A Baroclinic Model of the Atmosphere Applicable to the Problem of Numerical Forecasting in Three Dimensions.[1] I

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Abstract

A three-dimensional baroclinic model of the atmosphere is presented. This model is completely determined by 4 parameters, three of which specify the field of temperature. The main difference between this model and other models so far published is that it allows for a non-uniform temperature lapse-rate in the horizontal. Moreover, it takes into account the observed division of the atmosphere into a troposphere and a stratosphere.

A simplification is introduced by utilizing the synoptic experience that the horizontal temperature gradient is usually reversed when traversing the tropopause. This essentially reduces the number of parameters to three.

This simplified version of the model is used for integrating the vorticity equation along the vertical, and the coefficients of the resulting two-dimensional equation are discussed in some detail. Examination of the coefficients give some valuable information regarding the levels of mean wind and non-divergence.

The main results are that in middle latitudes the level of mean wind may vary from 2.5—6.5 km and that the level of of non-divergence is always higher than the former, the difference amounting to 0.5—3.0 km. On the average, the level of mean wind lies below and the level of non-divergence above the 500 mb surface. One is therefore lead to the conclusion that this surface might be used to approximate each of the two levels mentioned.

The consequence of this for the practical procedures for numerical forecasting is that the height changes at this particular level could alternatively be predicted in the two following ways:

a) By use of the barotropic vorticity equation, thereby assuming that the level of non-divergence can be replaced by the 500 mb surface.
b) By use of an integrated vorticity equation applicable to the level of mean wind. In this case one replaces the level of mean wind with the 500 mb surface.

1. Introduction

The remarkable fact that temperature is nearly a linear function of height is used in this paper as a guiding principle for constructing an atmospheric model which is believed to take into account the essential features of the actual atmosphere. The main divisions, the troposphere and the stratosphere, are kept in the model, but the numerous irregularities in the observed soundings are eliminated through a smoothing procedure. An objective smoothing may be achieved by approximating an observed temperature height curve through a polynomial in z in

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which the coefficients are functions of x, y and t. Any required degree of accuracy can be obtained in this way; however, because of the well established linearity in temperature with height, a first order polynomial is thought to be sufficient and has been chosen for the model presented in this paper.

Models based upon simplifications in the theoretical representation of the field of temperature are certainly not uncommon; to the contrary, such simplifications are found explicitly or implicitly in a number of papers. We may mention the advective model of ROSSBY (1942), CHARNEY'S (1947) and EADY'S (1949) basic current models, and the model underlying the forecasting method developed by the Russian meteorologist J. A. Kibel in the early forties [see IZVEKOW (1946) and HAURWITZ (1946)]. Moreover, we will mention the so-called $2^1/_2$ dimensional models presented by SUTCLIFFE (1947), FJÖRTOFT (see CHARNEY 1951), PHILLIPS (1951), EADY (1952) and ELIASSEN (1952).

Common to most of the works referred to above is the assumption that the horizontal temperature gradient is independent of height. The model presented in this paper goes one step further inasmuch as it contains a tropospheric temperature gradient which varies linearly with height in accordance with the equation

$$\nabla \tau = \nabla \tau_0 - z \nabla k \qquad (1, 1)$$

Here $-\nabla \tau_0$ is the horizontal temperature gradient at $z = o$, and $-\nabla k$ the horizontal gradient in the temperature lapse-rate. It is firmly believed that the introduction of a non-uniform lapse-rate in the field of temperature is a considerable improvement over models based on a temperature gradient which is independent of height. This improvement appears clearly when one deals with the adiabatic equation. In the case of temperature gradient independent of height this equation admits only a trivial solution for the vertical velocity, whereas the linear temperature gradient gives a parabolic distribution of the vertical velocity with height. Moreover, an application of the model to numerical forecasting has shown that local changes in the average (with respect to height) temperature lapse rate may contribute considerably to the local pressure changes.

Section 2 of this paper is devoted to the detailed presentation of the equations describing the fields of pressure, geostrophic wind and geostrophic vorticity corresponding to the smoothed field of temperature. The knowledge of the functional relationship of these elements with respect to the height z makes the model particularly well suited for integration of the vorticity equation along a vertical.

The integration of this equation is dealt with in sections 3 and 4, and rather definite information regarding the levels of mean wind and non-divergence is gained. It turns out that these two levels are primarily determined by the vertical structure of the atmosphere, and that the particular distribution of temperature and pressure is not essential for their determination. A considerable influence upon the height of these levels is exerted, however, by the horizontal distribution of temperature in the stratosphere.

A continuation of this article, to be published in a later issue of Tellus, will be devoted to the application of the model to numerical prediction of the fields of temperature and pressure.

2. Specification of the model with respect to height

The field of temperature. The model consists of a troposphere, in which the temperature τ is given as function of the height z by the equation:

$$\tau = \tau_0 - kz \qquad (2, 1)$$

and a stratosphere where the temperature is independent of height. In equ. (2, 1) k is the temperature lapse-rate and is independent of z. Both τ_0 (surface temperature) and k are functions of the horizontal coordinates x, y and the time t. Since the stratosphere by definition is characterized by isothermal conditions in the z-direction, its temperature is obviously obtained from (2, 1) by putting $z = h$, where h is the height of the tropopause, which may vary with x, y and t. Thus:

$$\tau_s = \tau_h = \tau_0 - kh \qquad (2, 2)$$

where the subscripts s and h refer to the values of τ in the stratosphere and at the tropopause level respectively.

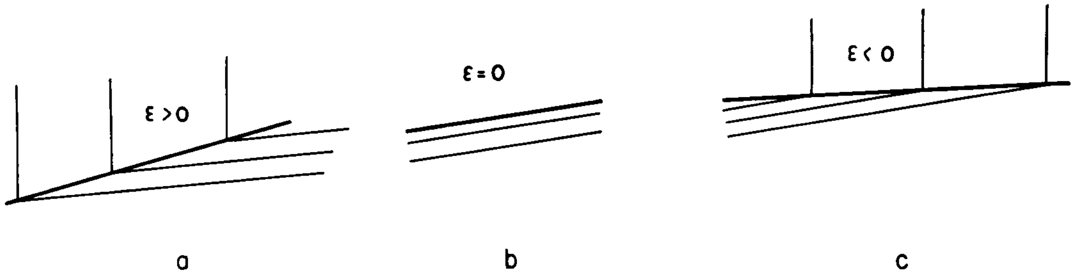

Fig. 1. The thick line is the tropopause and the thin lines are isotherms. In a) the slope of the tropopause is greater than that of the isotherms in the troposphere and the tropospheric and the stratospheric temperature gradients have opposite directions. In b) the isotherms and the tropopause have the same slope and $\nabla \tau_s = 0$, and in c) the two temperature gradients have the same direction.

From (2, 1) and (2, 2) the following expressions for $\nabla \tau$ — the horizontal temperature ascendent in the troposphere — and $\nabla \tau_s$ — the horizontal temperature ascendent in the stratosphere — are obtained through derivation:

$$\nabla \tau = \nabla \tau_0 - z \nabla k \qquad (2, 3)$$

$$\nabla \tau_s = \nabla \tau_0 - \nabla (kh) = (\nabla \tau)_{z=h} - k \nabla h \qquad (2, 4)$$

Three-dimensional nabla operators will not occur in this paper, and the symbol ∇ will therefore be used throughout to denote the two-dimensional nabla operator referring to a horizontal surface.

It is obvious from (2, 4) that knowledge of the topography of the tropopause is needed in order to determine the jump in the horizontal temperature gradient, when passing from the troposphere to the stratosphere.

For later reference and for comparison with the model, the general expression for the relationship between the tropospheric and the stratospheric temperature gradients is given below:

$$(\nabla \tau_s)_{z=h} = (\nabla \tau)_{z=h} - (k - k) \nabla h \qquad (2, 5)$$

where k_s is the stratospheric temperature lapse-rate at the level of the tropopause and k is the tropospheric one at the same level. Putting $k_s = 0$ and assuming k and $\nabla \tau_0$ to be independent of z, equ. (2, 5) becomes identical with (2, 4).

It is frequently observed that, at least in middle latitudes, the direction of the horizontal temperature gradient is reversed when passing through the tropopause. This implies that $(k - k_s) \nabla h$ has the same direction as $(\nabla \tau)_{z=h}$ and is greater in magnitude. Moreover, experience indicates that a strong tropospheric temperature gradient is positively correlated to a great slope of the tropopause. Since the details of the stratospheric field of temperature are presumably of minor importance for the solution of a great many problems in dynamic and synoptic meteorology, these empirical findings will be used to simplify the model. This is done by replacing equ. (2, 4) by the simpler one:

$$\nabla \tau_s = -\varepsilon (\nabla \tau)_{z=h} \qquad (2, 6)$$

where ε is a parameter of the model, the value of which, however, is sofar unknown. Fig. 1 is a vertical cross-section showing how the slope of the tropopause in relation to that of the isotherms of the troposphere depends upon the sign of ε. By use of the daily tropopause charts published by the British Meteorological Office, London, and the associated aerological data, values of the parameter ε have been computed and their variability studied to a certain extent. The results will be discussed in some detail further below.

Returning to equ. (2, 3), the tropospheric temperature gradient is obviously a linear function of height since ∇k is independent of z. An observed temperature gradient alternatively veering and backing with height could therefore not be approximated through (2, 3). Observations indicate that such a variation with height is not very common, and the average veering or backing with height can always be accounted for by the model, since ∇k is independenτt of $\nabla \tau_0$.

It is convenient to distinguish between the following special cases:

a) $\nabla k = 0$ The horizontal temperature gradient is independent of height. As will appear below this implies a linear variation in the geostrophic wind with height.

b) $\nabla k = \varkappa \Delta \tau_0$. The direction of $\Delta \tau$ is the same throughout the troposphere, with a possible exception for a change of 180 degrees. The magnitude will decrease or increase with height depending on the sign of the proportionality factor $\varkappa$.

c) $\nabla k = \varkappa \nabla \tau_0 \times \mathbf{k}$ where $\mathbf{k}$ is a unit vector directed along the positive vertical. In this case the temperature gradient rotates at a constant rate throughout the troposphere.

The field of pressure. Having expressed the temperature and the temperature gradient as known functions of height, the pressure field is easily determined from the hydrostatic equation, which integrated is most conveniently used in the form:

$$p_2 = p_1 \cdot e^{-\frac{g}{R}\int_{z_1}^{z_2} \frac{dz}{\tau}} \qquad (2, 7)$$

where g is the acceleration of gravity and R the gas constant per unit mass.

Applied to the model equ. (2, 7) gives the following expressions for the pressure, the horizontal pressure gradient and the density:

$$p = p_0\left(\frac{\tau}{\tau_0}\right)^n; \quad \varrho = \varrho_0\left(\frac{\tau}{\tau_0}\right)^{n-1} \qquad (2, 8)$$

$$\nabla p = \left(\frac{\tau}{\tau_0}\right)^n \left[\nabla p_0 + \frac{g\varrho_0}{\tau} z \left(\nabla \tau_0 - \frac{z}{2}\nabla k\right)\right] \qquad (2, 9)$$

$$p_s = p_h \cdot e^{-mz_s}; \quad \varrho_s = \varrho_h e^{-mz_s} \qquad (2, 10)$$

$$\nabla p_s = e^{-mz_s}\left(\nabla p_h + \frac{g\varrho_h}{\tau_h} z_s \nabla \tau_s\right) \qquad (2, 11)$$

where $n = \frac{g}{Rk}$ and $m = \frac{g}{R\tau_h}$

Equations (2, 8) and (2, 9) apply to the troposphere and (2, 10) and (2, 11) to the stratosphere. A slight approximation has been made in (2, 9).

Applying the geostrophic equation we arrive at the following expressions for the geostrophic wind in the troposphere and the stratosphere respectively:

$$\mathbf{v} = \frac{\tau}{\tau_0}\mathbf{v}_0 + \mathbf{v}'z - \mathbf{v}''z^2 \qquad (2, 12)$$

$$\mathbf{v}_s = \mathbf{v}_h + \mathbf{v}'_s z_s \qquad (2, 13)$$

where $\mathbf{v}' = \frac{g}{f\tau_0}\mathbf{k} \times \nabla \tau_0$, $\quad \mathbf{v}'' = \frac{g}{2f\tau_0}\mathbf{k} \times \nabla k$,

$\mathbf{v}_s' = \frac{g}{f\tau_h}\mathbf{k} \times \nabla \tau_s$, and f is the Coriolis parameter.

In virtue of (2, 6),

$$\mathbf{v}_s' = -\varepsilon\frac{\tau_0}{\tau_h}(\mathbf{v}' - 2h\mathbf{v}'') \qquad (2, 14)$$

which inserted in (2, 13) gives:

$$\mathbf{v}_s = \mathbf{v}_h - \varepsilon\frac{\tau_0}{\tau_h}(\mathbf{v}' - 2h\mathbf{v}'')z_s \qquad (2, 15)$$

It follows from (2, 12) that the wind in the troposphere is a polynomial of second order in z. There is no relationship between the constant vectors $\mathbf{v}_0$, $\mathbf{v}'$ and $\mathbf{v}''$ and the wind may decrease or increase with height and change direction as well. In the stratosphere the wind is a linear function of height only, corresponding to the simpler field of temperature.

Fig. 2 gives a sample of possible wind profiles in the case of uniform wind direction with height. It may of course also be interpreted as the profile of the wind component in a certain direction.

Neglecting the variation in $\frac{\tau}{\tau_0}$, $\frac{1}{\tau_0}$ and $\frac{1}{\tau_h}$ compared with that of $\mathbf{v}_0$, $\nabla \tau_0$, ∇k and $\nabla \tau_s$ respectively, the following expressions for the relative vorticity of the troposphere and the stratosphere are obtained from (2, 12) and (2, 13):

$$\zeta = \frac{\tau}{\tau_0}\zeta_0 + \zeta' z - \zeta'' z^2 \qquad (2, 16)$$

$$\zeta_s = \zeta_h + \zeta'_s z_s \qquad (2, 17)$$

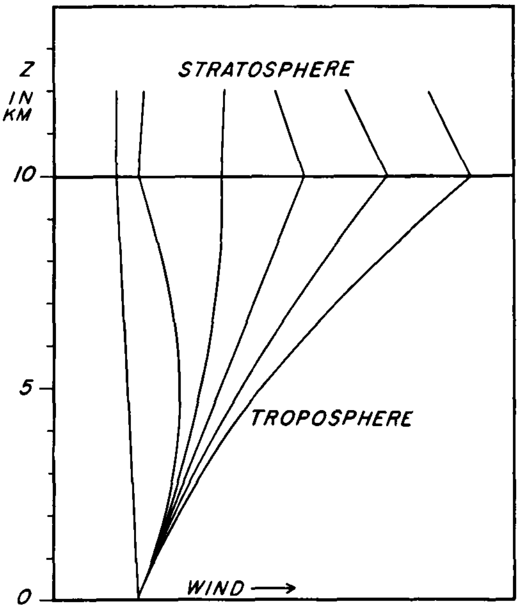

Fig. 2. Examples of possible wind profiles in the model. The first profile from the left corresponds to no temperature gradient and the fourth profile corresponds to a constant temperature gradient throughout the troposphere.

where

$$\zeta_0 = \frac{1}{f\varrho_0}\nabla^2 p_0,\ \zeta' = \frac{g}{f\tau_0}\nabla^2 \tau_0,\ \zeta'' = \frac{g}{2f\tau_0}\nabla^2 k,$$

$$\zeta_s' = \frac{g}{f\tau_h}\nabla^2 \tau_s \qquad (2,\ 18)$$

and ∇^2 is a two-dimensional Laplacian referring to a horizontal surface. The expressions for the wind and the relative vorticity are analogous, and the profiles depicted in fig. 2 may therefore just as well represent a sample of vorticity profiles.

Derivating equ. (2, 6) and neglecting the variation in h and ε compared with that of $\nabla \tau_0$ and ∇k, we may write

$$\nabla^2 \tau_s = -\varepsilon\,(\nabla^2 \tau_0 - h\nabla^2 k) \qquad (2,\ 19)$$

This inserted in the expression for ζ_s' gives:

$$\zeta_s' = -\varepsilon\frac{\tau_0}{\tau_h}(\zeta' - 2h\zeta'') \qquad (2,\ 20)$$

Substituting for ζ_s' in (2, 17) we then arrive at the following expression for the stratospheric vorticity:

$$\zeta_s = \zeta_h - \varepsilon\frac{\tau_0}{\tau_h}(\zeta' - 2h\zeta'')z_s \qquad (2,\ 21)$$

The assumptions underlying equation (2, 6) may for some purposes prove to be too crude. We will therefore go back to (2, 4) and give the corresponding expressions for the wind and relative vorticity of the stratosphere:

$$\mathbf{v}_s = \mathbf{v}_h + \left[\frac{\tau_0}{\tau_h}(\mathbf{v}' - 2h\mathbf{v}'') - \frac{gk}{f\tau_h}\mathbf{k}\times\nabla h\right]z_s \quad (2,22)$$

$$\zeta_s = \zeta_h + \left[\frac{\tau_0}{\tau_h}(\zeta' - 2h\zeta'') - \frac{gk}{f\tau_h}\nabla^2 h\right]z_s \qquad (2,\ 23)$$

In deriving (2, 23) from (2, 22) similar approximations have been done as in the derivation of (2, 16) and (2, 17).

It is clear from (2, 22) and (2, 23) that the topography of the tropopause gives rise to stratospheric wind and vorticity. When replacing the atmosphere by this model, this contribution of the tropopause can be measured from a chart of tropopause contours in the same way as is geostrophic wind and vorticity measured from ordinary upper air maps.

Computed values of the parameter ε based on observations. On the preceding pages the fields of temperature, pressure, geostrophic wind and geostrophic vorticity have been described in terms of quantities, the magnitude and range of variability of which are well known from surface and upper air observations. The single exception is the parameter ε, which is not readily obtained from current data.

In order to investigate the magnitude and variability of this parameter some studies have been made based on the tropopause maps and the aerological data in the Daily Aerological Record, published by the Meteorological Office, London. One value of ε was computed for nearly each day of a period covering the months Febr., July, Aug. 1950 and Nov., Dec. 1951.

As mentioned earlier the horizontal temperature gradient of the stratosphere at $z = h$ is related to that of the troposphere at the same level and the gradient of the contour lines of the tropopause in accordance with the following equation:

$$(\nabla \tau_s)_{z=h} = (\nabla \tau)_{z=h} - (k - k_s)\nabla h \qquad (2,\ 5)$$

The British material quoted above shows that in the majority of cases the three gradients in this equation are approximately directed along the same line. With this in mind two sounding stations were chosen such that their line of connection, as far as possible, was perpendicular to the contour lines of the tropopause, and tropospheric and stratospheric temperature differences were computed as sketched in fig. 3.

Altogether 147 values of ε were computed, and the results are given in table 1 and fig. 4.

As one might expect, the two temperature gradients are in general of the opposite direction. The recorded exceptions were associated with very weak temperature gradient in the layer 500—300 mb and the same applies to the cases with high positive values of ε. It is obvious from (2, 5) that when $(\nabla \tau)_{z=h}$ is zero, $(\nabla \tau_s)_{z=h}$ is proportional to $-\nabla h$. The latter may of course differ from zero, in which case ε approaches $\pm \infty$. Study of the tropopause charts in relation to the relative thickness lines 500—300 mb have shown that the proportionality between the tropospheric and the stratospheric temperature gradients in the vicinity of the tropopause is rather well established, whenever the temperature distribution is not too far from the normal one. The parameter ε is then positive. When, however, the temperature distribution is greatly disturbed, the direction of the two temperature gradients may form a rather arbitrary angle and the assumption of proportionality breaks down. In such cases there will be regions with a very weak temperature gradient in the troposphere, whereas the slope of the tropopause may be rather pronounced, giving rise to a stratospheric temperature gradient which has no simple relationship to the tropospheric one. Determination of ε will then give values within a very great range.

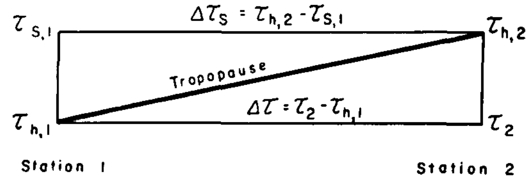

Fig. 3. A vertical cross-section at approximately right angle to the contour lines of the tropopause. The components in this plane of the tropospheric and the stratospheric horizontal temperature gradients at the tropopause are represented through $\triangle \tau$ and $\triangle \tau_s$ respectively. Subscript h refers to the tropopause level, and subscript s to the stratosphere.

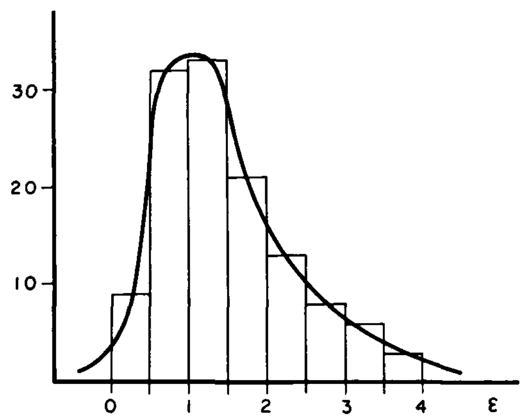

Fig. 4. Frequency distribution for ε. The number of cases within each of the intervals 0.0—0.5, 0.5—1.0 etc. up to 3.5—4.0 are given by means of a histogram. The total number of cases is 147.

In the model one has assumed that the stratospheric temperature gradient is independent of height, whereas the magnitude of the observed one in general decreases with height, and a change in direction may also occur. This actual decrease in magnitude can be accounted for in the model by proper choice

Table 1. Mean values of the parameter ε

Month	Total number of cases	Number of cases with:			Mean values
		$0 \leq \varepsilon \leq 3.99$	$\varepsilon \geq 4.00$	$\varepsilon < 0$	$0 \leq \varepsilon \leq 3.99$
Febr. 1950	28	23	0	5	1.23
July »	29	26	2	1	1.26
Aug. »	30	28	1	1	1.67
Nov. 1951	29	25	1	3	1.50
Dec. »	31	23	4	4	1.48
Period	147	125	8	14	1.43

of the parameter ε. In order to study this decrease to a certain extent, the temperature difference at 100 mb between the two stations in fig. 3 — whenever available — was computed. In the average this difference was 40 % of that at the tropopause level. Altogether 137 cases were treated and in 15 of these, change in the sign of the temperature difference was observed. In 3 cases it increased with height. These 18 cases were associated with weak to moderate temperature differences.

In spite of the very limited region (the British Isles and the two British weather ships in the Atlantic), to which the aerological material used here refers, the results are presumably valid for middle latitudes. In high latitudes negative values of ε will — at least in wintertime — occur more frequently than in middle latitudes, and the assumption of proportionality between the two temperature gradients may turn out to be in less agreement with observations.

3. Integration of the vorticity equation along the vertical

The integrated vorticity equation. When friction is neglected the vorticity equation may be written as follows:

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta =$$

$$- \left(\eta \operatorname{div} \mathbf{v} + \frac{f}{\varrho} \mathbf{v}_g \cdot \nabla \varrho + w \frac{\partial \zeta}{\partial z} + \nabla w \times \frac{\partial \mathbf{v}}{\partial z} \cdot \mathbf{k}\right)$$

where $\mathbf{v}$ is the horizontal wind vector, $\eta = \zeta + f$ the absolute vorticity component along the vertical, w the vertical velocity, and $\mathbf{v}_g$ the geostrophic wind. Elimination of div $\mathbf{v}$ by use of the equation of continuity

$$\varrho \operatorname{div} \mathbf{v} + \frac{\partial}{\partial z} \varrho w + \mathbf{v} \cdot \nabla \varrho + \frac{\partial \varrho}{\partial t} = 0$$

gives:

$$\varrho \frac{\partial \eta}{\partial t} + \varrho \mathbf{v} \cdot \nabla \eta = \eta \left[\frac{\partial}{\partial z} \varrho w + \mathbf{v} \cdot \nabla \varrho + \frac{\partial \varrho}{\partial t}\right] -$$

$$- f \mathbf{v}_g \cdot \nabla \varrho - \varrho w \frac{\partial \zeta}{\partial z} - \varrho \nabla w \times \frac{\partial \mathbf{v}}{\partial z} \cdot \mathbf{k}$$

It is generally assumed that the last term on the right hand side is comparatively small, and it is therefore neglected here. Furthermore the terms $\eta \mathbf{v} \cdot \nabla \varrho$ and $\eta \frac{\partial \varrho}{\partial t}$ are small compared with the terms $\varrho \mathbf{v} \cdot \nabla \eta$ and $\varrho \frac{\partial \eta}{\partial t}$ because the relative change $\frac{\triangle \varrho}{\varrho}$ in the density is in all cases of practical interest very much smaller than the relative change $\frac{\nabla \eta}{\eta}$ in the absolute vorticity. In virtue of this the vorticity equation is reduced to:

$$\varrho \frac{\partial \eta}{\partial t} + \varrho \mathbf{v} \cdot \nabla \eta = \eta \frac{\partial}{\partial z} (\varrho w) - \varrho w \frac{\partial \zeta}{\partial z} = \eta^2 \frac{\partial}{\partial z} \left(\frac{\varrho w}{\eta}\right) \tag{3, 1}$$

Equation (3, 1) will now be integrated with respect to z for the boundary conditions $w = 0$ for $z = 0$ and $\varrho = 0$ for $z = \infty$ $\left(\frac{w}{\eta}\right.$ finite at infinity) by use of the model described in the preceding section. This implies that $\mathbf{v}$ and η are to be interpreted as the geostrophic wind and the geostrophic vorticity of the model. Moreover, we will assume that the integrated contribution due to the variation in η^2 with height is so small that it can be neglected. This together with the boundary conditions makes the right hand side of (3, 1) disappear when integrated from $z = 0$ to $z = \infty$, and we obtain:

$$\int_0^\infty \varrho \frac{\partial \zeta}{\partial t} dz + \beta \int_0^\infty \varrho v dz + \int_0^\infty \varrho \mathbf{v} \cdot \nabla \zeta dz = 0 \tag{3, 2}$$

where $\beta = \frac{df}{dy}$ and $v = \frac{1}{f\varrho} \frac{\partial p}{\partial x}$ is the y-component of the geostrophic wind. It is possible to make a certain statement regarding the sign of the error introduced by neglecting the variation in η^2 with height. Fig. 5 gives a schematic picture of η^2, $\frac{\varrho w}{\eta}$, and $\frac{\partial}{\partial z}\left(\frac{\varrho w}{\eta}\right)$ as functions of z.

Obviously, η^2 decreasing with height will make the $\int_0^\infty \eta^2 \frac{\partial}{\partial z}\left(\frac{\varrho w}{\eta}\right) dz$ positive when associated with positive vertical velocity and negative when associated with negative vertical velocity. This statement is to be reversed when η^2 increases with height. Apart from this qualitative consideration we will not at this stage try to evaluate the integrated contribution of the right hand side of (3, 1), but it is assumed to be comparatively small in most of cases.

Substituting in (3, 2) for ϱ, $\mathbf{v}$ and ζ the expressions given in the preceding section, the integration can be carried out by elementary methods; but, since a considerable amount of calculation is involved, the details are not given here. From the expressions quoted it can be seen that all the integrals to be evaluated are of the two types $\int_0^h \left(\frac{\tau}{\tau_0}\right)^n z^q dz$ and $\int_0^\infty e^{-mz} z^q dz$ where m, n and q are independent of z, τ is the temperature at the height z and τ_0 the temperature at the zero level.

Having carried out the integration of (3, 2) the following equation is obtained:

$$A_1\frac{\partial \zeta_0}{\partial t} + A_2\frac{\partial \zeta'}{\partial t} - A_3\frac{\partial \zeta''}{\partial t} + \frac{\beta}{f}\left(\frac{1}{\varrho_0}A_1\frac{\partial p_0}{\partial x} + \frac{g}{\tau_0}A_2\frac{\partial \tau_0}{\partial x} - \frac{g}{2\tau_0}A_3\frac{\partial k}{\partial x}\right) + \mathbf{v}_0\cdot(A_4\nabla\zeta_0 + A_5\nabla\zeta' - A_6\nabla\zeta'') + \mathbf{v}'\cdot(A_5\nabla\zeta_0 + A_7\nabla\zeta' - A_8\nabla\zeta'') - \mathbf{v}''\cdot(A_6\nabla\zeta_0 + A_8\nabla\zeta' - A_9\nabla\zeta'') = 0 \tag{3, 3}$$

The coefficients A_1, A_2, ... A_9 are in general positive and are functions of the parameters k, h, ε, p_0 and τ_0 of the model. Except for A_1 and A_4, which do not contain ε, each of the coefficients depends upon all the parameters. This functional relationship will be discussed at some length later on.

The considerable number of terms makes (3, 3) somewhat unattractive to deal with, and before attempting a general discussion two special cases will be mentioned:

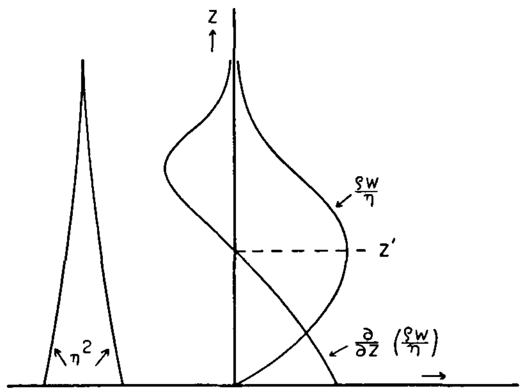

Fig. 5. A schematic picture of the curves η^2, $\frac{\varrho w}{\eta}$ and $\frac{\partial}{\partial z}\left(\frac{\varrho w}{\eta}\right)$. The area between the $\frac{\partial}{\partial z}\left(\frac{\varrho w}{\eta}\right)$ curve and the z-axis is divided in two equal parts by the level $z = z'$.

a. No horizontal temperature gradient, i.e. $\nabla\tau = \nabla k = 0$. Equation (3, 3) reduces to:

$$\frac{\partial \zeta_0}{\partial t} + \mathbf{v}_0\cdot\nabla f + \frac{A_4}{A_1}\mathbf{v}_0\cdot\nabla\zeta_0 = 0 \quad \text{where} \quad \frac{\beta}{f\varrho_0}\frac{\partial p_0}{\partial x} \tag{3, 4}$$

has been replaced by $\mathbf{v}_0\cdot\nabla f$. Except for the factor $\frac{A_4}{A_1}$, (3, 4) is identical with the barotropic vorticity equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{v}\cdot\nabla(f+\zeta) = 0 \tag{3, 5}$$

where $\mathbf{v}$ and ζ are independent of height. While in the baroropic case it is assumed that the vertical velocity is identically zero, no such assumption underlies the derivation of equ. (3, 4). This difference, however, is only of academic interest, since an identically vanishing temperature gradient is physically not compatible with non-vanishing vertical velocity, except in connection with adiabatic processes in an atmosphere with adiabatic temperature lapse-rate.

The ratio $\frac{A_4}{A_1}$ depends practically on the lapse-rate k only; it is equal to unity for $k=0$ and drops to 0.86 for $k = 0.75\cdot 10^{-2}$ °C/meter. The difference between the case considered here ($\nabla \tau = 0$) and the barotropic one is explained by the fact that in the latter case the temperature is constant along a pressure surface while in the former it is prescribed to stay constant in a horizontal plane. Only for $k = 0$, or when the pressure surfaces are horizontal, are these two cases identical.

b. Uniform temperature lapse-rate, i.e. $\nabla k = 0$. In this case equation (3, 3) reduces to:

$$A_1 \frac{\partial \zeta_0}{\partial t} + A_2 \frac{\partial \zeta'}{\partial t} + \frac{\beta}{f}\left(\frac{A_1}{\varrho_0}\frac{\partial p_0}{\partial x} + \frac{g}{\tau_0} A_2 \frac{\partial \tau_0}{\partial x}\right) + \mathbf{v}_0 \cdot$$
$$\cdot (A_4 \nabla \zeta_0 + A_5 \nabla \zeta') + \mathbf{v}' \cdot (A_5 \nabla \zeta_0 + A_7 \nabla \zeta') = 0 \qquad (3, 6)$$

The geostrophic wind is now a linear function of height, and as is well known, models with this property have been used by many authors when dealing with small perturbations, and with considerable success. In a paper not yet published equ. (3, 6) has been applied to investigate the speed, vertical velocity, and, to a certain extent, the stability of spherical harmonic wave disturbances superimposed upon a uniform basic current. The results were promising and it would seem that (3, 6) is an approximation good enough to give reasonably correct answers to certain types of problems.

Discussion of the coefficients of the integrated vorticity equation in relation to the parameters of the model. Returning to the integrated vorticity equation (3, 3) we will now insert $\frac{1}{f\varrho_0}\nabla^2 p_0$, $\frac{g}{\tau_0}\nabla^2 \tau_0$ and $\frac{g}{2f\tau_0}\nabla^2 k$ for ζ_0, ζ' and ζ'' respectively. After the substitution and a multiplication of all the terms with the Coriolis parameter f, equ. (3, 3) takes the form:

$$B_1 \nabla^2 \frac{\partial p_0}{\partial t} + B_2 \nabla^2 \frac{\partial \tau_0}{\partial t} - B_3 \nabla^2 \frac{\partial k}{\partial t} + \beta \left(B_1 \frac{\partial p_0}{\partial x} + \right.$$
$$\left. + B_2 \frac{\partial \tau_0}{\partial x} - B_3 \frac{\partial k}{\partial x}\right) + \mathbf{v}_0 \cdot (B_4 \nabla \nabla^2 p_0 +$$
$$+ B_5 \nabla \nabla^2 \tau_0 - B_6 \nabla \nabla^2 k) + \mathbf{v}' \cdot$$
$$\cdot \left(\frac{\tau_0}{g\varrho_0} B_5 \nabla \nabla^2 p_0 + B_7 \nabla \nabla^2 \tau_0 - B_8 \nabla \nabla^2 k\right) -$$
$$- \mathbf{v}'' \left(\frac{2\tau_0}{g\varrho_0} B_6 \nabla \nabla^2 p_0 + 2B_8 \nabla \nabla^2 \tau_0 - \right.$$
$$\left. - B_9 \nabla \nabla^2 k\right) = 0 \qquad (3, 7)$$

The relation between the two sets of coefficients is as follows:

$$\frac{f \cdot A_1}{B_1} = \frac{f \cdot A_4}{B_4} = \varrho_0 ; \frac{f \cdot A_2}{B_2} = \frac{f \cdot A_5}{B_5} = \frac{f \cdot A_7}{B_7} = \frac{\tau_0}{g} ;$$
$$\frac{f \cdot A_3}{B_3} = \frac{f \cdot A_6}{B_6} = \frac{f \cdot A_8}{B_8} = \frac{f \cdot A_9}{B_9} = \frac{2\tau_0}{g} \qquad (3, 8)$$

Each of the coefficients $B_1, B_2, \ldots B_9$ is a function of all the parameters k, h, ε, p_0 and τ_0 except for B_1 and B_4, which do not contain ε.

Analytical expressions for the coefficients in terms of the parameters of the model are found in the appendix.

We will now investigate to what extent the coefficients are sensitive for variation in the parameters, since this is a question of great practical importance when applying the equation (3, 7). It appears when studying the expressions for the coefficients, that p_0 is a common factor for all of them expect B_1 and B_4, where we, however, can introduce p_0 by replacing $R\tau_0$ by $\frac{p_0}{\varrho_0}$. Thus a change in the surface pressure does not directly influence the relative magnitude of the coefficients in (3, 7). After division with the common factor p_0, B_1 and B_4 and the coefficients for $\mathbf{v}' \cdot \nabla \nabla^2 p_0$ and $\mathbf{v}'' \cdot \nabla \nabla^2 p_0$ are proportional to the surface value $\alpha_0 = \frac{1}{\varrho_0}$ of the specific volume and will therefore vary in the same manner. The relative change in α_0 is, however, so small, that for practical purposes it may generally be neglected.

The relation to the surface temperature τ_0 is considerably more complicated and cannot easily be surveyed without making computations for different values of τ_0. This has therefore been done for $\tau_0 = 255$, 270, 285 and 300 degrees absolute, keeping the parameters k, h and ε constant and equal to $0.65 \cdot 10^{-2}$ °C/meter, 10 km and 0.50 respectively. The results are given in table 2, where values of all the coefficients in equ. (3, 7) have been computed for the four different values of τ_0 and for $p_0 = 1{,}000$ mb. For 15° C variation in τ_0 the percental change in the coefficients is from 2—10 %. It is worth-while to notice that some of the coefficients increase with increasing τ_0 while others decrease. Although all the terms of equ. (3, 7) are of the same

Table 2. Values of the coefficients of equ. (3, 7) for four different values of τ_0, and for k, p_0, h and ε equal to 0.65 · 10^{-2} °C/m, 1,000 mb, 10 km and 0.50 respectively

	$\tau_0 = 255°$	$\tau_0 = 270°$	$\tau_0 = 285°$	$\tau_0 = 300°$
$B_1 \cdot 10^{-4}$	0.6457	0.6862	0.7270	0.7679
$B_2 \cdot 10^{-4}$	0.1759	0.1677	0.1596	0.1520
$B_3 \cdot 10^{-8}$	0.0472	0.0434	0.0395	0.0359
$B_4 \cdot 10^{-4}$	0.5647	0.6017	0.6392	0.6769
$B_5 \cdot 10^{-4}$	0.1439	0.1384	0.1330	0.1277
$B_6 \cdot 10^{-8}$	0.0380	0.0354	0.0327	0.0301
$\frac{\tau_0}{g\varrho_0} \cdot B_5 \cdot 10^{-8}$	0.2736	0.2952	0.3160	0.3362
$B_7 \cdot 10^{-8}$	0.1176	0.1139	0.1103	0.1074
$B_8 \cdot 10^{-12}$	0.0439	0.0431	0.0424	0.0423
$\frac{2\tau_0}{g\varrho_0} \cdot B_6 \cdot 10^{-12}$	0.1444	0.1510	0.1556	0.1583
$B_9 \cdot 10^{-16}$	0.0459	0.0475	0.0495	0.0525

order of magnitude it turns out that those with B_1, B_2, B_4 and B_5 are in general the most important. These coefficients show a variation of around 1 % for a 5° C change in τ_0, but for B_1 and B_4 the variation is opposite to that of B_2 and B_5.

Next we shall consider variations merely due to the lapse-rate k. Table 3 gives the values of the B:s for $k=0.55$, 0.65 and $0.75 \cdot 10^{-2}$ °C/meter. The other parameters τ_0, p_0, h and ε have been asigned the values: 285° absolute, 1,000 mb, 10 km and 0.50 respectively.

Within this range of k none of the coefficients shows a variation exceeding 5 %. Some of the coefficients, however, increase whereas other decrease for the same change in k.

The last column in table 3 gives the values of the coefficients in the case of isothermalcy in the z-direction. This very special case implies no tropopause and consequently no discontinuity in the derivatives of the temperature. The values given correspond to $\tau = 220$ degrees absolute. Analytical expressions for the B:s in this special case are given in the appendix.

Finally the sensitivity with regard to variation in h and ε will be discussed. It appears when one compares reasonable ranges for the parameters that the sensitivity of the coefficients is somewhat greater for changes in h and ε than for changes in the parameters discussed so far. As earlier B_1 and B_4 change in a way different from the rest of the coefficients.

Values of all the coefficients for $h=8$, 10, 12, 14 and 16 km and ε = 1.00, 0.75, 0.50, 0.25, 0.0, − 0.25, − 0.50, − 0.75 and − 1.00 are given in the appendix. The remaining parameters have been chosen as follows: $\tau_0=285°$ absolute, $p_0=1{,}000$ mb, $k=0.65 \cdot 10^{-2}$ °C/meter. B_1 and B_4 have been computed for three values of k.

Above there has been given a brief discussion of the coefficients of the integrated vorticity equation in relation to the parameters of the model. For practical reasons, and also to be consistent with the basic equations describing the model, one will when applying equ. (3, 7) have to work with certain mean values for the parameters p_0, τ_0, k, h and ε. Obviously this implies a certain inaccuracy in the results, since the coefficients vary with the parameters as shown above. Although this investigation regarding the sensitivity of the coefficients with respect to the parameters is somewhat incomplete, it is quite clear that a proper choice of the parameters h and ε is important. This applies particularly to the latter, the variability of which, in space and time, imposes a considerable restriction upon the applicability of equ. (3, 7). Applied to a

Table 3. Values of the coefficients of equ. (3, 7) for $k = 0.55, 0.65, 0.75 \cdot 10^{-2}$ °C/meter and for τ_0, p_0, h and ε equal to 285°, 1,000 mb, 10 km and 0.50 respectively. The last column gives the coefficients in the case of isothermalcy in the vertical ($\tau = 220°$ abs.)

k	$0.55 \cdot 10^{-2}$	$0.65 \cdot 10^{-2}$	$0.75 \cdot 10^{-2}$	0.00
$B_1 \cdot 10^{-4}$	0.7429	0.7270	0.7112	0.6436
$B_2 \cdot 10^{-4}$	0.1598	0.1596	0.1596	0.2926
$B_3 \cdot 10^{-8}$	0.0394	0.0395	0.0399	0.1883
$B_4 \cdot 10^{-4}$	0.6657	0.6392	0.6138	0.6436
$B_5 \cdot 10^{-4}$	0.1371	0.1330	0.1290	0.2926
$B_6 \cdot 10^{-8}$	0.0337	0.0327	0.0320	0.1883
$\frac{\tau_0}{g\varrho_0} \cdot B_5 \cdot 10^{-8}$	0.3258	0.3160	0.3065	0.4143
$B_7 \cdot 10^{-8}$	0.1110	0.1103	0.1101	0.3766
$B_8 \cdot 10^{-12}$	0.0429	0.0424	0.0423	0.1818
$\frac{2\tau_0}{g\varrho_0} \cdot B_6 \cdot 10^{-12}$	0.1599	0.1556	0.1519	0.5332
$B_9 \cdot 10^{-16}$	0.0505	0.0495	0.0491	0.1560

limited region and using carefully chosen mean values of the parameters the errors involved can certainly in most cases be kept below 20 %. In favourable cases the error may amount to a few per cent only.

4. The levels of mean wind and non-divergence

The level of mean wind. It is common when dealing with numerical forecasting to replace the actual atmosphere by a barotropic one, which has the same horizontal pressure distribution as the former when averaged with respect to height. These mean conditions will in general be found to be the same as those observed at some middle level — for inst. 500 mb — and the problem is thereby reduced to two dimensions.

It is of great practical importance to investigate to what extent the level representing the mean conditions depend upon the particular distribution of temperature and pressure, and whether or not it can be assumed to be approximately at a constant height or a constant pressure in time and space. If such a constancy is found to exist, it may be a great advantage in the theory of three-dimensional numerical forecasting to choose this level as a level of reference, rather than the earths surface or some other levels. This particular level will here be referred to as the *level of mean wind.*

Study of observed wind profiles (CHARNEY 1947) has revealed that the zonal wind at around 600 mb is in the average equal to the mean zonal wind defined by the equation:

$$\bar{U} = \frac{1}{p_0} \int_0^{p_0} U dp \qquad (4, 1)$$

where U is the zonal wind component, and the bar refers to an average with respect to a vertical coordinate. Applying this concept of a mean wind to the geostrophic wind of our model, we will now attempt to determine a tropospheric level, at which the wind is equal to a mean wind defined in accordance with equ. (4, 1).

Utilizing expressions for $\mathbf{v}$ as a function of z given earlier in this paper, the integration is easily carried out, and the following result is obtained:

$$\bar{\mathbf{v}} = \frac{g}{p_0}\left(\varrho_0 B_1 \mathbf{v}_0 + \frac{\tau_0}{g} B_2 \mathbf{v}' - \frac{2\tau_0}{g} B_3 \mathbf{v}''\right) \qquad (4, 2)$$

Table 4. Computed values for z (in km) in equ. (4, 4) for $\tau_0 = 285°$ and $k = 0.65 \cdot 10^{-2}$ °C/meter, and h and ε ranging from 8—16 km and 1.00 to —1.00 resp.

	h / ε	1.00	0.75	0.50	0.25	0.00	— 0.25	— 0.50	— 0.75	— 1.00
$z_2 = \frac{\tau_0}{p_0} \cdot B_2$	8 km	2.1	2.9	3.6	4.3	5.0	5.7	6.5	7.2	7.9
	10 »	3.5	4.0	4.6	5.1	5.6	6.2	6.7	7.2	7.8
	12 »	4.5	4.9	5.3	5.7	6.1	6.5	6.8	7.2	7.6
	14 »	5.3	5.5	5.8	6.1	6.4	6.7	6.9	7.2	7.5
	16 »	5.8	6.0	6.2	6.4	6.6	6.8	7.0	7.2	7.4
$z_3 = \sqrt{\frac{2\tau_0}{p_0} \cdot B_3}$	8 km	Not real		3.1	4.7	5.8	6.7	7.5	8.2	8.9
	10 »	1.1	3.4	4.7	5.8	6.6	7.4	8.1	8.7	9.3
	12 »	4.4	5.1	5.9	6.7	7.3	7.9	8.5	9.0	9.5
	14 »	5.6	6.2	6.8	7.4	7.9	8.3	8.8	9.2	9.6
	16 »	6.6	7.1	7.5	7.9	8.3	8.6	9.0	9.3	9.6

where the notations are the same as used earlier.

Assuming that there is at least one level within the troposphere at which the wind is equal to $\bar{\mathbf{v}}$, this level is in virtue of (2, 12) determined from:

$$\left(\frac{\tau}{\tau_0} - \frac{g\varrho_0}{p_0} B_1\right) \mathbf{v}_0 + \left(z - \frac{\tau_0}{p_0} B_2\right) \mathbf{v}' - \left(z^2 - \frac{2\tau_0}{p_0} B_3\right) \mathbf{v}'' = 0 \qquad (4, 3)$$

where z is the height of this level and τ the corresponding temperature. If all the coefficients of (4, 3) are zero for the same z, this is independent of the particular distribution of pressure and temperature; if not, the level of mean wind will be different from one case to another depending upon the special features of the fields of pressure and temperature.

Putting the coefficients in (4, 3) equal to nought, the following three independent equations for determining z are obtained:

$$\frac{\tau}{\tau_0} = 1 - \frac{k}{\tau_0} z = \frac{g}{R\tau_0} B_1; \quad z = \frac{\tau_0}{p_0} B_2; \quad z^2 = \frac{2\tau_0}{p_0} B_3 \qquad (4, 4)$$

It appears from the expressions for B_1, B_2 and B_3 given in the appendix that z is independent of p_0 and from earlier discussion on the B:s that for practical purposes variation in τ_0 and k may be neglected. The first equation (4, 4) gives $z = 5.0$ km for $h = 8$ km and increasing almost linearly to 6.6 km for $h = 16$ km (B_1 is indep. of the parameter ε).

The values of z in km obtained from the second and third equation of (4, 4) are given in table 4 as z_2 and z_3. It is interesting to note that these two are for most values of h and ε rather close to each other, the latter being (in general by $^1/_2$—1 km) the highest.

Although the three equ. (4, 4) do not give identically the same values for z, we still may conclude, that the particular horizontal distribution of temperature and pressure is rather unimportant for the determination of this level. Obviously the level of mean wind is highest when the horizontal temperature gradient is caused by non-uniformity in lapse-rate only. In the case of reasonably strong and uniform temperature gradient this level is very close to the level given by z_2, which is the lowest one of those three determined from (4, 4).

The results given here are in very close agreement with the value 4.0—4.5 km for mean zonal wind at middle latitudes given by CHARNEY (1947).

Due to the increase in the height of the tropopause from the pole to the equator, one would expect that the level of mean wind increases in height towards low latitudes. The parameter ε, however, presumably decreases towards higher latitudes and this effects the height in an opposite direction. Assuming h and ε to be 14 km and 0.75 respectively at 30° latitude and $h = 9$ km and $\varepsilon = 0.25$ at 70° latitude the level of the mean wind will be around only 1 km higher at 30° than at 70° lat. This corresponds approximately to the slope of the 500 mb surface.

The mean relative vorticity ζ is obviously given by an equation analogous to (4, 2), namely:

$$\bar{\zeta} = \frac{g}{p_0}\left(\varrho_0 B_1 \zeta_0 + \frac{\tau_0}{g} B_2 \zeta' - \frac{2\tau_0}{g} B_3 \zeta''\right) \quad (4, 5)$$

Taking the level of mean wind as our level of reference, the wind velocity $\mathbf{v}$ and the relative vorticity ζ may be expressed as follows:

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}^{\star} \text{ and } \zeta = \bar{\zeta} + \zeta^{\star} \quad (4, 6)$$

where $\mathbf{v}^{\star}$ and $\zeta^{\star}$ are the deviations from the respective means. If this is introduced into the vorticity equation (3, 2) it is possible to compute the vorticity advection $\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}}$. Obviously,

$$\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}} = \overline{\mathbf{v} \cdot \nabla \zeta} - \bar{\mathbf{v}} \cdot \nabla \bar{\zeta} \quad (4, 7)$$

The first term on the right hand side of this equation is known from (3, 3) and inserting for $\bar{\mathbf{v}}$ and $\bar{\zeta}$ from (4, 2) and (4, 5) resp. the following expression for $\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}}$ is arrived at:

$$\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}} = \mathbf{v}_0 \cdot (C_4 \nabla \zeta_0 + C_5 \nabla \zeta' - C_6 \nabla \zeta'') + \\ + \mathbf{v}' \cdot (C_5 \nabla \zeta_0 + C_7 \nabla \zeta_1 - C_8 \nabla \zeta'') - \mathbf{v}'' \cdot \\ \cdot (C_6 \nabla \zeta_0 + C_8 \nabla \zeta' - C_9 \nabla \zeta'') \quad (4, 8)$$

Expressions for $\bar{\mathbf{v}} \cdot \nabla \bar{\zeta}$ and the coefficients C_4, ... C_9 in terms of B_1, ... B_9 are given in the appendix. Having computed all the C:s comparison is made between $\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}}$ and $\bar{\mathbf{v}} \cdot \nabla \bar{\zeta}$ written in a form analogous to (4. 8). We find then that C_4 is practically zero and that C_5 and C_6 amount to approximately 5 % and 10 % resp. of the corresponding coefficients in the expression for $\bar{\mathbf{v}} \cdot \nabla \bar{\zeta}$. The corresponding figures for C_7, C_8 and C_9 are 50 %, 100 % and 100—500 % resp.

In the case of uniform lapse-rate ($\nabla k = 0$) the vorticity advection would seem to be determined mainly by the advection of mean vorticity by the mean wind, but the more complicated the temperature distribution becomes, the more important becomes the "eddy" term $\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}}$. When judging, however, the importance of this compared with $\bar{\mathbf{v}} \cdot \nabla \bar{\zeta}$, the relative magnitude of the various terms in (4, 8) or the corresponding expression for $\bar{\mathbf{v}} \cdot \nabla \bar{\zeta}$, has to be taken into consideration. Rather often is it so, that the terms containing a derivative of the temperature lapse-rate are considerably smaller than the others.

We will now neglect the terms with C_4, C_5 and C_6 in the expression for $\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}}$, which is then reduced to:

$$\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}} = C_7 \mathbf{v}' \cdot \nabla \zeta' - C_8 \mathbf{v}' \cdot \nabla \zeta'' - C_8 \mathbf{v}'' \cdot \\ \cdot \nabla \zeta' + C_9 \mathbf{v}'' \cdot \nabla \zeta'' \quad (4, 9)$$

and try to determine a coefficient γ such that

$$\overline{\mathbf{v}^{\star} \cdot \nabla \zeta^{\star}} = \gamma \, \mathbf{v}_T(\bar{z}) \cdot \nabla \zeta_T(\bar{z})$$

where $\bar{z}$ is the level of mean wind and $\mathbf{v}_T$ and ζ_T the thermal wind and the thermal relative vorticity respectively for the layer $z = 0$ to $z = \bar{z}$. But,

$$\mathbf{v}_T(\bar{z}) \cdot \nabla \zeta_T(\bar{z}) = \bar{z}^2 \mathbf{v}' \cdot \nabla \zeta' - \bar{z}^3 \mathbf{v}' \cdot \nabla \zeta'' - \\ \bar{z}^3 \mathbf{v}'' \cdot \nabla \zeta' + \bar{z}^4 \mathbf{v}'' \cdot \nabla \zeta''$$

Comparison with (4, 9) shows that γ will somewhat depend upon the particular distribution of temperature and pressure. In the case of uniform lapse-rate, i.e. $\mathbf{v}'' = \zeta'' = 0$ γ is determined from: $\gamma = \frac{C_7}{\bar{z}^2}$

One finds that γ is nearly constant within a reasonable range of the parameters h and ε of the model and can in the average be put equal to 0.5.

In the very special case of a horizontal temperature distribution which is exclusively due to non-uniformity in the temperature lapse-rate, γ is determined by the following equation:

$$\gamma = \frac{C_9}{\bar{z}^4}$$

For the most common values of h and ε, γ is within the range 1—4. As an overall value for γ in the general case we will suggest the number 0.6.

The integrated vorticity equation may then be written

$$\frac{\partial \bar{\zeta}}{\partial t} = -\bar{\mathbf{v}} \cdot \nabla \bar{\eta} - 0.6 \cdot \mathbf{v}_T \cdot \nabla \zeta_T \quad (4, 10)$$

The second term on the right hand side of (4, 10) is sometimes called the "development term". By this is meant that this term admits non-conservative changes of the vorticity in the mean motion. In practice one may possibly identify the level of mean wind with the 500 mb surface. From (4, 10) we then learn that the tendency at this level can be computed by means of two maps, namely the 500 mb map and the map for the relative topography 1,000—500 mb.

The level of non-divergence. Besides the level of mean wind, the so-called level of non-divergence plays an important role in numerical forecasting. By definition, the barotropic vorticity equation

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla \eta = 0$$

shall apply at the latter level, and consequently the left hand side of equ. (3, 1) has to be nought there.

Because of the great simplification the knowledge of this level means to the theory and practical procedures for numerical forecasting, an attempt will be made to gather some information regarding this level by consideration similar to that given above for the level of mean wind.

Substituting for the wind velocity and relative vorticity from (2, 12) and (2, 16) p. ... into $\frac{\partial \zeta}{\partial t} + \beta v + \mathbf{v} \cdot \nabla \zeta = 0$ and comparing with the integrated vorticity equation (3, 3), the following equation for determining the height z of the level of non-divergence is arrived at:

$$\left(A_1 - \frac{p_0}{g}\frac{\tau}{\tau_0}\right)\frac{\partial \zeta_0}{\partial t} + \left(A_2 - \frac{p_0}{g} z\right)\frac{\partial \zeta'}{\partial t} -$$
$$\left(A_3 - \frac{p_0}{g} z^2\right)\frac{\partial \zeta''}{\partial t} + \frac{\beta}{\varrho_0 f}\left(A_1 - \frac{p_0}{g}\frac{\tau}{\tau_0}\right)\frac{\partial p_0}{\partial x} +$$
$$+ \frac{\beta}{f}\cdot\frac{g}{\tau_0}\left(A_2 - \frac{p_0}{g} z\right)\frac{\partial \tau_0}{\partial x} - \frac{\beta}{f}\cdot\frac{g}{2\tau_0}\left(A_3 - \right.$$
$$\left. - \frac{p_0}{g} z^2\right)\frac{\partial k}{\partial x} + \left(A_4 - \frac{p_0}{g}\left(\frac{\tau}{\tau_0}\right)^2\right)\mathbf{v}_0 \cdot \nabla \zeta_0 +$$
$$+ \left(A_5 - \frac{p_0}{g}\frac{\tau}{\tau_0} z\right)\mathbf{v}_0 \cdot \nabla \zeta' - \left(A_6 - \frac{p_0}{g}\frac{\tau}{\tau_0} z^2 \cdot\right)$$
$$\cdot \mathbf{v}_0 \cdot \nabla \zeta'' + \left(A_5 - \frac{p_0}{g}\frac{\tau}{\tau_0} z\right)\mathbf{v}' \cdot \nabla \zeta_0 +$$
$$+ \left(A_7 - \frac{p_0}{g} z^2\right)\mathbf{v}' \cdot \nabla \zeta' - \left(A_8 - \frac{p_0}{g} z^3\right)\mathbf{v}' \cdot$$
$$\cdot \nabla \zeta'' - \left(A_6 - \frac{p_0}{g}\frac{\tau}{\tau_0} z^2\right)\mathbf{v}'' \cdot \nabla \zeta_0 -$$
$$- \left(A_8 - \frac{p_0}{g} z^3\right)\mathbf{v}'' \cdot \nabla \zeta' + \left(A_9 - \frac{p_0}{g} z^4\right) \cdot$$
$$\cdot \mathbf{v}'' \cdot \nabla \zeta'' = 0 \qquad (4, 11)$$

By putting all the coefficients in (4, 11) equal to nought the following 9 independent equations for determinating z are obtained:

$$\frac{\tau}{\tau_0} = 1 - \frac{k}{\tau_0} z = \frac{g}{R\tau_0} B_1; \quad z = \frac{\tau_0}{p_0} B_2$$
$$z^2 = \frac{2\tau_0}{p_0} B_3; \quad \left(\frac{\tau}{\tau_0}\right)^2 = \frac{g}{R\tau_0} B_4$$
$$\frac{\tau}{\tau_0} z = \frac{\tau_0}{p_0} B_5; \quad \frac{\tau}{\tau_0} z^2 = \frac{2\tau_0}{p_0} B_6; \quad z^2 = \frac{\tau_0}{p_0} B_7$$
$$z^3 = \frac{2\tau_0}{p_0} B_8; \quad z^4 = \frac{2\tau_0}{p_0} B_9 \qquad (4, 12)$$

where the A:*s* have been replaced by B:*s* from (3, 8). The three first equations are identical with (4,4) and the corresponding values of z were given in table 4 above.

The values for the remaining z:s are not given here, but it appears from them that the level of non-divergence is somewhat higher than the level of mean wind. How much, depends upon the particular horizontal distribution of temperature and pressure. When non-uniformity in the lapse-rate is the predominant feature of the field of temperature, the height difference between these two levels is slightly larger than in the case of uniform lapse-rate.

In fig. 6 we have given the height of the level of mean wind and the above mentioned height difference for the latter case. It appears from the figure that in middle latitudes, the level of non-divergence is from 0.7 to 3.0 km higher than the level of mean wind, and that the most frequent values for this height difference are below 1.5 km. Since ε is usually positive in middle latitudes and the most frequent values of k lie between 9 and 11 km, it is obvious from figure 6 that the level of mean wind is generally somewhat below the 500 mb surface and the

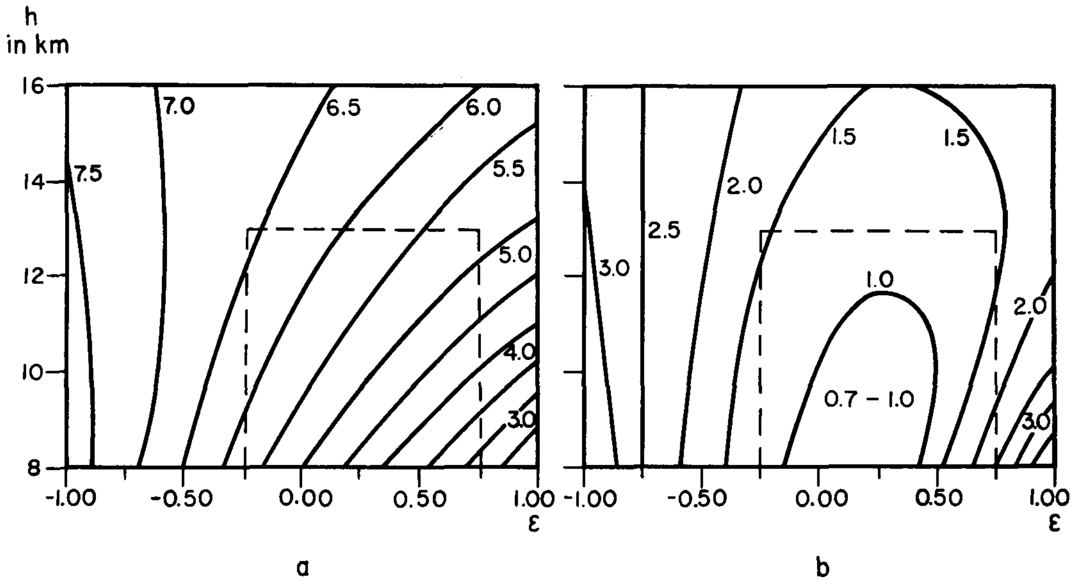

Fig. 6. The curved lines in a) are isopleths for the level of mean wind for the case of uniform lapse-rate. In b) the curves show how much the level of non-divergence is higher than the level of mean wind, also for the case of uniform lapse-rate. The numbers refer to km as a unit. The dashed rectangle gives the range of h and ε which is thought to cover middle latitudes.

level of non-divergence slightly above the same. *This is of considerable practical interest, since the 500 mb level may be used as an approximation for both of them.*

The results given here are in fair agreement with the statement (CHARNEY, FJÖRTOFT, v. NEUMANN, 1950) that the level of non-divergence is in the average between 400—500 mb.

So far considerations have only been given to the dynamical aspects connected with the model. In a following article the thermodynamics will be consulted, and a complete set of equations for the purpose of numerical forecasting of the fields of temperature and pressure will be derived.

APPENDIX

A. Expressions for the coefficients B_1, B_2 . . . . B_9 of the equation (3, 7) p . . . in terms of the parameters of the model:

$$B_1 = \frac{R\tau_0}{g} \cdot \frac{n}{n+1} \left[1 + \frac{1}{n} \left(\frac{\tau_h}{\tau_0} \right)^{n+1} \right]$$

$$B_2 = \frac{p_0}{\tau_0} \left[\int_0^h \left(\frac{\tau}{\tau_0} \right)^n dz - \varepsilon \frac{R\tau_0}{g} \left(\frac{\tau_h}{\tau_0} \right)^n \right]$$

$$B_3 = \frac{p_0}{\tau_0} \left[\int_0^h \left(\frac{\tau}{\tau_0} \right)^n z dz - \varepsilon h \cdot \frac{R\tau_0}{g} \cdot \left(\frac{\tau_h}{\tau_0} \right)^n \right]$$

$$B_4 = \frac{R\tau_0}{g} \cdot \frac{n}{n+2} \left[1 + \frac{2}{n} \left(\frac{\tau_h}{\tau_0} \right)^{n+2} \right]$$

$$B_5 = \frac{p_0}{\tau_0} \left[\frac{n}{n+1} \int_0^h \left(\frac{\tau}{\tau_0} \right)^{n+1} dz + \left(\frac{h}{n+1} - \varepsilon \frac{R\tau_0}{g} \right) \cdot \left(\frac{\tau_h}{\tau_0} \right)^{n+1} \right]; \quad B_6 = \frac{p_0}{\tau_0} \left[\frac{n}{n+1} \int_0^h \left(\frac{\tau}{\tau_0} \right)^{n+1} \cdot z dz + h \left(\frac{h}{2(n+1)} - \varepsilon \frac{R\tau_0}{g} \right) \left(\frac{\tau_h}{\tau_0} \right)^{n+1} \right]$$

$$B_7 = \frac{2p_0}{\tau_0} \left[\int_0^h \left(\frac{\tau}{\tau_0} \right)^n z dz - \varepsilon \frac{R\tau_0}{g} \left(h - \varepsilon \frac{R\tau_0}{g} \right) \cdot \left(\frac{\tau_h}{\tau_0} \right)^n \right]; \quad B_8 = \frac{3}{2} \frac{p_0}{\tau_0} \left[\int_0^h \left(\frac{\tau}{\tau_0} \right)^n z^2 dz - \varepsilon \frac{R\tau_0}{g} \cdot h \left(h - \frac{4}{3} \varepsilon \frac{R\tau_0}{g} \right) \left(\frac{\tau_h}{\tau_0} \right)^n \right]$$

$$B_9 = \frac{2p_0}{\tau_0}\left[\int_0^h \left(\frac{\tau}{\tau_0}\right)^n z^3 dz - \varepsilon\frac{R\tau_0}{g}h^2 \cdot \right.$$
$$\left. \cdot \left(h - 2\varepsilon\frac{R\tau_0}{g}\right)\left(\frac{\tau_h}{\tau_0}\right)^n\right] \quad (1)$$

The notations are the same as earlier. The integrals involved in the expressions for the coefficients are easily evaluated by use of the recurrence formula:

$$I_{n,q} = \int_0^h \left(\frac{\tau}{\tau_0}\right)^n z^q dz = \frac{\tau_0}{k(n+1)}\left[qI_{n+1,\,q-1} - \right.$$
$$\left. - h^q\left(\frac{\tau_h}{\tau_0}\right)^{n+1}\right] \quad (2)$$

together with

$$I_{n,0} = \int_0^h \left(\frac{\tau}{\tau_0}\right)^n dz = \frac{\tau_0}{k(n+1)}\left[1 - \left(\frac{\tau_h}{\tau_0}\right)^{n+1}\right] \quad (3)$$

B. Expressions for the coefficients B_1, B_2, ... B_9 in the case of isothermalcy in the z-direction.

$$B_1 = B_4 = \frac{R\tau_0}{g}, \quad B_2 = B_5 = \frac{Rp_0}{g}$$
$$B_3 = B_6 = B_2 \cdot B_1, \quad B_7 = 2B_6 = 2B_2 \cdot B_1 \quad (4)$$
$$B_8 = \frac{3}{2}B_3 \cdot B_1 = \frac{3}{2}B_2 \cdot B_1^2, \quad B_9 = 2B_2 \cdot B_1^3$$

C. Numerical values for the coefficients B_1, B_2, ... B_9.

The tables 5 and 6 below give the values of B_1 and B_4 for $p_0 = 1{,}000$ mb, $\tau_0 = 285°$ absolute, the three k-values $0.55 \cdot 10^{-2}$, $0.65 \cdot 10^{-2}$ and $0.75 \cdot 10^{-2}$ °C/meter and $h = 8$, 10, 12, 14 and 16 km. The remaining coefficients are given in table 7 for the same p_0, τ_0 and h values, for $k = 0.65 \cdot 10^{-2}$ °C/meter only and ε ranging from -1.00 to $+1.00$.

Table 5. Computed values for $B_1 \cdot 10^{-4}$

h \ k	$0.55 \cdot 10^{-2}$	$0.65 \cdot 10^{-2}$	$0.75 \cdot 10^{-2}$
8 km ...	0.7527	0.7383	0.7241
10 » ...	5.7429	0.7270	0.7112
12 » ...	0.7355	0.7186	0.7019
14 » ...	0.7302	0.7126	0.6954
16 » ...	0.7263	0.7083	0.6909

Table 6. Computed values for $B_4 \cdot 10^{-4}$

h \ k	$0.55 \cdot 10^{-2}$	$0.65 \cdot 10^{-2}$	$0.75 \cdot 10^{-2}$
8 km ...	0.6820	0.6573	0.6335
10 » ...	0.6657	0.6392	0.6138
12 » ...	0.6541	0.6266	0.6006
14 » ...	0.6461	0.6181	0.5919
16 » ...	0.6406	0.6126	0.5866

D. Expressions for the term $\bar{\mathbf{v}} \cdot \nabla\bar{\zeta}$ and the coefficients C_4 ...... C_9 in equ. (4, 8)

$$\bar{\mathbf{v}} \cdot \nabla\bar{\zeta} = \left(\frac{g}{p_0}\right)^2\left[\varrho_0 B_1 \mathbf{v}_0 + \frac{\tau_0}{g}B_2\mathbf{v}' - \frac{2\tau_0}{g}B_3\mathbf{v}''\right] \cdot$$
$$\cdot \nabla\left[\varrho_0 B_1\zeta_0 + \frac{\tau_0}{g}B_2\zeta' - \frac{2\tau_0}{g}B_2\zeta''\right] =$$
$$= \mathbf{v}_0 \cdot \left[\left(\frac{g}{R\tau_0}\right)^2 B_1^2 \nabla\zeta_0 + \left(\frac{g}{R\tau_0}B_1\right) \cdot \right.$$
$$\left. \cdot \left(\frac{\tau_0}{p_0}B_2\right)\nabla\zeta' - \left(\frac{g}{R\tau_0}B_1\right) \cdot \left(\frac{2\tau_0}{p_0}B_3\right)\nabla\zeta''\right]$$
$$+ \mathbf{v}' \cdot \left[\left(\frac{g}{R\tau_0}B_1\right)\left(\frac{\tau_0}{p_0}B_2\right)\nabla\zeta_0 + \left(\frac{\tau_0}{p_0}B_2\right)^2\nabla\zeta' - \right.$$
$$\left. - \left(\frac{\tau_0}{p_0}B_2\right)\left(\frac{2\tau_0}{p_0}B_3\right)\nabla\zeta''\right]$$
$$- \mathbf{v}'' \cdot \left[\left(\frac{g}{R\tau_0}B_1\right)\left(\frac{2\tau_0}{p_0}B_3\right)\nabla\zeta_0 + \left(\frac{\tau_0}{p_0}B_2\right) \cdot \right.$$
$$\left. \cdot \left(\frac{2\tau_0}{p_0}B_3\right)\nabla\zeta' - \left(\frac{2\tau_0}{p_0}B_3\right)^2\nabla\zeta''\right] \quad (5)$$

Comparing with (3, 3) and utilizing (3, 8) the coefficients C_4 .... C_9 in (4, 8) are found to be:

$$C_4 = \frac{g}{R\tau_0}B_4 - \left(\frac{g}{R\tau_0}B_1\right)^2$$
$$C_5 = \frac{\tau_0}{p_0}B_5 - \frac{g}{R\tau_0}B_1 \cdot \frac{\tau_0}{p_0}B_2$$
$$C_6 = \frac{2\tau_0}{p_0}B_6 - \frac{g}{R\tau_0}B_1 \cdot \frac{2\tau_0}{p_0}B_3$$
$$C_7 = \frac{\tau_0}{p_0}B_7 - \left(\frac{\tau_0}{p_0}B_2\right)^2$$
$$C_8 = \frac{2\tau_0}{p_0}B_8 - \frac{\tau_0}{p_0}B_2 \cdot \frac{2\tau_0}{p_0}B_3$$
$$C_9 = \frac{2\tau_0}{p_0}B_9 - \left(\frac{2\tau_0}{p_0}B_3\right)^2 \quad (6)$$

Table 7. Computed values for B_2, B_3, B_5, ... B_9

	h \ ε	1.00	0.75	0.50	0.25	0.00	— 0.25	— 0.50	— 0.75	— 1.00
$B_2 \cdot 10^{-4}$	8 km	0.0746	0.1000	0.1254	0.1507	0.1761	0.2015	0.2269	0.2522	0.2776
	10 »	0.1220	0.1408	0.1596	0.1783	0.1971	0.2159	0.2346	0.2534	0.2722
	12 »	0.1581	0.1717	0.1853	0.1990	0.2126	0.2262	0.2398	0.2534	0.2670
	14 »	0.1850	0.1946	0.2043	0.2140	0.2237	0.2333	0.2430	0.2527	0.2624
	16 »	0.2046	0.2113	0.2180	0.2247	0.2315	0.2382	0.2449	0.2516	0.2583
$B_3 \cdot 10^{-8}$	8 km	- 0.0229	- 0.0026	0.0177	0.0380	0.0583	0.0786	0.0989	0.1192	0.1395
	10 »	0.0020	0.0207	0.0395	0.0583	0.0771	0.0958	0.1146	0.1334	0.1522
	12 »	0.0287	0.0450	0.0613	0.0777	0.0940	0.1104	0.1267	0.1430	0.1593
	14 »	0.0542	0.0678	0.0813	0.0948	0.1084	0.1219	0.1355	0.1490	0.1626
	16 »	0.0770	0.0877	0.0985	0.1092	0.1200	0.1308	0.1415	0.1523	0.1630
$B_5 \cdot 10^{-4}$	8 km	0.0666	0.0873	0.1080	0.1288	0.1495	0.1703	0.1910	0.2117	0.2325
	10 »	0.1040	0.1185	0.1330	0.1475	0.1620	0.1764	0.1909	0.2054	0.2199
	12 »	0.1301	0.1400	0.1499	0.1598	0.1697	0.1796	0.1895	0.1994	0.2092
	14 »	0.1479	0.1545	0.1611	0.1677	0.1743	0.1808	0.1874	0.1940	0.2006
	16 »	0.1596	0.1639	0.1682	0.1725	0.1767	0.1810	0.1853	0.1896	0.1938
$B_6 \cdot 10^{-8}$	8 km	- 0.0177	- 0.0011	0.0155	0.0321	0.0487	0.0653	0.0819	0.0985	0.1151
	10 »	0.0038	0.0183	0.0327	0.0472	0.0617	0.0762	0.0907	0.1052	0.1197
	12 »	0.0248	0.0367	0.0485	0.0604	0.0723	0.0841	0.0960	0.1079	0.1197
	14 »	0.0434	0.0526	0.0618	0.0711	0.0803	0.0895	0.0987	0.1080	0.1172
	16 »	0.0586	0.0654	0.0723	0.0791	0.0859	0.0927	0.0996	0.1064	0.1133
$\frac{\tau_0}{g\varrho_0} B_5 \cdot 10^{-8}$	8 km	0.1582	0.2075	0.2567	0.3060	0.3553	0.4046	0.4538	0.5031	0.5524
	10 »	0.2471	0.2816	0.3160	0.3504	0.3849	0.4192	0.4537	0.4881	0.5225
	12 »	0.3093	0.3328	0.3563	0.3797	0.4032	0.4267	0.4503	0.4738	0.4972
	14 »	0.3514	0.3670	0.3827	0.3984	0.4141	0.4297	0.4453	0.4610	0.4767
	16 »	0.3793	0.3895	0.3996	0.4098	0.4199	0.4301	0.4402	0.4504	0.4605
$B_7 \cdot 10^{-8}$	8 km	0.1235	0.0900	0.0778	0.0866	0.1166	0.1678	0.2401	0.3335	0.4482
	10 »	0.1291	0.1119	0.1103	0.1244	0.1541	0.1995	0.2605	0.3371	0.4294
	12 »	0.1481	0.1411	0.1453	0.1610	0.1880	0.2263	0.2761	0.3371	0.4095
	14 »	0.1729	0.1718	0.1787	0.1937	0.2168	0.2479	0.2871	0.3343	0.3897
	16 »	0.1988	0.2007	0.2082	0.2213	0.2400	0.2643	0.2943	0.3298	0.3709
$B_8 \cdot 10^{-12}$	8 km	0.0801	0.0452	0.0273	0.0263	0.0422	0.0749	0.1247	0.1913	0.2749
	10 »	0.0800	0.0534	0.0424	0.0471	0.0674	0.1034	0.1550	0.2223	0.3051
	12 »	0.0867	0.0684	0.0637	0.0727	0.0953	0.1315	0.1814	0.2448	0.3219
	14 »	0.0998	0.0888	0.0890	0.1005	0.1233	0.1574	0.2028	0.2595	0.3275
	16 »	0.1178	0.1122	0.1156	0.1280	0.1493	0.1796	0.2189	0.2670	0,3243
$\frac{2\tau_0}{g\varrho_0} B_6 \cdot 10^{-12}$	8 km	- 0.0832	- 0.0050	0.0737	0.1526	0.2315	0.3102	0.3891	0.4679	0.5468
	10 »	0.0180	0.0869	0.1556	0.2245	0.2933	0.3622	0.4311	0.4999	0.5688
	12 »	0.1179	0.1743	0.2306	0.2870	0.3434	0.3999	0.4563	0.5126	0.5690
	14 »	0.2061	0.2500	0.2938	0.3377	0.3815	0.4254	0.4693	0.5131	0.5570
	16 »	0.2783	0.3108	0.3434	0.3759	0.4082	0.4407	0.4733	0.5058	0.5383
$B_9 \cdot 10^{-16}$	8 km	0.1445	0.0757	0.0340	0.0194	0.0318	0.0713	0.1379	0.2315	0.3523
	10 »	0.1623	0.0903	0.0495	0.0401	0.0620	0.1152	0.1997	0.3155	0.4526
	12 »	0.1764	0.1090	0.0743	0.0723	0.1030	0.1664	0.2625	0.3912	0.5527
	14 »	0.1922	0.1345	0.1086	0.1143	0.1515	0.2204	0.3210	0.4532	0.6170
	16 »	0.2126	0.1671	0.1506	0.1626	0.2033	0.2727	0.3708	0.4975	0.6532

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