Starr's Invariant

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In a recent communication to this Journal, STARR (1950) pointed out that for two-dimensional flow of an inviscid incompressible fluid between parallel rigid walls (which coincide, say, with latitude circles) on a sphere, the integral

$$\int \delta\zeta \oint \frac{1}{2}\, \varepsilon^2\, \delta\lambda \qquad (1)$$

is invariant, and from this, commented on the stability of such flows, in particular on results obtained earlier by FJØRTOFT (1950).

The notation is as follows: λ is the longitude, ζ the (absolute) vorticity; the integration covers the entire *cyclic* region occupied by fluid, and this region is regarded as composed of strips cut out by lines of equal vorticity ζ, $\zeta + \delta\zeta$, etc.; the line integral follows a line of equal vorticity, and

$$\gamma \equiv a^2 \sin \Phi$$
$$\varepsilon \equiv \gamma - \bar{\gamma},$$

where a in the radius of the sphere, Φ the latitude, and

$$\bar{\gamma} \equiv \frac{1}{2\pi} \oint \gamma\, \delta\lambda \qquad (2)$$

is the average value of γ for all particles composing a given line of equal vorticity.

It is tacit in Starr's discussion that each line of equal vorticity encircles the sphere, as pictured in figure 1 a. Accordingly, $\bar{\gamma}$ corresponds to the mean latitude of the ζ-line, and ε to the departure of a particle from this mean position. A curve of the type pictured in figure 1 a divides the whole doubly-connected region of fluid into two subregions, each doubly connected; we will call this a *cyclic* type of curve.

To provide for more general species of two-dimensional flow, we may consider three other distinct types of curves (vorticity lines); these are illustrated in figures 1 b, c, d. The following "topological" classification of vorticity lines is implied:

	Original space divided into:
a. Cyclic type	— two doubly-connected regions
b. Closed type	— one simply, one triply-connected region
c. Boundary type I	— one simply, one doubly-connected region
d. Boundary type II	— one simply-connected region.

Since vorticity is conserved, and assuming particles never leave or enter the boundaries, the topological classification of any vorticity line is invariant with respect to all dynamically and kinematically possible motions of the fluid.

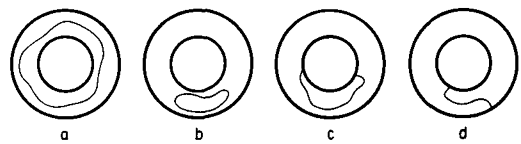

Fig. 1. Each of the four schematic illustrations represents a polar stereographic projection of the spherical sheet of fluid, in which the flow is contained in the zone between two latitude circles. The lightly-drawn curve in each case depicts one of the four topologically distinct types of curves of equal vorticity.

Under these conditions, it is not difficult to prove that the integral in (1) is invariant also when the more general types of vorticity lines, classified above, are permitted, provided $\bar{y}$ in (2) is suitably defined.

For what we have called the *cyclic* type of curve, the line integral (2) is determined by the area between this curve and the pole. Since the fluid is incompressible and vorticity is conserved, $\bar{y}$ thus defined is invariant for any vorticity line of this type. Likewise, for the *closed* type of curve (fig. 1 b), the line integral (2) is determined by the area enclosed by the curve, so for the same reasons $\bar{y}$ is invariant.

We start with the general statement that

$$\iint y\zeta \, dS \tag{3}$$

is invariant (FJØRTOFT, 1950), where dS is an element of surface area of the sphere; for, it can be shown that this integral is determined by the total angular momentum of the fluid. Thus, let ψ denote the stream function and write $\zeta = \operatorname{div} \nabla\psi$ so that

$$y\zeta = \operatorname{div}(y \nabla \psi) - \nabla y \cdot \nabla \psi.$$

If this identity is integrated over a region between two latitude circles corresponding to y_2 (on the north) and y_1 (on the south), we get the result (STARR, 1950) that (3) is equal to $y_1 \Gamma_1 - y_2 \Gamma_2 +$ {total angular momentum}, where Γ_1 and Γ_2 are the fluid circulations along the boundaries. The latter, as well as the total angular momentum, are conserved.[1]

Consider first a flow in which the "boundary types" of vorticity lines are excluded, and the boundaries coincide with lines of equal vorticity. Imagine the entire region divided into strips by vorticity lines ζ, $\zeta + \delta\zeta$, etc., and then into elementary parallelograms by meridians λ, $\lambda + \delta\lambda$, etc. (figure 2). The element of area ds is then $\delta y \delta\lambda$, and the contribution of one of these vorticity strips to (3) is[2]

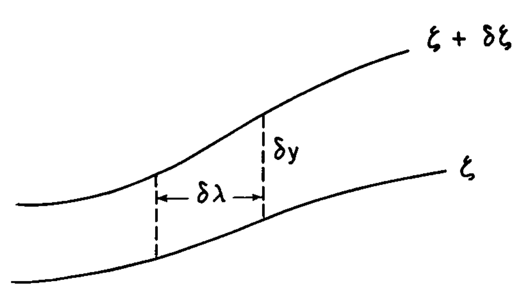

Fig. 2. The strip between vorticity lines ζ and $\zeta + \delta\zeta$ is divided into elementary parallelograms by meridians λ and $\lambda + \delta\lambda$.

$$\zeta y \, \delta y \, \delta\lambda = \oint\oint \delta\left(\frac{1}{2} y^2 \zeta\right) \delta\lambda - \delta\zeta \oint \frac{1}{2} y^2 \, \delta\lambda.$$

In adding the contributions from each strip we can integrate the first term on the right partially for a fixed value of λ, from wall to wall (y_1 to y_2), and get the result that *minus* (3) is equal to

$$\frac{1}{2} y_1^2 \oint \zeta_1 \, \delta\lambda - \frac{1}{2} y_2^2 \oint \zeta_2 \, \delta\lambda + \tag{4}$$

$$+ \int \delta\zeta \oint \frac{1}{2} y^2 \, \delta\lambda. \tag{5}$$

In (5) substitute $y = \bar{y} + \varepsilon$; the line integral is then equal to

$$\frac{1}{2} \bar{y}^2 \oint \delta\lambda + \bar{y} \oint \varepsilon \, \delta\lambda + \tag{6}$$

$$+ \oint \frac{1}{2} \varepsilon^2 \, \delta\lambda. \tag{7}$$

In view of the definition of $\bar{y}$ in (2), we find

$$\oint \varepsilon \, \delta\lambda = 2\pi \bar{y} (1 - \tau), \text{ where}$$

$$\tau \equiv \frac{1}{2\pi} \oint \delta\lambda = \begin{matrix} 1 \text{ for cyclic curve} \\ 0 \text{ for closed curve.} \end{matrix}$$

It follows that (6) is equal to

$$2\pi \bar{y}^2 \left(1 - \frac{1}{2}\tau\right).$$

[1] The boundary effects disappear when the flow covers the entire sphere; or when viscosity is invoked to provide no slip at the boundaries (in which case, of course, the total angular momentum may not be conserved).

[2] Note that $\delta\zeta$ must be taken in such a sense that the corresponding δy is positive wherever $\delta\lambda$ is positive, because $\delta y \, \delta\lambda = dS$ is inherently positive.

Collecting these results, we see that *minus* (3) is equal to

$$\frac{1}{2}\,\gamma_1^2 \oint \zeta_1\,\delta\lambda - \frac{1}{2}\,\gamma_2^2 \oint \zeta_2\,\delta\lambda + \qquad (4)$$

$$+\,2\pi \int \bar{y}^2 \left(1 - \frac{1}{2}\,\tau\right) \delta\zeta + \qquad (8)$$

$$+ \int \delta\zeta \oint \frac{1}{2}\,\varepsilon^2\,\delta\lambda. \qquad (1)$$

If all vorticity strips are either cyclic or closed, the boundaries must coincide with vorticity lines; in this case each term in (4) is invariant. Further, (8) is invariant in any case because not only are ζ and $\bar{y}$ conservative quantities, but also τ is conservative because of the invariant character of the topological classification of ζ-lines. The invariance of (4) and (8) therefore leads directly to the invariance of (1).

The preceding argument depends upon exclusion of "boundary types" of vorticity lines, since otherwise (4) is not invariant in general. For such curves the integral defining $\bar{y}$ in (2) must be interpreted as a *closed* line integral (for example, for "boundary type I") by arbitrarily incorporating a segment of the boundary joining the two end points of the vorticity line. The integral in (2) is then determined by the area occupied by the fluid thus enclosed, and therefore $\bar{y}$ still is a conservative quantity. With this provision it can be proved that the contribution to (3) represented by (4) *vanishes* if the line integral in (1) is interpreted as traversing in each case a closed curve, by including, where necessary, appropriate portions of the boundary. The invariance of (1) then follows from the invariance of (8).[3]

Of the three conclusions formulated by Starr from the invariance of (1), the first particularly calls for some discussion. He states, in effect, that if the vorticity distribution is *montone*, so that $\delta\zeta$ is everywhere positive or everywhere negative, then "zonal flow cannot develop into disturbed motion." This inference is based on the idea that while (1) vanishes for zonal flow, it cannot vanish for a disturbed monotone vorticity distribution unless $\varepsilon \equiv 0$; but then there can be no disturbance.

If this statement were taken at face value, it would reduce to an obvious truism because zonal flow can never "develop" into disturbed motion unless a disturbance of some kind is supplied; for, regardless of *stability*, zonal flow is an equilibrium state and therefore cannot change without the action of a disturbance. But if a disturbance is postulated, the flow is automatically disturbed *ab initio*.

Evidently, the statement must be interpreted as an inference regarding the *stability* of zonal equilibrium flow, implying, in other words, that if the postulated zonal flow is slightly disturbed, then in some sense (suitably defined) the disturbance will not grow. Since (1) is not restricted in any way to *small* disturbances, it is essential to use the word "stability" with caution. The harmonic-time-dependent perturbation method provides clear-cut analytical procedures for investigating this question, and at the same time establishes a precise operational basis for a *definition* of stability (a definition that cannot, however, be abstracted readily from its operational context).

If the term is used in this sense, the statement in question coincides with the well-known Rayleigh criterion; but it is not evident that there is an equivalence between the concept of stability involved in Rayleigh's criterion and that which is tacit in Starr's statement.

A rather general formal stability criterion in accord with intuition is as follows. Let P and Q be suitable parameters characteristic of the *deformation* of an equilibrium state and let P_0 be the initial value of P corresponding to a prescribed initial disturbance.

> The equilibrium state is *stable* if, regardless of the initial *form* of the disturbance, it is always possible to assign a finite value to P_0 that will make Q smaller than *any* preassigned value *for all time*.

In principle the qualification "for all time" is essential (as intuition suggests), yet this proviso probably prohibits the use of perturbation theory, or in fact any approximation theory whose validity is inherently limited to a restricted interval of time. Since we are dealing with a nonlinear system, the preceding formal definition of stability suggests an investigation of the integral invariants of the system.

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[3] If the lateral boundaries are dispensed with and the flow covers the entire sphere, the "boundary types" of vorticity curves do not occur and the integral (1) requires no further elaboration.

The properties P and Q need not be distinct; for example, both may be identified with the energy of the disturbance. It should also be remarked that the choice of P and Q may to some extent restrict the scope of the criterion; in other words stability with respect to one pair of parameters may not always imply stability with respect to a different pair. For example, in connection with the Rayleigh criterion, FJØRTOFT (1950) has stressed that stability can be proved only if P is identified with disturbance *vorticities* (assuming Q is identified with the sum of squares of meridional displacements).

Returning to an interpretation of Starr's inference from the invariance of (1), suppose we identify (1) with the parameter P, and Q with the kinetic energy of the disturbance. This means that we must establish a connection between (1) and the disturbance energy, in such a way as to satisfy the stability criterion. A rigorous proof of stability in this sense would be of considerable interest from the standpoint of theoretical fluid mechanics.

One of the perplexities of this problem arises from the consideration that there may be *certain* parameters Q with respect to which stability can never be proved. Possibly one such inherently "unstable" parameter is the magnitude $|\nabla\zeta|$ of the vorticity gradient. Apart from highly exceptional cases (for example, steady wave propagation), it seems probable that by and large, within the nonlinear framework of the vorticity equation, a given initial flow will evolve toward greater and greater complexity: the vorticity distribution, at least in some regions of the flow, must become progressively distorted, individual lines exhibiting more and more involved convolutions and tending to "fill the space." It seems inevitable that in certain critical regions vorticity gradients must ultimately attain large values (for which viscosity may begin to dominate), regardless of any considerations of the *stability* of the system.[4]

To the writer, an exacting and highly idealized criterion such as involved in the preceding formal definition of stability does not seem to be especially relevant in connection with the behavior of large-scale atmospheric flow. Probably no equilibrium states exist in the atmosphere; and because of the complexity of this system, the inherent oversimplifications in any model render extremely dubious any attempt to investigate the behavior of the model *for all time*.

Starr's invariant is a highly ingenious formulation of the momentum principle for two-dimensional flows; an integral invariant of the hydrodynamical equations, it is of interest *per se*. The writer hopes that the remarks made above will call attention to the need for clarifying some of the inferences made by Starr in his note.

[4] It may be of interest to point out that because it is responsible for the transference of vorticity between fluid particles, viscosity will render nonconservative the "topological" parameter τ defined above. Each such nonconservative event, altering the connectivity of a vorticity strip, corresponds to a transition of τ from 1 to 0 (or vice versa).

REFERENCES

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