravariant and components of the air velocity $\mathbf{v}$ with respect to the earth, ω the angular speed of the earth, $\emptyset$ the geopotential, T^{ij}, T_{ij}, T^i_j respectively the contravariant, covariant and mixed components of the internal stresses. We know that $T_{ij} \equiv -p\gamma_{ij} + P_{ij} + R_{ij}$, where p is the pressure, $P_{ij} \equiv P_{ji}$ the Stokes (1842)—Navier (1822) viscosity stresses and $R_{ij} \equiv R_{ji}$ the Reynolds stresses (1895) due to small-scale turbulence.

It should be noted that the coefficients γ_{12} and γ_{13} are identically equal to zero, and that the four other γ's (namely: γ_{11}, γ_{22}, γ_{23}, γ_{33}) and the geopotential $\emptyset$ are independent of time t and longitude $x^1 \equiv \lambda$.

The coefficient γ_{11} represents the square of the distance to the earth's axis; the coefficients γ_{22} and γ_{33} define the linear elements $\sqrt{\gamma_{22}}\,\delta x^2$ and $\sqrt{\gamma_{33}}\,\delta x^3$ of the coordinate lines $x^3 =$ constant and $x^2 =$ constant in the meridional planes and γ_{23} the angle α between these lines $(\sqrt{\gamma_{22}\,\gamma_{23}}\cos\alpha = \gamma_{23})$.

In spherical coordinates: $x^1 \equiv \lambda$, $x^2 \equiv \varphi$, $x^3 \equiv r$, so that we have $\gamma_{11} \equiv r^2\cos^2\varphi$, $\gamma_{22} \equiv r^2$, $\gamma_{33} \equiv 1$, $\gamma_{ij} \equiv 0$ when $i \neq j$, and $\gamma \equiv r^4\cos^2\varphi$. The usual zonal, meridional and vertical components (u, v, w) of the air velocity $\mathbf{v}$ are:

$$u \equiv r\cos\varphi\,\mathrm{v}^1 \equiv r\cos\varphi\frac{d\lambda}{dt} \equiv \frac{\mathrm{v}_1}{r\cos\varphi},\quad v \equiv r\mathrm{v}^2 \equiv r\frac{d\varphi}{dt} \equiv \frac{\mathrm{v}_2}{r},\quad w \equiv \mathrm{v}^3 \equiv \frac{dr}{dt} \equiv \mathrm{v}_3.$$

The coordinate components of the tensor $\mathbf{T}$ in the usual zonal, meridional and vertical axis x, y, z in an arbitrary point are:

$$T^x_x \equiv T^1_1,\ T^y_y \equiv T^2_2,\ T^z_z \equiv T^3_3,\ T^x_y \equiv T^y_x \equiv \cos\varphi\ T^1_2 \equiv \frac{T^2_1}{\cos\varphi},\ T^y_z \equiv T^z_y \equiv r\ T^2_3 \equiv \frac{T^3_2}{r},$$

$$T^z_x \equiv T^x_z \equiv r\cos\varphi\ T^1_3 \equiv \frac{T^3_1}{r\cos\varphi}.$$

In axially symmetric coordinates (x^1, x^2, x^3) the equations of motion assume the form:

$$\begin{cases} \varrho \dfrac{d\mathrm{v}_1}{dt} = -\omega\, \gamma_{11,\alpha}\, \varrho\, \mathrm{v}^\alpha + T^i_{1\cdot i}, & (1) \\ \varrho \dfrac{d\mathrm{v}_\delta}{dt} - \dfrac{1}{2}\gamma_{\alpha\beta,\delta}\, \varrho\, \mathrm{v}^\alpha \mathrm{v}^\beta = \dfrac{1}{2}\gamma_{11,\delta}\,(2\omega + \mathrm{v}^1)\, \varrho\, \mathrm{v}^1 + T^i_{\delta\cdot i} - \varrho \dfrac{\partial \varnothing}{\partial x^\delta}, & (2) \end{cases}$$

$$(i = 1, 2, 3;\ \alpha, \beta\ \delta = 2, 3)$$

where the point before the suffix i indicates the covariant derivation with respect to x^i and where $\dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} + \mathrm{v}^i \dfrac{\partial}{\partial x^i}$ and $\gamma_{ij,\alpha} \equiv \dfrac{\partial \gamma_{ij}}{\partial x^\alpha}$.

In spherical coordinates we have

$$\begin{cases} \dfrac{1}{r\cos\varphi} T^i_{1\cdot i} \equiv \dfrac{\partial T^x_x}{\partial x} + \dfrac{1}{\cos^2\varphi}\dfrac{\partial(\cos^2\varphi\, T^y_x)}{\partial y} + \dfrac{1}{r^3}\dfrac{\partial(r^3 T^z_x)}{\partial z}, \\ \dfrac{1}{r} T^i_{2\cdot i} \equiv \dfrac{\partial T^x_y}{\partial x} + \dfrac{1}{\cos\varphi}\dfrac{\partial(\cos\varphi\, T^y_y)}{\partial y} + \dfrac{1}{r^3}\dfrac{\partial(r^3 T^z_y)}{\partial z} + T^x_x \dfrac{\operatorname{tg}\varphi}{r}, \\ T^i_{3\cdot i} \equiv \dfrac{\partial T^x_z}{\partial x} + \dfrac{1}{\cos\varphi}\dfrac{\partial(\cos\varphi\, T^y_z)}{\partial y} + \dfrac{1}{r^2}\dfrac{\partial(r^2 T^z_z)}{\partial z} - (T^z_z + T^y_y)\dfrac{1}{r}, \end{cases}$$

where $\dfrac{\partial}{\partial x} \equiv \dfrac{1}{r\cos\varphi}\dfrac{\partial}{\partial\lambda}$, $\dfrac{\partial}{\partial y} \equiv \dfrac{1}{r}\dfrac{\partial}{\partial\varphi}$, $\dfrac{\partial}{\partial z} \equiv \dfrac{\partial}{\partial r}$.

In order to make a clear distinction between "zonal" and "meridional" quantities, two sets of indices are used systematically: $i, j, k, \ldots = 1, 2, 3$ and $\alpha, \beta, \delta, \ldots = 2, 3$, the last set designating only coordinates and components in the meridional planes.

After multiplication of (1) by v^1, of (2) by v^δ, summation with respect to δ $(= 2, 3)$ and substitution of the equation of continuity

$$\frac{\partial \varrho}{\partial t} + \frac{1}{\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\,\varrho\,\mathrm{v}^i)}{\partial x^i} = 0 \qquad \text{or} \qquad \frac{\partial\varrho}{\partial t} + (\varrho\,\mathrm{v}^i)_{\cdot i} = 0, \tag{3}$$

one obtains the equations of the "zonal" $\left(k_\xi \equiv \dfrac{1}{2}\mathrm{v}_1 \mathrm{v}^1\right)$ and "meridional" $\left(k_\mu \equiv \dfrac{1}{2}\mathrm{v}_\alpha \mathrm{v}^\alpha\right)$ kinetic energies

$$\begin{cases} \dfrac{\partial(\varrho k_\xi)}{\partial t} + \dfrac{1}{\sqrt{\gamma}}\dfrac{\partial}{\partial x^i}\left[\sqrt{\gamma}\,(\varrho k_\xi \mathrm{v}^i - T^i_1 \mathrm{v}^1)\right] = -\dfrac{1}{2}\gamma_{11,\alpha}(2\omega + \mathrm{v}^1)\,\varrho\,\mathrm{v}^1\mathrm{v}^\alpha - T^i_1 \dfrac{\partial \mathrm{v}^1}{\partial x^i}, & (4) \\ \dfrac{\partial(\varrho k_\mu)}{\partial t} + \dfrac{1}{\sqrt{\gamma}}\dfrac{\partial}{\partial x^i}\left[\sqrt{\gamma}\,(\varrho k_\mu \mathrm{v}^i - T^i_\alpha \mathrm{v}^\alpha\right] = & \\ \qquad \dfrac{1}{2}\gamma_{11,\alpha}\mathrm{v}^\alpha\left[(2\omega + \mathrm{v}^1)\,\varrho\,\mathrm{v}^1 - T^{11}\right] - T^\beta_\alpha \mathrm{v}^\alpha_{\cdot\beta} - T^1_\alpha \dfrac{\partial \mathrm{v}^\alpha}{\partial x^1} - g\varrho w, & (5) \end{cases}$$

$$(i = 1, 2, 3;\ \alpha = 2, 3)$$

and by addition of (4) and (5), the equation of balance of the total kinetic energy $k \equiv k_\xi + k_\mu \equiv \frac{1}{2} v_i v^i$,

$$\frac{\partial(\varrho k)}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} [\sqrt{\gamma} (\varrho k v^i - T^i_j v^j)] = - T^i_j v^j_{.i} - g \varrho w. \tag{6}$$

Further, the equation of balance for the potential energy may be written

$$\frac{\partial(\varrho \varnothing)}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} (\sqrt{\gamma} \varrho \varnothing v^i) = \varrho \frac{\partial \varnothing}{\partial x^\alpha} v^\alpha \cong g \varrho w, \tag{7}$$

where $\varnothing$ is the geopotential (or the potential energy per unit of mass), g the acceleration of gravity and w the vertical cartesian component of the air velocity $\mathbf{v}$.

Finally, multiplying (1) by v_β and (2) by v_1 adding and substituting (3), the equation of the tensor component $\varrho v_1 v_\delta$ is easily obtained:

$$\frac{\partial}{\partial t} (\varrho v_1 v_\delta) + [\varrho v_1 v_\delta v^i - T^i_1 v_\delta - T^i_\delta v_1]_{.\,i} = - \omega (\gamma_{11,\alpha} \varrho v^\alpha v_\delta - \gamma_{11,\delta} \varrho v^1 v_1) -$$
$$- (T^i_1 v_{\delta \cdot i} + T^i_\delta v_{1 \cdot i}) - \varrho \frac{\partial \varnothing}{\partial x^\delta} v_1. \tag{8}$$

In the same way, an equation for the tensor component $\varrho v_\alpha v_\beta$ may be established.

3. Equations for the averaged properties

The averaged motion being defined by the zonal mean values $\bar{\varrho}$ and $\overline{\varrho v^i}$ of the specific mass ϱ and of the contravariant components of the relative linear momentum $\varrho \mathbf{v}$, is independent of longitude λ or x^1. In order to take full advantage of this circumstance, we shall consider all the possible transformations of coordinates in the meridional plane only, thus keeping the longitude λ as independent variable. Then it is obvious that v^1, v_1, γ_{11}, γ^{11}, T_{11}, T^{11} and T^1_1 are invariants, v^α, T^α_1, $\gamma^{\alpha\beta}$ and $T^{\alpha\beta}$ contravariant, and v_α, T^1_α, $\gamma_{\alpha\beta}$ and $T_{\alpha\beta}$ covariant quantities with respect to any change of variables x^2 and x^3. Hence, both $k_{\xi m} \equiv \frac{1}{2} \hat{v}_1 \hat{v}^1$ and $k_{\mu m} \equiv \frac{1}{2} \hat{v}_\alpha \hat{v}^\alpha$ are independent of longitude and invariant with respect to any change of meridional variables; this justifies the introduction of the "zonal" and "meridional" energies of the mean motion.

Every equation in connection with the zonally averaged motion will be written in such a way that all terms have a variance with respect to the transformations of the meridional coordinates only.

From the averaged equation (7), we deduce the components $C^\alpha(\varnothing)$ of the meridional flux and the rate of production $\sigma(\varnothing)$ of the potential energy $\varnothing$ of the mean motion:

$$C^\alpha(\varnothing) \equiv \varnothing \overline{\varrho v^\alpha} = \varnothing (\bar{\varrho} \bar{v}^\alpha + \overline{\varrho' v'^\alpha}) \quad \text{and} \quad \sigma(\varnothing) \equiv g \overline{\varrho w} = g (\bar{\varrho} \bar{w} + \overline{\varrho' w'}). \tag{9}$$

In most cases, the correlation between ϱ' and w' is negative, and the correlation between ϱ' and v' is negative in the troposphere, but positive in the stratosphere. Consequently, the flux $\overline{\varrho' \mathbf{v}'}$ is generally directed toward the earth, both in the stratosphere and in the troposphere, toward the pole in the stratosphere, and toward the equator in the troposphere. The second term $g \overline{\varrho' w'}$ of $\sigma(\varnothing)$ is generally negative so that it represents a destruction of potential energy of the mean motion. From the first term of $\sigma(\varnothing)$, it follows that production of potential energy of the mean motion occurs in the ascending branch ($\overline{w} > 0$), and destruction of potential energy in the descending branch ($\overline{w} < 0$) of the mean meridional circulation ($\bar{v}$, $\overline{w}$).

Using the equations of the zonally averaged motion [instead of (1) and (2)] and the equation of continuity of the mean motion

$$\frac{\partial \bar{\varrho}}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} (\sqrt{\gamma}\, \bar{\varrho}\, \hat{v}^\alpha) = 0, \tag{10}$$

instead of (3), equations of balance for the "zonal" $\left(k_{\xi m} \equiv \frac{1}{2} \hat{v}_1 \hat{v}^1\right)$ and "meridional" $\left(k_{\mu m} \equiv \frac{1}{2} \hat{v}_\alpha \hat{v}^\alpha\right)$ kinetic energies of the mean motion, similar to (4) and (5), are obtained:

$$\frac{\partial}{\partial t} (\bar{\varrho}\, k_{\xi m}) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} [\sqrt{\gamma}\, \bar{\varrho}\, k_{\xi m}\, \hat{v}^\alpha - \overline{T}{}_1^\alpha \hat{v}^1 + \overline{\varrho\, v_1'' v''^\alpha}\, \hat{v}^1] = \tag{11}$$
$$= -\frac{1}{2} \gamma_{11,\alpha} (\hat{v}^1 + 2\omega)\, \bar{\varrho}\, \hat{v}^1 \hat{v}^\alpha - \overline{T}{}_1^\alpha \frac{\partial \hat{v}^1}{\partial x^\alpha} + \overline{\varrho\, v_1'' v''^\alpha} \frac{\partial \hat{v}^1}{\partial x^\alpha},$$

and

$$\frac{\partial}{\partial t} (\bar{\varrho}\, k_{\mu m}) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} [\sqrt{\gamma} (\bar{\varrho}\, k_{\mu m}\, \hat{v}^\alpha - \overline{T}{}_\beta^\alpha \hat{v}^\beta + \overline{\varrho\, v_\beta'' v''^\alpha}\, \hat{v}^\beta] = \tag{12}$$
$$= \frac{1}{2} \gamma_{11,\alpha} \hat{v}^\alpha \left[(\hat{v}^1 + 2\omega)\, \bar{\varrho}\, \hat{v}^1 - \overline{T}{}^{11} + \overline{\varrho\, v''^1 v''^1}\right] - \overline{T}{}_\alpha^\beta\, \hat{v}^\alpha_{\cdot\beta} + \overline{\varrho\, v''^\beta v_\alpha''}\, \hat{v}^\alpha_{\cdot\beta} - g\, \overline{\varrho w},$$

where $-\overline{\varrho\, v_i'' v''^j}$ represent the Jeffreys-stresses (1926) due to the large-scale eddies.

Averaging equations (4) and (5) along the latitude circles and subtracting from them the equations (11) and (12), one obtains the equations of balance of the "zonal" $\left(k_{\xi e} \equiv \frac{1}{2} v''^1 v_1''\right)$ and "meridional" $\left(k_{\mu e} \equiv \frac{1}{2} v''^\alpha v_\alpha''\right)$ kinetic energies of the large-scale eddying motion:

$$\frac{\partial}{\partial t} (\overline{\varrho\, k_{\xi e}}) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} [\sqrt{\gamma} (\overline{\varrho\, k_{\xi e}}\, \hat{v}^\alpha + \overline{\varrho\, k_{\xi e}\, v''^\alpha})] =$$
$$= -\frac{1}{2} \gamma_{11,\alpha} \left\{\left[2 (\omega + \hat{v}^1)\, \overline{\varrho\, v''^1 v''^\alpha} + \overline{\varrho\, v''^1 v''^1 v''^\alpha}\right] + \overline{\varrho\, v''^1 v''^1}\, \hat{v}^\alpha\right\} - \tag{13}$$
$$- \overline{\varrho\, v_1'' v''^\alpha} \frac{\partial \hat{v}^1}{\partial x^\alpha} + \overline{T^i_{1\cdot i}\, v''^1},$$

and

$$\frac{\partial}{\partial t} (\overline{\varrho\, k_{\mu e}}) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} [\sqrt{\gamma} (\overline{\varrho\, k_{\mu e}}\, \hat{v}^\alpha + \overline{\varrho\, k_{\mu e}\, v''^\alpha})] = \tag{14}$$
$$= \frac{1}{2} \gamma_{11,\alpha} \left[2 (\omega + \hat{v}^1)\, \overline{\varrho\, v''^1 v''^\alpha} + \overline{\varrho\, v''^1 v''^1 v''^\alpha}\right] - \overline{\varrho\, v''^\alpha v_\beta''}\, \hat{v}^\beta_{\cdot\alpha} + \overline{T^i_{\alpha\cdot i}\, v''^\alpha}.$$

Finally, addition of (11) and (12), and of (13) and (14), leads to the equations of balance of the kinetic energy $\left(k_m \equiv k_{\xi m} + k_{\mu m} \equiv \frac{1}{2}\, \hat{v}_i\, \hat{v}^i\right)$ of the mean motion and of the kinetic energy $\left(k_e \equiv k_{\xi e} + k_{\mu e} \equiv \frac{1}{2}\, v''_i\, v''^i\right)$ of the large-scale eddying motion:

$$\frac{\partial}{\partial t}(\bar{\varrho}\, k_m) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}[\sqrt{\gamma}\ (\bar{\varrho}\, k_m\, \hat{v}^\alpha - \overline{T}{}^\alpha_i\, \hat{v}^i + \overline{\varrho\, v''_i v''^\alpha}\, \hat{v}^i] = -\overline{T}{}^i_j\, \hat{v}^j_{\cdot i} + \overline{\varrho\, v''^i v''_j}\, \hat{v}^j_{\cdot i} - g\, \bar{\varrho}\, \bar{w}, \quad (15)$$

and

$$\frac{\partial}{\partial t}\, (\overline{\varrho\, k_e}) + \frac{1}{\sqrt{\gamma}}\, \frac{\partial}{\partial x^\alpha}\, [\sqrt{\gamma}\, (\overline{\varrho\, k_e}\, \hat{v}^\alpha + \overline{\varrho\, k_e\, v''^\alpha})] = \overline{T^i_{j\cdot i} v''^j} - \overline{\varrho\, v''^i v''_j}\, \hat{v}^j_{\cdot i}. \quad (16)$$

$$(\alpha = 2,\ 3;\ i, j = 1,\ 2,\ 3)$$

All these equations assume the form of a balance:

$$\frac{\partial}{\partial t}\, (\bar{\varrho}\, k) + \frac{1}{\sqrt{\gamma}}\, \frac{\partial}{\partial x^\alpha}\, [\sqrt{\gamma}\, C^\alpha (k)] = \sigma(k),$$

where the meridional flux $C^\alpha (k)$ and the rate of production $\sigma (k)$ of the different energy forms k appear in the divergence term and in the second member of equations (11) to (16) respectively. These equations may be given the following interpretation.

The kinetic energies $\bar{\varrho}\, k_{\xi m}$, $\bar{\varrho}\, k_{\mu m}$ and $\bar{\varrho}\, k_m$ in a fixed unit of volume may change in consequence of:

a) transport of air across the boundary: $\bar{\varrho}\, k_{\xi m}\, \hat{v}^\alpha$, $\bar{\varrho}\, k_{\mu m}\, \hat{v}^\alpha$ and $\bar{\varrho}\, k_m\, \hat{v}^\alpha$;

b) performance of boundary work by the internal stresses: $-(\overline{P}{}^\alpha_1 + \overline{R}{}^\alpha_1)\, \hat{v}^1, \bar{p}\, \hat{v}^\alpha - (\overline{P}{}^\alpha_\beta + \overline{R}{}^\alpha_\beta)\, \hat{v}^\beta$ and $\bar{p}\, \hat{v}^\alpha - (\overline{P}{}^\alpha_i + \overline{R}{}^\alpha_i)\, \hat{v}^i$;

c) large-scale eddy flux of momentum through the boundary:

$\overline{\varrho\, v''_1\, v''^\alpha}\, \hat{v}^1$, $\overline{\varrho\, v''_\beta\, v''^\alpha}\, \hat{v}^\beta$, $\overline{\varrho\, v''_i\, v''^\alpha}\, \hat{v}^i$, (VAN MIEGHEM, 1949 a);

d) production within the volume at the rate:

$$\left\{\begin{aligned}
\sigma (k_{\xi m}) &\equiv \frac{1}{2}\, \gamma_{11,\alpha}\, (2\,\omega + \hat{v}^1)\, \bar{\varrho}\, \hat{v}^1\, \hat{v}^\alpha - (\overline{P}{}^\alpha_1 + \overline{R}{}^\alpha_1)\, \frac{\partial \hat{v}^1}{\partial x^\alpha} \\
&\qquad + \overline{\varrho\, v''^\alpha v''_1}\, \frac{\partial \hat{v}^1}{\partial x^\alpha}, \\
\sigma (k_{\mu m}) &\equiv -\frac{1}{2}\gamma_{11,\alpha}\, (2\omega + \hat{v}^1)\bar{\varrho}\, \hat{v}^1 \hat{v}^\alpha - (\overline{P}{}^\beta_\alpha + \overline{R}{}^\beta_\alpha)\, \hat{v}^\alpha_{\cdot\beta} - \frac{1}{2}\, \gamma_{11,\alpha}\, \hat{v}^\alpha\, (\overline{P}{}^{11} + \overline{R}{}^{11}) + \\
&\qquad + \bar{p}\, \operatorname{div} \hat{v} + \overline{\varrho\, v''^\beta v''_\alpha}\, \hat{v}^\alpha_{\cdot\beta} + \frac{1}{2}\, \gamma_{11,\alpha}\, \hat{v}^\alpha\, \overline{\varrho\, v''^1 v''^1} - g\, \bar{\varrho}\, \bar{w}, \\
\sigma (k_m) &\equiv \qquad - (\overline{P}{}^i_j + \overline{R}{}^i_j)\, \hat{v}^j_{\cdot i} \\
&\qquad + \bar{p}\, \operatorname{div} \hat{v} + \overline{\varrho\, v''^i v''_j}\, \hat{v}^j_{\cdot i} \qquad - g\, \bar{\varrho}\, \bar{w},
\end{aligned}\right. \quad (17)$$

where $\operatorname{div}\, \hat{v} \equiv \frac{1}{\sqrt{\gamma}}\, \frac{\partial}{\partial x^\alpha}\, (\sqrt{\gamma}\, \hat{v}^\alpha) \equiv \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} - \frac{\hat{v}}{r}\, \operatorname{tg}\, \varphi + 2\, \frac{\hat{w}}{r}.$

The mode of redistribution of kinetic energy due to the boundary work of the Stokes-Navier and Reynolds mean stresses $(\overline{P}^i_j + \overline{R}^i_j)$ may be disregarded outside the boundary layer.

The production of kinetic energy $k_{\xi m}$ of the zonal mean motion results from the meridional eddy flux $\overline{\varrho\, \mathrm{v}''_1 \mathrm{v}''^\alpha}$ of relative angular momentum $\varrho\, \mathrm{v}_1$ and the meridional shears $\frac{\partial \hat{\mathrm{v}}^1}{\partial x^\alpha}$ of the mean zonal wind (Kuo, 1951). The kinetic energy of the meridional mean motion is produced by:

a) the meridional eddy flux $\overline{\varrho\, \mathrm{v}''_\alpha \mathrm{v}''^\beta}$ of relative meridional mean momentum $\varrho\, \mathrm{v}_\alpha$ and the shears of the mean wind in the meridional planes;

b) the expansion (div $\hat{\mathrm{v}} > 0$) of the air (Starr, 1948);

c) the destruction of potential energy, $(g\, \overline{\varrho\, w} < 0)$.

It should be noted that the destruction (production) of a certain amount of potential energy of the mean motion entails *ipso facto* the production (destruction) of an equal amount of kinetic energy. For instance, as a consequence of the negative correlation between the fluctuations of density and vertical velocity, the term $-g\, \overline{\varrho' w''}$ represents a rate of effective production of kinetic energy of the mean motion.

The action of the viscous and turbulent frictional stresses $\left[\text{represented by } (\overline{P}^\alpha_1 + \overline{R}^\alpha_1)\, \frac{\partial \hat{\mathrm{v}}^1}{\partial x^\alpha},\right.$ $\left.(\overline{P}^\beta_\alpha + \overline{R}^\beta_\alpha)\, \hat{\mathrm{v}}^\alpha_{\cdot\beta} + \frac{1}{2}\, \gamma_{11,\alpha}\, (\overline{P}^{11} + \overline{R}^{11})\, \hat{\mathrm{v}}^\alpha \text{ and } (\overline{P}^i_j + \overline{R}^i_j)\, \hat{\mathrm{v}}^j_{\cdot i}\right]$ consists of a dissipation. It is well known that the Rayleigh (1873) dissipation function $P^i_j\, \mathrm{v}^j_{\cdot i}$ is a positive quadratic form o the wind shears and that the same is *generally* true for the dissipation function $R^i_j\, \mathrm{v}^j_{\cdot i}$ for small-scale turbulence.

Finally the term $\frac{1}{2}\, \gamma_{11,\alpha}\, (2\,\omega + \hat{\mathrm{v}}^1)\, \overline{\varrho}\, \hat{\mathrm{v}}^1 \hat{\mathrm{v}}^\alpha$ represents the rate of conversion of kinetic energy of the zonal mean motion into kinetic energy of the meridional mean motion. This term drops out in the expression of the rate of production of the kinetic energy $k_m \equiv k_{\xi m} + k_{\mu m}$ of the mean motion.

The kinetic energies $\overline{\varrho\, k_{\xi e}}$, $\overline{\varrho\, k_{\mu e}}$ and $\overline{\varrho\, k_e}$ in a fixed unit of volume may change in consequence of:

a) meridional transport of air across the boundary: $\overline{\varrho\, k_{\xi e}}\, \hat{\mathrm{v}}^\alpha$, $\overline{\varrho\, k_{\mu e}}\, \hat{\mathrm{v}}^\alpha$ and $\overline{\varrho\, k_e}\, \hat{\mathrm{v}}^\alpha$;

b) large-scale meridional eddy diffusion across the boundary: $\overline{\varrho\, k_{\xi e}\, \mathrm{v}''^\alpha}$, $\overline{\varrho\, k_{\mu e}\, \mathrm{v}''^\alpha}$ and $\overline{\varrho\, k_e\, \mathrm{v}''^\alpha}$;

c) production within the volume at the rate:

$$
\left\{
\begin{aligned}
\sigma(\hat{k}_{\xi e}) &\equiv -\frac{1}{2}\, \gamma_{11\,\alpha} \left[2\,(\omega + \hat{\mathrm{v}}^1)\, \overline{\varrho\, \mathrm{v}''^1 \mathrm{v}''^\alpha} + \overline{\varrho\, \mathrm{v}''^1 \mathrm{v}''^1 \mathrm{v}''^\alpha}\right] - \overline{\varrho\, \mathrm{v}''_1 \mathrm{v}''^\alpha}\, \frac{\partial \hat{\mathrm{v}}^1}{\partial x^\alpha} - \\
&\quad - \frac{1}{2}\, \gamma_{11,\alpha}\, \hat{\mathrm{v}}^\alpha\, \overline{\varrho\, \mathrm{v}''^1 \mathrm{v}''^1} + \overline{T^i_{1\cdot i}\, \mathrm{v}''^1}, \\
\sigma(\hat{k}_{\mu e}) &\equiv \frac{1}{2}\, \gamma_{11,\alpha} \left[2\,(\omega + \hat{\mathrm{v}}^1)\, \overline{\varrho\, \mathrm{v}''^1 \mathrm{v}''^\alpha} + \overline{\varrho\, \mathrm{v}''^1 \mathrm{v}''^1 \mathrm{v}''^\alpha}\right] - \overline{\varrho\, \mathrm{v}''_\beta\, \mathrm{v}''^\alpha}\, \hat{\mathrm{v}}^\beta_{\cdot\alpha} \\
&\quad + \overline{T^i_{\alpha\cdot i}\, \mathrm{v}''^\alpha}, \\
\sigma(\hat{k}_e) &\equiv \qquad - \overline{\varrho\, \mathrm{v}''^i \mathrm{v}''_j}\, \hat{\mathrm{v}}^j_{\cdot i} \\
&\quad + \overline{T^i_{j\cdot i}\, \mathrm{v}''^j}.
\end{aligned}
\right. \tag{18}
$$

The terms $\overline{T^i_{1\cdot i}\mathrm{v}''^1}$, $\overline{T^i_{\alpha\cdot i}\mathrm{v}''^\alpha}$ and $\overline{T^i_{j\cdot i}\mathrm{v}''^j}$ represent the mean values of the work, per unit volume and time, performed by the pressure forces and by the viscous and turbulent frictional forces along the large-scale eddying motion parallel to the latitude circles, in meridional planes and in the three dimensional space respectively. The energy per unit volume and time expressed by $\overline{T^i_{j\cdot i}\mathrm{v}''^j}$ may be interpreted as an *instability energy* (VAN MIEGHEM, 1949 b). According to the sign of the instability energy, the large-scale eddying motion is stable or unstable. When $\overline{T^i_{j\cdot i}\mathrm{v}''^j} > 0$, the large-scale eddies do work: the large-scale eddying motion is unstable and $\overline{T^i_{j\cdot i}\mathrm{v}''^j}$ represents, per unit volume and time, the instability energy released by the large-scale eddies. On the other hand, when $\overline{T^i_{j\cdot i}\mathrm{v}''^j} < 0$, the large-scale eddies have work done on them at the expense of the energy of the surroundings and therefore the large-scale eddying motion is stable in that case. It should be noted that the inequality $\sigma(\hat{k}_e) > 0$ expresses a necessary condition for the maintenance of the large-scale eddying motion (generalization of the Richardson criterion).

Finally, the term $\frac{1}{2}\gamma_{11,\alpha}\left[2(\omega+\hat{\mathrm{v}}^1)\overline{\varrho\mathrm{v}''^1\mathrm{v}''^\alpha} + \overline{\varrho\mathrm{v}''^1\mathrm{v}''^1\mathrm{v}''^\alpha}\right]$ represents the rate of conversion of kinetic energy $k_{\xi e}$ of the zonal eddying motion into kinetic energy $k_{\mu e}$ of the meridional eddying motion. This term drops out in the expression of the rate of production of the eddy kinetic energy $k_e \equiv k_{\xi e} + k_{\mu e}$.

4. Conversions involving mechanical energies only

In the formulas (17) and (18) of the rates of production, certain terms appear twice with opposite signs. As already mentioned in the introduction such a term may be considered as a rate of energy conversion. In spherical coordinates, these terms assume the form:

$$\Gamma_m \equiv -\frac{1}{2}\gamma_{11,\alpha}(2\omega+\hat{\mathrm{v}}^1)\hat{\mathrm{v}}^1\overline{\varrho\mathrm{v}^\alpha} = \left(2\omega\sin\varphi + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)\hat{u}\overline{\varrho v} - \left(2\omega\cos\varphi + \frac{\hat{u}}{r}\right)\hat{u}\overline{\varrho w}$$

$$\Gamma_e \equiv -\frac{1}{2}\gamma_{11,\alpha}\left[2(\omega+\hat{\mathrm{v}}^1)\overline{\varrho\mathrm{v}''^1\mathrm{v}''^\alpha} + \overline{\varrho\mathrm{v}''^1\mathrm{v}''^1\mathrm{v}''^\alpha}\right] =$$

$$= 2\left(\omega + \frac{\hat{u}}{r\cos\varphi}\right)\left(\overline{\varrho u''v''}\sin\varphi - \overline{\varrho u''w''}\cos\varphi\right) + \frac{1}{r\cos\varphi}\left(\overline{\varrho u''u''v''}\sin\varphi - \overline{\varrho u''u''w''}\cos\varphi\right),$$

$$\chi_\mu \equiv \overline{\varrho\mathrm{v}''^\beta\mathrm{v}''_\alpha}\,\hat{\mathrm{v}}^\alpha_{\cdot\beta} = \overline{\varrho v''v''}\left(\frac{\partial\hat{v}}{\partial y} + \frac{\hat{w}}{r}\right) + \overline{\varrho w''w''}\frac{\partial\hat{w}}{\partial z} + \overline{\varrho v''w''}\left(\frac{\partial\hat{v}}{\partial z} + \frac{\partial\hat{w}}{\partial y} - \frac{\hat{v}}{r}\right),$$

$$\chi_\xi \equiv \overline{\varrho\mathrm{v}''_1\mathrm{v}''^\alpha}\frac{\partial\hat{\mathrm{v}}^1}{\partial x^\alpha} = \overline{\varrho u''v''}\left(\frac{\partial\hat{u}}{\partial y} + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right) + \overline{\varrho u''w''}\left(\frac{\partial\hat{u}}{\partial z} - \frac{\hat{u}}{r}\right),$$

$$\chi_{\mu\xi} \equiv \frac{1}{2}\gamma_{11,\alpha}\hat{\mathrm{v}}^\alpha\,\overline{\varrho\mathrm{v}''^1\mathrm{v}''^1} = \overline{\varrho u''u''}\left(-\frac{\hat{v}}{r}\operatorname{tg}\varphi + \frac{\hat{w}}{r}\right).$$

Hence,

$$\left\{\begin{array}{llllllll}
\sigma(\varnothing) \equiv & g\bar{\varrho}\bar{w} & +\, g\overline{\varrho' w''}, & & & & & \\
 & \updownarrow & \downarrow & & & & & \\
\sigma(k_m) \equiv & -g\bar{\varrho}\bar{w} & -\, g\overline{\varrho' w''} & +\,\chi_{\mu\xi} & +\,\chi_\xi & +\,\chi_\mu & +\,\ldots, & \\
 & & & \updownarrow & \downarrow\uparrow & \updownarrow & & \\
\sigma(\hat{k}_e) \equiv & & & -\,\chi_{\mu\xi} & -\,\chi_\xi & -\,\chi_\mu & +\,\ldots, &
\end{array}\right. \tag{19}$$

and

$$\left\{\begin{array}{lllll}
\sigma(k_{\xi m}) \equiv & \Gamma_m & + \chi_\xi & & + \ldots, \\
\sigma(k_{\mu m}) \equiv & -\Gamma_m + \chi_{\mu\xi} & & + \chi_\mu - g\,\varrho\,\overline{w} & + \ldots, \\
\sigma(\hat{k}_{\xi e}) \equiv & \Gamma_e - \chi_{\mu\xi} & - \chi_\xi & & + \ldots, \\
\sigma(\hat{k}_{\mu e}) \equiv & -\Gamma_e & & -\chi_\mu & + \ldots,
\end{array}\right. \tag{20}$$

where the dots indicate terms which, as we shall see in section 6, correspond to conversions between kinetic energy and internal energy as well as kinetic energy of microturbulence.

The arrows in (19) and (20) indicate all the possible conversions between the mechanical energies. As a rule, in low and middle latitudes: $\overline{\varrho u'' v''} > 0$ and $\overline{\varrho u'' w''} < 0$. Consequently $\Gamma_e > 0$, so that Γ_e expresses always a conversion of $k_{\mu e}$-energy into $k_{\xi e}$-energy, the terms $\overline{\varrho u'' u'' v''}$ and $\overline{\varrho u'' u'' w''}$ being presumably of less importance. Kuo (1951) has shown that the two terms of χ_ξ have opposite signs, the first being positive and of the right order of magnitude to account for the frictional dissipation of kinetic energy at the earth's surface. The rate of conversion $\chi_{\mu\xi}$ between the $\hat{k}_{\xi e}$ and the $\hat{k}_{\mu m}$-energy is very small because it depends only on the normal and geodesic curvatures of the parallel circles and may therefore be neglected ($\chi_{\mu\xi} \cong 0$). Finally, there is no conversion possible between the $k_{\xi m}$ and the $k_{\mu e}$ energy.

5. Influence of the inertial stability on the conversion of "meridional" eddy-kinetic energy into "zonal" eddy-kinetic energy

Averaging the equation (8) along the latitude circles and subtracting from this averaged equation the corresponding equation for the mean motion, one finds, after some calculations,

$$\frac{\partial}{\partial t}\left(\overline{\varrho\, \mathrm{v}''_1 \mathrm{v}''_\delta}\right) = -\left(\omega\,\gamma_{11,\beta} + \frac{\partial \hat{\mathrm{v}}_1}{\partial x^\beta}\right) \overline{\varrho\, \mathrm{v}''_\delta \mathrm{v}''^\beta} + \ldots\ldots\ldots, \tag{21}$$

$$(\beta,\ \delta = 2,\ 3)$$

where the dots designate terms which do not contain the Jeffreys stresses $-\overline{\varrho\, \mathrm{v}''_\delta \mathrm{v}''^\beta}$.

The local change $\dfrac{\partial \Gamma_e}{\partial t}$ of the rate of conversion $\Gamma_e\,(> 0)$ of "meridional" kinetic energy into "zonal" kinetic energy of the large-scale eddying motion has the following expression:

$$\frac{\partial \Gamma_e}{\partial t} = -\gamma_{11,\alpha}\,(\omega + \hat{\mathrm{v}}^1)\,\frac{\partial}{\partial t}\left(\overline{\varrho\, \mathrm{v}''^1 \mathrm{v}''^\alpha}\right) + \ldots\ldots\ldots \tag{22}$$

Multiplication of (21) by $\gamma^{11}\,\gamma^{\alpha\delta}$ and subsequent substitution in (22), give

$$\frac{\partial \Gamma_e}{\partial t} = \gamma_{11,\alpha}\,\gamma^{11}\left(\omega\,\gamma_{11,\beta} + \frac{\partial \hat{\mathrm{v}}_1}{\partial x^\beta}\right)(\omega + \hat{\mathrm{v}}^1)\,\overline{\varrho\, \mathrm{v}''^\alpha \mathrm{v}''^\beta} + \ldots\ldots\ldots, \tag{23}$$

or, in spherical coordinates,

$$\frac{\partial \Gamma_e}{\partial t} = 2\left\{\left(\omega \sin\varphi + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)\left(2\omega\sin\varphi - \frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)\overline{\varrho v'' v''} - \right.$$

$$-\left[\left(\omega\sin\varphi + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)\left(2\omega\cos\varphi + \frac{\partial\hat{u}}{\partial z} + \frac{\hat{u}}{r}\right) + \left(\omega\cos\varphi + \frac{\hat{u}}{r}\right)\left(2\omega\sin\varphi - \right.\right. \tag{23 a}$$

$$\left.\left.\left. - \frac{\partial\hat{u}}{\partial y} + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)\right]\overline{\varrho v'' w''} + \left(\omega\cos\varphi + \frac{\hat{u}}{r}\right)\left(2\omega\cos\varphi + \frac{\partial\hat{u}}{\partial z} + \frac{\hat{u}}{r}\right)\overline{\varrho w'' w''}\right\} + \ldots,$$

or, in vector form,

$$\frac{\partial \Gamma_e}{\partial t} = 2\,\overline{\varrho\,(\mathbf{v}''\times\Omega)_x\cdot(\mathbf{v}''\times\operatorname{curl}\mathbf{U})_x} + \ldots, \tag{23 b}$$

where Ω represents the absolute angular speed about the earth's axis and $\mathbf{U}$ the absolute linear velocity along the latitude circles.

In the atmosphere, $(\mathbf{v}''\times\Omega)_x$ and $(\mathbf{v}''\times\operatorname{curl}\mathbf{U})_x$ have the sign of $\left(\omega\sin\varphi + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)v''$ and $\left(2\omega\sin\varphi - \frac{\partial\hat{u}}{\partial y} + \frac{\hat{u}}{r}\operatorname{tg}\varphi\right)v''$ respectively.

When the large-scale eddying motion is inertially stable, $\frac{\partial\hat{u}}{\partial y} < 2\omega\sin\varphi + \frac{\hat{u}}{r}\operatorname{tg}\varphi$, or $(\mathbf{v}''\times\Omega)_x\cdot(\mathbf{v}''\times\operatorname{curl}\mathbf{U})_x > 0$, (VAN MIEGHEM, 1951); consequently the inertial stability contributes to the local increase with time of the rate of conversion Γ_e. On the other hand, the inertial instability contributes to the local decrease with time of the rate of conversion Γ_e of "meridional" into "zonal" eddy kinetic energy.

6. The first law

The first law of thermodynamics expresses the conservation of the *total energy* of the system envisaged. In the case considered here, the total energy u, per unit of mass, is equal to the sum of the potential energy $\varnothing$, the kinetic energy k, the kinetic energy $\varkappa$ of microturbulence and the internal energy e per unit of mass $(u \equiv \varnothing + k + \varkappa + e)$.

The equation of balance of the kinetic energy $\varkappa$ of microturbulence assumes the form (VAN MIEGHEM, 1949 b):

$$\frac{\partial}{\partial t}(\varrho\varkappa) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left[\sqrt{\gamma}\,(\varrho\varkappa\mathrm{v}^i + K^i)\right] = \varrho\triangle + R^i_j\,\mathrm{v}^j_{.;i} > 0, \tag{24}$$

$$(i, j = 1, 2, 3)$$

where K^i designates the flux due to small-scale turbulent diffusion, $\triangle$ the instability energy released per unit mass and time along the small-scale eddying motion, and $R^i_j\,\mathrm{v}^j_{.;i}$ the amount of kinetic energy available per unit of volume for conversion into kinetic energy of small-scale turbulence. It is *generally* accepted that $R^i_j\,\mathrm{v}^j_{.;i} > 0$. When the small-scale eddying motion is stable, $\triangle < 0$ and $\triangle$ represents the amount of kinetic energy of small-scale turbulence converted per unit mass into heat; when $\triangle > 0$, the small-scale eddying motion is unstable and $\triangle$ represents the amount of heat converted per unit mass and time into kinetic energy of small scale turbulence. The Richardson number h^2 of the small-scale turbulence may be defined as $h^2 \equiv \varrho\,|\triangle| : (R^i_j\,\mathrm{v}^j_{.;i})$ (VAN MIEGHEM 1949 b, 1950 a). The inequality in (24) expresses the necessary condition for the maintenance of small-scale turbulence $(h^2 < 1)$.

In virtue of (6), (7) and (24), the flux $C^i(u)$ of the total energy u is given by

$$C^i(u) \equiv \varrho u v^i - T^i_j v^j + K^i + W^i,$$

where W^i represents the flux of heat due to radiation, conduction and convection (the effect of conduction is negligible in the atmosphere), hence the balance

$$\frac{\partial(\varrho u)}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\, C^i(u))}{\partial x^i} = 0, \tag{25}$$

the rate of production of the total energy being equal to zero as a result of the conservation of the total energy, ($\sigma(u) = 0$, VAN MIEGHEM, 1949).

Averaging equation (25) along latitude circles and subtracting the equations (15), (16) and the averaged equations (7) and (24), one obtains the equation of balance of the averaged internal energy

$$\frac{\partial \overline{\varrho e}}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left[\sqrt{\gamma}\left(\overline{\varrho e}\, \hat{v}^\alpha + \overline{\varrho h v''^\alpha} - \overline{P^\alpha_i v''^i} + \overline{W^\alpha}\right)\right] = \tag{26}$$

$$= -\overline{\varrho \triangle} - \bar{p} \operatorname{div} \hat{\mathbf{v}} + \bar{P}^i_j \hat{v}^j_{.i} - \overline{\left(-\frac{\partial p}{\partial x^i} v''^i + P^i_{j\cdot i} v''^j\right)},$$

where h designates the specific enthalpy ($\varrho h \equiv \varrho e + p$).

It should be noted that the averaged equation of balance (24) may be written

$$\frac{\partial(\overline{\varrho \varkappa})}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left[\sqrt{\gamma}\left(\overline{\varrho \varkappa}\, \hat{v}^\alpha + \overline{\varrho \varkappa v''^\alpha} + \overline{K}^\alpha - \overline{R^\alpha_j v''^j}\right)\right] = \overline{\varrho \triangle} + \bar{R}^i_j \hat{v}^j_{.i} - \overline{R^i_{j\cdot i} v''^j}. \tag{27}$$

The zonally averaged internal energy $\overline{\varrho e} = \bar{\varrho} \hat{e}$ in a fixed unit of volume changes in consequence of:

a) transport of air across the boundary: $\bar{\varrho}\, \hat{e} \hat{v}^\alpha$;
b) large-scale eddy-diffusion of heat: $W^\alpha_{(e)} \equiv \overline{\varrho h v''^\alpha} - \overline{P^\alpha_i v''^i}$;
c) heat flux $\overline{W^\alpha}$ (radiation + convection);
d) production within the volume at the rate:

$$\sigma(\hat{e}) \equiv -\bar{p} \operatorname{div} \hat{\mathbf{v}} + \bar{P}^i_j \hat{v}^j_{.i} - \overline{\left(-\frac{\partial p}{\partial x_i} + P^j_{i\cdot j}\right) v''^i} - \overline{\varrho \triangle}. \tag{28}$$

Finally, grouping in one table the rates of energy production $\sigma(\emptyset)$, $\sigma(k_m)$, $\sigma(\hat{k}_e)$, $\sigma(\hat{\varkappa})$ and $\sigma(\hat{e})$, the possible conversions between the five considered energy forms appear immediately:

$$\left.\begin{aligned}
\sigma(\Phi) &\equiv g\overline{\varrho w}, \\
\sigma(k_m) &\equiv -g\overline{\varrho w} + \bar{p} \operatorname{div} \hat{\mathbf{v}} - \bar{P}^i_j \hat{v}^j_{.i} - \bar{R}^i_j \hat{v}^j_{.i} + \overline{\varrho v''_j v''^i}\, \hat{v}^j_{.i}, \\
\sigma(\hat{k}_e) &\equiv -\overline{\varrho v''_j v''^i}\, \hat{v}^j_{.i} + \overline{\left(-\frac{\partial p}{\partial x_i} + P^j_{i\cdot j}\right) v''^i} + \overline{R^i_{j\cdot i} v''^j}, \\
\sigma(\hat{\varkappa}) &\equiv \bar{R}^i_j \hat{v}^j_{.i} + \overline{\varrho \triangle} - \overline{R^i_{j\cdot i} v''^j}, \\
\sigma(\hat{e}) &\equiv -\bar{p} \operatorname{div} \hat{\mathbf{v}} + \bar{P}^i_j \hat{v}^j_{.i} - \overline{\varrho \triangle} - \overline{\left(-\frac{\partial p}{\partial x^i} + P^j_{i\cdot j}\right) v''^i}
\end{aligned}\right\} \tag{29}$$

$(i, j = 1, 2, 3)$

The corresponding energy fluxes in the meridional planes assume the form:

$$\left.\begin{aligned}
C^{\alpha}(\varnothing) &\equiv \varnothing\,\overline{\varrho \mathrm{v}^{\alpha}}, \\
C^{\alpha}(k_m) &\equiv (k_m + R\hat{T})\,\overline{\varrho \mathrm{v}^{\alpha}} \qquad - \left(\overline{P_i^{\alpha}} + \overline{R_i^{\alpha}} - \overline{\varrho \mathrm{v}_i'' \mathrm{v}''^{\alpha}}\right)\hat{\mathrm{v}}^i, \\
C^{\alpha}(\hat{k}_e) &\equiv \hat{k}_e\,\overline{\varrho \mathrm{v}^{\alpha}} + \overline{\varrho\, k_e\, \mathrm{v}''^{\alpha}}, \\
C^{\alpha}(\hat{\varkappa}) &\equiv \hat{\varkappa}\,\overline{\varrho \mathrm{v}^{\alpha}} + \overline{\varrho\, \varkappa\, \mathrm{v}''^{\alpha}} - \overline{R_i^{\alpha}\, \mathrm{v}''^{i}} + \overline{K}^{\alpha}, \\
C^{\alpha}(\hat{e}) &\equiv \hat{e}\,\overline{\varrho \mathrm{v}^{\alpha}} + \overline{\varrho\, h\, \mathrm{v}''^{\alpha}} - \overline{P_i^{\alpha}\, \mathrm{v}''^{i}} + \overline{W}^{\alpha},
\end{aligned}\right\} \quad (30)$$

$$(i = 1, 2, 3;\ \alpha = 2, 3)$$

where $R\hat{T} = \bar{p} : \bar{\varrho}$, R being the specific ideal gas constant for air and T the absolute temperature.

7. Large-scale eddy-diffusion of heat

In section 6, we have established that the large-scale eddy-diffusion is responsible for the heat flux

$$W_{(e)}^{\alpha} \equiv \overline{\varrho\, h\, \mathrm{v}''^{\alpha}} - \overline{P_i^{\alpha}\, \mathrm{v}''^{i}} \qquad (31)$$

$$(\alpha = 2, 3;\ i = 1, 2, 3)$$

in the meridional planes. The last term in (31) due to the viscosity may be neglected, consequently

$$W_{(e)}^{\alpha} \cong \overline{\varrho\, h\, \mathrm{v}''^{\alpha}} = \overline{\varrho\, h''\, \mathrm{v}''^{\alpha}}, \qquad (31\text{ a})$$

where (VAN MIEGHEM and DUFOUR 1948)

$$h = \tau_a\, h_a + \tau_{\mathrm{v}}\, h_{\mathrm{v}} + \tau_w\, h_w, \qquad (32)$$

h_a, h_{v}, h_w, being the specific enthalpies of dry air, water vapour and liquid water respectively and τ_a, τ_{v}, τ_w the corresponding mass-proportions. It is easily seen that

$$\tau_{\mathrm{v}} = \varepsilon_{\mathrm{v}} \frac{1 - \varepsilon}{1 - \varepsilon_{\mathrm{v}}}, \qquad (33)$$

where $\varepsilon_{\mathrm{v}} \equiv \dfrac{\tau_{\mathrm{v}}}{\tau_a + \tau_{\mathrm{v}}}$ is the specific humidity and $\varepsilon \equiv \tau_{\mathrm{v}} + \tau_w \equiv 1 - \tau_a$. Finally substituting (33) and the classical formula $L_{\mathrm{v}} = h_{\mathrm{v}} - h_w$ for the heat of vaporization of water into (32), one obtains for the fluctuation h'' of h the following expression:

$$h'' \equiv (1 - \varepsilon)\, h_a'' + \varepsilon\, h_w'' + (1 - \varepsilon)\left(\frac{\varepsilon_{\mathrm{v}} L_{\mathrm{v}}}{1 - \varepsilon_{\mathrm{v}}}\right)'', \qquad (34)$$

account being taken of the fact that the fluctuation ε'' of ε is practically equal to zero ($\varepsilon'' \cong 0$).

In the atmosphere, one may, with a very good approximation, neglect ε and ε_{v} with respect to 1 and assume the constancy of the specific heats c_{pa}, $c_{p\mathrm{v}}$ and c_w; hence, the approximate relation

$$h'' \cong c_p\, T'' + (\varepsilon_{\mathrm{v}}\, L_{\mathrm{v}})'', \qquad (34\text{ a})$$

where T'' designates the fluctuation of the absolute temperature T with respect to its weighted zonal mean value $\hat{T} = \overline{\varrho T} : \bar{\varrho}$, and where $c_p = c_{pa} + \varepsilon\, c_w \cong c_{pa}$. Moreover for the fluctuations ε_v'' and L_v'' of ε_v and L_v with respect to the weighted zonal mean values $\hat{\varepsilon}_v$ and $\hat{\Gamma}_v$, it is easy to verify the inequality[2] $(L_v'' : \hat{L}_v) \ll \varepsilon_v'' : \hat{\varepsilon}_v$, so that one obtains the approximate formula (PRIESTLEY 1948)

$$h'' \cong c_p\, T'' + L_v\, \varepsilon_v'' \tag{34 b}$$

in which L_v is considered as being a constant.

Finally, substitution of (34 b) in (31 a) gives the expression of the heat flux due to the correlation between the fluctuations h'' and v''^α of the enthalpy h and the meridional components (v^2, v^3) of the air velocity $\mathbf{v}$, namely

$$W^\alpha_{(e)} \cong \overline{\varrho\, h'' v''^\alpha} \cong c_p\, \overline{\varrho\, T'' v''^\alpha} + L_v\, \overline{\varrho\, \varepsilon_v'' v''^\alpha}, \qquad (\alpha = 2, 3) \tag{31 b}$$

where, according to PRIESTLEY (1948), the first term represents the flux of "sensible" heat and the second, the flux of "latent" heat.

8. Application to the large disturbances

Neglecting viscosity and small-scale turbulence, the expressions (29) and (30) for the rates of energy production and for the energy fluxes assume simplified forms. Thus for the rates:

$$\left\{\begin{array}{ll} \sigma(\emptyset) \equiv & g\,\overline{\varrho w} \\ \sigma(k_m) \equiv & -g\,\overline{\varrho w} + \chi + \bar{p}\,\mathrm{div}\,\hat{\mathbf{v}}, \\ \sigma(\hat{k}_e) \equiv & -\chi \qquad - \overline{\nabla p \cdot \mathbf{v}''}, \\ \sigma(\hat{e}) \equiv & -\bar{p}\,\mathrm{div}\,\hat{\mathbf{v}} + \overline{\nabla p \cdot \mathbf{v}''}, \end{array}\right. \tag{35}$$

where

$$\chi \equiv \overline{\varrho u'' u''}\left(-\frac{\hat{v}}{r}\,\mathrm{tg}\,\varphi + \frac{\hat{w}}{r}\right) + \overline{\varrho v'' v''}\left(\frac{\partial \hat{v}}{\partial y} + \frac{\hat{w}}{r}\right) + \overline{\varrho w'' w''}\,\frac{\partial \hat{w}}{\partial z} +$$
$$+ \overline{\varrho v'' w''}\left(\frac{\partial \hat{v}}{\partial z} + \frac{\partial \hat{w}}{\partial y} - \frac{\hat{w}}{r}\right) + \overline{\varrho u'' w''}\left(\frac{\partial \hat{u}}{\partial z} - \frac{\hat{u}}{r}\right) + \overline{\varrho u'' v''}\left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi\right) \cong$$
$$\cong \overline{\varrho u'' v''}\left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi\right) + \overline{\varrho u'' w''}\,\frac{\partial \hat{u}}{\partial z}, \tag{36}$$

and for the fluxes:

$$\left\{\begin{array}{lll} C^\alpha(\emptyset) \equiv \emptyset\, \overline{\varrho v^\alpha}, & & \\ C^\alpha(k_m) \equiv (k_m + R\hat{T})\, \overline{\varrho v^\alpha} & & + \overline{\varrho v_i'' v''^\alpha}\, \hat{v}^i, \\ C^\alpha(\hat{k}_e) \equiv \hat{k}_e\, \overline{\varrho v^\alpha} & + \overline{\varrho k_e v''^\alpha}, & \\ C^\alpha(\hat{e}) \equiv \hat{e}\, \overline{\varrho v^\alpha} & + \overline{\varrho h v''^\alpha} + \overline{W}^\alpha. & \end{array}\right.$$

[2] This inequality is not necessarily satisfied in the case of small-scale turbulence.

Hence, for each of the above mentioned energy forms, the equation of balance for an axially symmetric volume may be written:

$$\frac{\partial}{\partial t}\iint\limits_{(\alpha)} r\cos\varphi\,\bar{\varrho} f\,\delta\alpha = -\oint\limits_{(s)} r\cos\varphi\, C_N(f)\,\delta s + \iint\limits_{(\alpha)} r\cos\varphi\,\sigma(f)\,\delta\alpha, \tag{37}$$

where α is the meridional section of the volume envisaged and s the closed curve limiting the area α, and N the external normal to s.

Applying the balance (37) to the kinetic energy of the zonally averaged motion, the friction at the earth's surface should not be overlooked when a portion of s coincides with the mean meridional profile of the earth's surface.

It should be noted that 1) $k_m \cong \frac{1}{2}(\hat{u})^2$ and 2) k_m may be neglected with respect to $R\hat{T}$ (STARR 1949).

For the very large atmospheric disturbances, the following assumptions are justified:

a) hydrostatic equilibrium along the vertical z: $\frac{\partial \bar{p}}{\partial z} + g\bar{\varrho} \cong 0$;

b) $\operatorname{div}\hat{\mathbf{v}} \cong -\frac{\hat{w}}{\bar{\varrho}}\frac{\partial\bar{\varrho}}{\partial z}$, (CHARNEY 1948, VAN MIEGHEM 1950 b);

c) quasi-homogeneity: $\left(\varrho'\frac{\hat{d}\hat{\mathrm{v}}_\delta}{dt} - \frac{1}{2}\gamma_{\alpha\beta,\delta}\,\varrho'\hat{\mathrm{v}}^\alpha\hat{\mathrm{v}}^\beta - \frac{1}{2}\gamma_{11,\delta}(2\omega+\hat{\mathrm{v}}^1)\,\varrho'\hat{\mathrm{v}}^1\right)$, (38)

where $\frac{\hat{d}}{dt} \equiv \frac{\partial}{\partial t} + \hat{\mathrm{v}}^\alpha\frac{\partial}{\partial x^\alpha}$, may be neglected, $(\alpha,\beta,\delta = 2, 3)$;

d) the terms $\varrho'\left[\overline{\varrho\mathrm{v}''^i\mathrm{v}''_\delta}\right]_{.i}$ and $\overline{\mathrm{v}''^i\frac{\partial p'}{\partial x^i}}$ may be deleted.

As a consequence of the assumptions (38) a and b and of the identity $\frac{1}{\hat{\Theta}}\frac{\partial\hat{\Theta}}{\partial z} \equiv \frac{c_v}{c_p}\frac{1}{\bar{p}}\frac{\partial\bar{p}}{\partial z} - \frac{1}{\bar{\varrho}}\frac{\partial\bar{\varrho}}{\partial z}$, where $\hat{\Theta}$ is the weighted zonally averaged value of the potential temperature $\hat{\Theta}$, one obtains

$$\bar{p}\operatorname{div}\hat{\mathbf{v}} \cong \left[\frac{\bar{p}}{\bar{\varrho}}\frac{1}{\hat{\Theta}}\frac{\partial\hat{\Theta}}{\partial z} + \frac{c_v}{c_p}g\right]\overline{\varrho w} \cong \frac{c_v}{c_p}g\,\overline{\varrho w}, \tag{39}$$

the first of the terms in brackets being one order of magnitude less than the other. On the other hand substitution of the equations of the zonally averaged meridional motion, namely

$$\bar{\varrho}\frac{\hat{d}\hat{\mathrm{v}}_\delta}{dt} - \frac{1}{2}\gamma_{\alpha\beta,\delta}\,\bar{\varrho}\hat{\mathrm{v}}^\alpha\hat{\mathrm{v}}^\beta - \frac{1}{2}\gamma_{11,\delta}(2\omega+\hat{\mathrm{v}}^1)\overline{\varrho\mathrm{v}^1} = -\frac{\partial\bar{p}}{\partial x^\delta} - \left(\overline{\varrho\mathrm{v}''^i\mathrm{v}''_\delta}\right)_{.i} - \bar{\varrho}\frac{\partial\Phi}{\partial x^\delta}$$

in

$$-\overline{\mathrm{v}''^i\frac{\partial p}{\partial x^i}} = -\overline{\mathrm{v}''^i\frac{\partial p'}{\partial x^i}} - \overline{\mathrm{v}''^\delta}\frac{\partial\bar{p}}{\partial x^\delta},$$

leads to

$$-\overline{\mathbf{v}''\nabla p} \equiv -\overline{\mathrm{v}''^i\frac{\partial p}{\partial x^i}} \cong -g\,\overline{\varrho' w'} \equiv -g\,\overline{\varrho' w''}, \tag{40}$$

in virtue of (38) c and d and of $-\bar{\varrho}\,\overline{\mathrm{v}''^\alpha} = \overline{\varrho'\mathrm{v}'^\alpha} = \overline{\varrho'\mathrm{v}''^\alpha}$.

After substitution of (36), (39) and (40) in (35), one obtains the approximate expressions of the rates of energy production, namely:

$$\left\{\begin{aligned}
\sigma(\emptyset) &\equiv g\,\overline{\varrho' w''} + g\,\bar{\varrho}\,\bar{w}\,, \\
\sigma(k_m) &\cong \overline{\varrho\, u'' v''}\left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi\right) - \frac{c_p - c_v}{c_p}\,g\,\overline{\varrho' w''} + \overline{\varrho\, u'' w''}\,\frac{\partial \hat{u}}{\partial z} - \frac{c_p - c_v}{c_p}\,g\,\bar{\varrho}\,\bar{w}\,, \\
\sigma(\hat{k}_e) &\cong -g\,\overline{\varrho' w''} - \overline{\varrho\, u'' v''}\left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi\right) - \overline{\varrho\, u'' w''}\,\frac{\partial \hat{u}}{\partial z} \\
\sigma(\hat{e}) &\cong + \frac{c_p - c_v}{c_p}\,g\,\overline{\varrho' w''} - \frac{c_v}{c_p}\,g\,\bar{\varrho}\,\bar{w}\,,
\end{aligned}\right. \tag{41}$$

where, as a rule, $\overline{\varrho' w'} = \overline{\varrho' w''} < 0$.

As already pointed out in section 4, the rate of conversion $\overline{\varrho u'' v''}\left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi\right)$ of k_e into $k_m \cong \frac{1}{2}\,(\hat{u})^2$ is more important than the rate of conversion $\overline{\varrho\, u'' w''}\,\frac{\partial \hat{u}}{\partial z}$ of k_m into k_e, the difference being presumably of about one order of magnitude.

As a result of the negative correlation between the fluctuations of specific mass ϱ and vertical velocity w, internal energy e is converted into kinetic energy k_m of the zonally averaged motion and simultaneously, at a rate at least three times $\left(\frac{c_p - c_v}{c_p} \cong 0.3\right)$ higher than the rate of conversion of e into k_m, potential energy $\emptyset$ is converted into kinetic energy k_e of the large-scale disturbances. Fjørtoft (1951) has recently stressed the importance of the conversion of potential energy into kinetic energy for the maintenance of the disturbances.

In the ascending branch ($\bar{w} > 0$) of the mean meridional circulation ($\bar{v}$, $\bar{w}$), internal energy is transformed into kinetic energy of the mean motion, and kinetic energy of the mean motion into potential energy. In the descending branch ($\bar{w} < 0$), however, the reversed conversions take place. As a consequence of these energy conversions "potential + internal" energy is transformed into kinetic energy of the mean motion, or vice versa, depending upon the sign of w. For these energy conversions, we may write

$$\sigma(k_m) \equiv -\sigma(\emptyset + \hat{e}) \cong -g\,\frac{c_p - c_v}{c_p}\,\bar{\varrho}\,\bar{w} > 0$$

when $\bar{w} < 0$, and $\sigma(k_m) < 0$ when $\bar{w} > 0$.

For the energy conversions due to the large-seale eddy motion only, we have

$$\sigma(\hat{k}) \equiv \sigma(k_m + \hat{k}_e) = -\sigma(\emptyset + \hat{e}) > 0.$$

This inequality indicates a permanent conversion of "potential + internal" energy into kinetic energy of the air flow. The necessary brake in the steady increase of kinetic energy in the atmosphere is provided by the friction at the earth's surface.

Combining the results (20) and (41), we finally obtain:

$$\sigma(\varnothing) \cong \qquad g\,\overline{\varrho' w''},$$

$$\sigma(k_m) \equiv \begin{cases} \sigma(k_{\xi m}) \cong \left(\dfrac{\partial \hat{u}}{\partial y} + \dfrac{\hat{u}}{r}\operatorname{tg}\varphi\right)\overline{\varrho u'' v''} + \left(\dfrac{\partial \hat{u}}{\partial z} - \dfrac{\hat{u}}{r}\right)\overline{\varrho u'' w''} + \left(2\omega + \dfrac{\hat{u}}{r\cos\varphi}\right)\left(\overline{\varrho' v''}\sin\varphi - \overline{\varrho' w''}\cos\varphi\right)\hat{u}, \\ + \\ \sigma(k_{\mu m}) \cong -\dfrac{c_p - c_v}{c_p}\, g\,\overline{\varrho' w''} \qquad\qquad -\left(2\omega + \dfrac{\hat{u}}{r\cos\varphi}\right)\left(\overline{\varrho' v''}\sin\varphi - \overline{\varrho' w''}\cos\varphi\right)\hat{u}, \end{cases}$$

$$\sigma(\hat{k}_e) \equiv \begin{cases} \sigma(\hat{k}_{\xi e}) \cong \left(2\omega\sin\varphi - \dfrac{\partial \hat{u}}{\partial y} + \dfrac{\hat{u}}{r}\operatorname{tg}\varphi\right)\overline{\varrho u'' v''} - \left(2\omega\cos\varphi + \dfrac{\partial \hat{u}}{\partial z} + \dfrac{\hat{u}}{r}\right)\overline{\varrho u'' w''}, \\ + \\ \sigma(\hat{k}_{\mu e}) \cong -g\,\overline{\varrho' w''} - 2\left(\omega + \dfrac{\hat{u}}{r\cos\varphi}\right)\left(\overline{\varrho u'' v''}\sin\varphi - \overline{\varrho u'' w''}\cos\varphi\right), \end{cases} \tag{42}$$

$$\sigma(\hat{e}) \cong \qquad \frac{c_p - c_v}{c_p}\, g\,\overline{\varrho' w''},$$

where the terms due to the mean meridional circulation $(\bar{v}, \bar{w})$ have been omitted consistently.

From (41) and (42), it seems reasonable to conclude that the two most important energy transformations on the scale of the general circulation are:

a) for the maintenance of the large disturbances, the conversion of potential energy into kinetic energy of the large-scale eddies, as a result of the negative correlation between density and vertical velocity fluctuations (or positive correlation between temperature and vertical velocity fluctuations);

b) for the maintenance of the zonally averaged motion ($\cong$ mean zonal motion), the conversion of kinetic energy of the large-scale disturbances into kinetic energy of the mean motion $\left(k_m \cong \frac{1}{2}(\hat{u})^2\right)$, due to the lateral Jeffreys-stress $-\overline{\varrho u'' v''}$ and the meridional shear of the mean zonal wind $\hat{u}$.

9. The second law

Let us assume that the zonally averaged physical state of the atmosphere has a mean specific entropy s_m which is a function of the mean internal energy $\hat{e}$ and of the mean specific mass $\bar{\varrho}$ $[s_m \equiv s_m(\hat{e}, \bar{\varrho})]$, satisfying the Gibbsian equation

$$\hat{T}\frac{\hat{d} s_m}{dt} = \frac{\hat{d}\hat{e}}{dt} - \frac{\bar{p}}{(\bar{\varrho})^2}\frac{\hat{d}\bar{\varrho}}{dt},$$

where $\dfrac{\hat{d}}{dt} \equiv \dfrac{\partial}{\partial t} + \hat{\mathrm{v}}^\alpha \dfrac{\partial}{\partial x^\alpha}$, $(\alpha = 2, 3)$. It should be noted that the functions $\bar{p}$, $\bar{\varrho}$, $\hat{e}$, $\hat{T}$ and s_m are independent of longitude λ. Upon substituting (26) and (10) into (43), we find the equation of balance

$$\frac{\partial \bar{\varrho} s_m}{\partial t} + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left[\sqrt{\gamma}\left(\bar{\varrho} s_m \hat{\mathrm{v}}^\alpha + \frac{\overline{W^\alpha} + W^\alpha_{(e)}}{\hat{T}}\right)\right] = \tag{44}$$

$$= -\frac{\overline{W^\alpha} + W^\alpha_{(e)}}{(\hat{T})^2}\frac{\partial \hat{T}}{\partial x^\alpha} + \frac{\overline{P^i_j}\,\hat{\mathrm{v}}^j_{.\,i} - \overline{\varrho\triangle}}{\hat{T}} - \frac{1}{\hat{T}}\overline{\left(-\frac{\partial p}{\partial x^i} + P^j_{i\cdot j}\right)\mathrm{v}''^i},$$

$$(\alpha = 2, 3;\ i, j = 1, 2, 3)$$

where the second member represents the rate of production $\sigma(s_m)$ of entropy. The second law states that the entropy production is real, $(\sigma(s_m) > 0)$.

The contribution of small-scale turbulence and thermo-convection to the entropy production has been examined elsewhere (VAN MIEGHEM 1949 b).

Neglecting viscosity and small-scale turbulence, we shall consider here the entropy production due to large-scale turbulence only. Large-scale turbulence generates entropy at the rate

$$\sigma_e(s_m) = -\frac{W^{\alpha}_{(e)}}{(\hat{T})^e}\frac{\partial \hat{T}}{\partial x^{\alpha}} + \frac{1}{\hat{T}}\overline{\frac{\partial p}{\partial x^i}\, v''^i}, \tag{45}$$

or, after substitution of (31 b),

$$\sigma_{(e)}(s_m) \cong \frac{1}{\hat{T}}\overline{\mathbf{v}''.\nabla p} - \frac{c_p}{(\hat{T})^2}\overline{\varrho\, T''\mathbf{v}''}.\nabla \hat{T} - \frac{L_v}{(\hat{T})^2}\overline{\varrho\,\varepsilon''_v\mathbf{v}''}.\nabla \hat{T}. \tag{45 a}$$

Introducing the fluctuation Θ'' of Θ with respect to the weighted mean value $\hat{\Theta}$ into (40) and (45 a),

$$\psi'' \equiv \frac{\Theta''}{\hat{\Theta}} \equiv -\frac{\varrho'}{\varrho} + \frac{c_v}{c_p}\frac{p'}{p} \equiv \frac{T''}{\hat{T}} - \frac{R}{c_p}\frac{p'}{p} \cong -\frac{\varrho'}{\varrho} \cong \frac{T''}{\hat{T}}. \tag{46}$$

where $\psi \equiv \lg \Theta$, we find

$$\overline{\mathbf{v}''.\nabla p} \cong g\,\overline{\varrho' w''} \cong -g\,\bar{\varrho}\,\overline{\psi'' w''} \cong -g\,\overline{\varrho\,\psi'' w''} \tag{40 a}$$

and

$$\sigma_e(s_m) \cong -c_p\overline{\varrho\,\psi''\mathbf{v}''}.\nabla\hat{\psi} - \frac{c_p - c_v}{\bar{p}}\frac{\partial \bar{p}}{\partial y}\overline{\varrho\,\psi'' v''} - \frac{L_v}{(\hat{T})^2}\overline{\varrho\,\varepsilon''_v\mathbf{v}''}.\nabla\hat{T}. \tag{47}$$

BJERKNES (1951) and WHITE (1951) have shown that $\overline{\varrho\,\psi'' v''} > 0$; consequently, the second term in (47) represents a positive contribution to the entropy production. Less is known regarding the flux of "latent" heat $\overline{\varrho\,\varepsilon''_v\mathbf{v}''}$, but it seems not unreasonable to assume that the last term represents also a positive contribution. Both correlations $\overline{\varrho\,\psi'' v''}$ and $\overline{\varrho\,\psi'' w''}$ being positive, the sign of the first term depends upon the slope of the meridional eddy-motion (v'', w'') with respect to the isentropic surfaces $\hat{\Theta}$ = constant or $\hat{\psi}$ = constant. When the large-scale eddy motion in the meridional planes takes place within the angle between the geopotential surfaces and the isentropic surfaces the first term represents also a positive contribution to the entropy production. Indeed, in the atmosphere, as a rule,

$$\operatorname{tg}\alpha = \frac{w''}{v''} > 0 \quad \text{and} \quad \operatorname{tg}\beta = -\frac{\partial\hat{\Theta}}{\partial y} : \frac{\partial\hat{\Theta}}{\partial z} = -\frac{\partial\hat{\psi}}{\partial y} : \frac{\partial\hat{\psi}}{\partial z} > 0;$$

hence,

$$-c_p\overline{\varrho\,\psi''\mathbf{v}''}.\nabla\hat{\psi} = \frac{c_p}{\hat{\Theta}}\frac{\partial\hat{\Theta}}{\partial z}\overline{\varrho\,\psi'' w''\left(\frac{\operatorname{tg}\beta}{\operatorname{tg}\alpha} - 1\right)} > 0,$$

when $\alpha < \beta$, the correlation between the fluctuations of temperature and vertical velocity being positive, $(\overline{\varrho\,\psi'' w''} > 0)$.

EADY (1949) has shown the existence of unstable cyclone waves such that, during their growth, potential energy is transformed into kinetic energy, the above mentioned condition for the meridional eddy motion (v'', w'') being fulfilled. Thus, we may conclude that the growth of cyclone waves is accompanied by a conversion of potential energy into kinetic energy and simultaneously, with a production of entropy.

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