Numerical Solution of Elliptic Difference Equations by Matrix Methods

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Abstract

A non-iterative method for solving boundary value problems of elliptic difference equations is developed that is based on the simple structure of the triangular matrices obtained by applying the well-known elimination procedure of Gauss to the matrix of the difference equation. It is proved that the procedure is numerically stable, i. e., there is no tendency to error-growth. The method seems to be particularly suited to such problems as weather forecasting on high-speed computing machines where one has to solve the same equations (e. g., Poisson's or Helmholtz's) a great number of times with the same net but different right member and boundary values. The number of stored constants as well as the computational work in the application of the method is only a fractional part of that required when using the inverse matrix (Green's function). Poisson's equation for a rectangular region is given a close study, but the method can be applied to irregular regions and to equations of higher order.

1. Introduction

For numerical solution of boundary value problems it is usual to replace the differential equation by a difference equation of the same or higher order. If the differential equation is linear we obtain a system of linear equations to solve that generally is of very high order. The methods for solving this system can be divided into iterative and direct methods (BODEWIG, 1947). To the former are counted the Liebmann process (STIEFEL, 1952), relaxation (SOUTHWELL, 1946), and steepest descent (STIEFEL, 1952). The direct methods consist either in forming the inverse matrix or in splitting the original matrix into triangular matrices according to Gauss or Choleski; the last-mentioned method can only be applied to symmetric matrices. Variants of the methods of Gauss and Choleski are given in the literature but common to for them all is some kind of elimination procedure.

The matrices appearing in connection with ordinary and partial differential equations have a simple structure. A splitting of the original matrix can be done once for all for a given region. Below it will be shown how this splitting process can be done and that the amount of work for calculating the unknowns from the triangular matrices often is considerably less than for calculating the unknowns from the inverse matrix. Both the decomposition procedure and the solving of the unknowns are extremely well suited for electronic computers, even those having a fixed binary point. From the tabulation point of view it is more economical to tabulate triangular matrices than inverse matrices.

2. Arrangement of the matrix of the Poisson's equation

Because Poisson's equation

$$\nabla^2 u = J(x, y), \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1)$$

has both general and meteorological interest this equation will be treated in detail. This equation occurs in the barotropic case where J essentially is a Jacobian of the absolute vorticity and the pressure height (CHARNEY, FJØRTOFT, VON NEUMANN, 1950).

Now we consider the corresponding difference equation of second order in two dimensions and with the u-values given on the boundary. The difference equation for quadratic nets with the mesh length h is

$$\triangle_x{}^2 u + \triangle_y{}^2 u = h^2 \cdot J(x, y) \qquad (2)$$

where $\triangle^2$ means central differences of the second order. We want a solution of (2) satisfying the boundary conditions. This is called the first boundary value problem and has been treated for difference equations from a purely mathematical standpoint by COURANT, FRIEDRICHS and LEWY (1928).

In a rectangular region (fig. 1) with r inner rows and s inner points per row the notation u_{jk} means u at $\varrho = j$ and $\sigma = k$. Using these subscripts we can form the following equations. (σ corresponds to the x-direction and ϱ to the y-direction).

$\varrho = 1$

$$\begin{aligned}
4u_{11} - u_{12} - u_{21} &= u_{01} + u_{10} - h^2 \cdot J_{11} \\
-u_{11} + 4u_{12} - u_{13} - u_{22} &= u_{02} - h^2 \cdot J_{12} \\
- u_{12} + 4u_{13} - u_{14} - u_{23} &= u_{03} - h^2 \cdot J_{13} \\
&\ \ - - - \\
-u_{1,s-1} + 4u_{1s} - u_{2s} &= u_{0s} + u_{1,s+1} - h^2 \cdot J_{1s}
\end{aligned}$$

$\varrho = 2$

$$\begin{aligned}
-u_{11} + 4u_{21} - u_{22} - u_{31} &= u_{20} - h^2 \cdot J_{21} \\
-u_{12} - u_{21} + 4u_{22} - u_{23} - u_{32} &= - h^2 \cdot J_{22} \\
&\ \ - - - \\
-u_{1s} - u_{2,s-1} + 4u_{2s} - u_{3s} &= u_{2,s+1} - h^2 \cdot J_{2s}
\end{aligned}$$

etc.

The equations above can be written in matrix notation

$$\mathbf{G} \cdot \mathbf{u} = \mathbf{t} \qquad (3)$$

where $\mathbf{u}$ is a column matrix partitioned in r columns $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ each holding the s u-values of row 1, 2, ..., r.

$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ \vdots \\ u_{1s} \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} u_{21} \\ u_{22} \\ u_{23} \\ \vdots \\ u_{2s} \end{bmatrix} \ \ldots \ \mathbf{u}_r = \begin{bmatrix} u_{r1} \\ u_{r2} \\ u_{r3} \\ \vdots \\ u_{rs} \end{bmatrix}$$

In the same way the column matrix $\mathbf{t}$ formed by the boundary values and the function J is partitioned.

$$\mathbf{t}_1 \begin{bmatrix} -h^2J_{11} + u_{01} + u_{10} \\ -h^2J_{12} + u_{02} \\ -h^2J_{13} + u_{03} \\ \vdots \\ -h^2J_{1s} + u_{0s} + u_{1,s+1} \end{bmatrix},$$

$$\mathbf{t}_2 = \begin{bmatrix} -h^2J_{21} + u_{20} \\ -h^2J_{22} \\ -h^2J_{23} \\ \vdots \\ -h^2J_{2s} + u_{2,s+1} \end{bmatrix} \ldots$$

$$\mathbf{t}_r = \begin{bmatrix} -h^2J_{r1} + u_{r+1,1} + u_{r0} \\ -h^2J_{r2} + u_{r+1,2} \\ -h^2J_{r3} + u_{r+1,3} \\ \vdots \\ -h^2J_{rs} + u_{r+1,s} + u_{r,s+1} \end{bmatrix}$$

The matrix $\mathbf{G}$ is partitioned in r^2 submatrices each of the order s.

$$\mathbf{G} = \begin{bmatrix} \mathbf{D} & -\mathbf{I} & \mathbf{O} & \mathbf{O} \ldots & & \\ -\mathbf{I} & \mathbf{D} & -\mathbf{I} & \mathbf{O} \ldots & & \\ \mathbf{O} & -\mathbf{I} & \mathbf{D} & -\mathbf{I} \ldots & & \\ & & & --- & & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \ldots & -\mathbf{I} & \mathbf{D} \end{bmatrix} \tag{4}$$

Here $\mathbf{I}$ denotes the unit matrix of the order s and the zeros are zero matrices of the same order. The submatrix $\mathbf{D}$ is defined as

$$\mathbf{D} = \begin{bmatrix} 4 & -1 & 0 & 0 \ldots & & \\ -1 & 4 & -1 & 0 \ldots & & \\ 0 & -1 & 4 & -1 \ldots & & \\ & & & --- & & \\ 0 & 0 & 0 & 0 \ldots & -1 & 4 \end{bmatrix} \tag{5}$$

From the structure of the matrix $\mathbf{G}$ it is clear that it is symmetric and contains at most 5 non-vanishing elements per row and that these are collected in diagonals, the outer lies s diagonals from the head diagonal.

3. General theory for splitting up a matrix into triangular matrices

Because of the symmetry of the matrix $\mathbf{G}$ in (4) this matrix can be split either according to Gauss or Choleski, but as the main procedure is the same in both cases, we treat the former method only. Gauss' method consists in forming a triangular matrix by elimination from which the unknowns can be solved by recurrence formula. Banachiewicz has shown that the matrix can be split up directly into two triangular matrices, a procedure called in German literature "Gauss' abgekürzte Verfahren" (ZURMÜHL, 1950). For manual solution this procedure is preferable to the pure elimination procedure because less writing is required. For electronic computers, however, the elimination is more suitable. The inverse matrix can be computed from triangular matrices. The number of arithmetic operations is the same for the two methods.

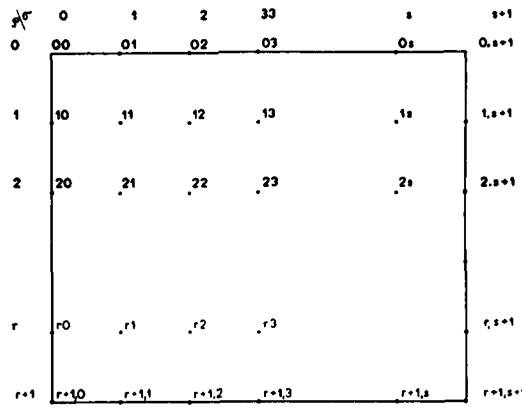

Fig. 1. Numbering in a rectangular region.

We now split up a *symmetric* matrix $\mathbf{A}$ into two triangular matrices $\mathbf{B}$ and $\mathbf{C}$. Hence

$$\mathbf{A} = \mathbf{B} \cdot \mathbf{C} \tag{6}$$

Here we have a certain freedom in the choice of the diagonal elements of the matrices $\mathbf{B}$ and $\mathbf{C}$. In this case the following arrangement seems to be convenient, for all elements except the diagonal elements in the important case of Poisson's equation will be positive.
Set

$$\mathbf{B} = \begin{bmatrix} b_{11} & 0 & 0 & 0 \ldots & 0 \\ b_{12} & b_{22} & 0 & 0 \ldots & 0 \\ b_{13} & b_{23} & b_{33} & 0 \ldots & 0 \\ --- & & & & \\ b_{1s} & b_{2s} & b_{3s} & b_{4s} \ldots & b_{ss} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} -1 & c_{12} & c_{13} & c_{14} \ldots & c_{1s} \\ 0 & -1 & c_{23} & c_{24} \ldots & c_{2s} \\ 0 & 0 & -1 & c_{34} \ldots & c_{3s} \\ --- & & & & \\ 0 & 0 & 0 & 0 \ldots & -1 \end{bmatrix}$$

We identify the two members in (6) and compute the elements b_{jk} and c_{jk} from the following relations

$$\begin{aligned} b_{1k} &= -a_{1k} & k &= 1, 2, \ldots, s \\ c_{1k} &= -b_{1k}/b_{11} & k &= 2, 3, \ldots, s \\ b_{2k} &= -a_{2k} + c_{12} \cdot b_{1k} & k &= 2, 3, \ldots, s \\ c_{2k} &= -b_{2k}/b_{22} & k &= 3, 4, \ldots, s \end{aligned} \tag{7}$$

$$---$$

$$b_{jk} = -a_{jk} + \sum_{n=1}^{j-1} c_n \cdot b_{nk} \quad k = j, j+1, \ldots, s$$

$$c_{jk} = -b_{jk}/b_{jj} \qquad k = j+1, j+2, \ldots, s$$

For checking purposes we can use a relation for the sum of the computed elements b_{jk} in the j_{th} column.

$$\sum_{n=j}^{s} b_{jn} = -\sum_{n=1}^{s} a_{nj} + \sum_{n=1}^{j-1} c_{nj} \cdot q_n$$

$$q_n = \sum_{m=n}^{s} b_{nm}$$

When the matrix **A** is symmetric as here, the matrix — **C** has originated from **B** by multiplying the transposed **B** by a diagonal matrix formed by the inverted values of **B**'s diagonal elements.

4. Splitting up the matrix G into triangular matrices

It is easy now to form triangular matrices for the matrix defined by (4). **G** is symmetric and beyond the $(s+1)$:th diagonal (the head diagonal is numbered one) **G**'s elements are zero.
We thus form

$$\mathbf{G} = \mathbf{B} \cdot \mathbf{C} \qquad g_{jk} = g_{kj}$$

The elements of the triangular matrices are computed from (7). In general

$b_{1k} \neq 0$ for $k = 1, 2, \ldots, s+1$

but

$b_{1k} = 0$ for $k > s + 1$

In the next b-column we have

$b_{2k} \neq 0$ for $k = 2, 3, \ldots, s+2$

$b_{2k} = 0$ for $k > s+2$

etc.

Generally we have

$b_{jk} \neq 0$ for $k = j, j+1, \ldots, j+s$

$b_{jk} = 0$ for $k > j + s$

From this we conclude the essential property of the triangular matrix **B** is that its diagonals $1, 2, \ldots, s+1$ (numbering as above) have non-vanishing elements but, the other diagonals contain zero elements only. For the matrix **C** we get similar results, but the non-vanishing diagonals lie to the right of the head diagonal.

The conclusions above are valid for general difference equations of any order in which the letter s does not mean the number of points per row in fig. 1. The difference equation can originate from ordinary or partial differential equations. The region need not to be rectangular, in non-rectangular regions let s denote the diagonal, usually not filled, beyond which we have zero elements only, then no non-vanishing elements will lie outside the s^{th} diagonal in the triangular matrix.

In Poisson's equation for a rectangular region the triangular matrices contain about rs^2 elements each, but the inverse matrix contains r^2s^2 elements, where r denotes the number of rows and s the number of points per row in the region. The triangular matrix has besides another delightful property, viz. we can add extra rows to the region without affecting the preceding elements in the triangular matrix. The inverse matrix on the other hand changes every element when one adds an extra row to the region. See also section 8.

5. Solution of the unknowns from the triangular matrices

In order to solve the function **u** from the equation

$$\mathbf{B} \cdot \mathbf{C} \cdot \mathbf{u} = \mathbf{t} \tag{8}$$

we first solve an auxiliary function **v** defined by

$$\mathbf{B} \cdot \mathbf{v} = \mathbf{t} \tag{9}$$

Then **u** is solved from

$$\mathbf{C} \cdot \mathbf{u} = \mathbf{v} \tag{10}$$

The function **v** is solved recurrently from (9) beginning with v_{11}.

$$\begin{aligned} v_{11} &= \frac{1}{b_{11}} \cdot t_{11} \\ v_{12} &= \frac{1}{b_{22}} (t_{12} - v_{11} \cdot b_{12}) \\ v_{13} &= \frac{1}{b_{33}} (t_{13} - v_{11} \cdot b_{13} - v_{12} \cdot b_{13}) \end{aligned} \tag{11}$$

etc.

Now we can solve **u** from (10) beginning with u_{rs}.

$$u_{rs} = - v_{rs}$$

$$u_{r,\,s-1} = - v_{r,\,s-1} + u_{rs} \cdot c_{rs-1,\,rs}$$

$$u_{r,\,s-2} = - v_{r,\,s-2} + u_{rs} \cdot c_{rs-2,\,rs} + $$

$$+ u_{r,\,s-1} \cdot c_{rs-2,\,rs-1}$$

etc.

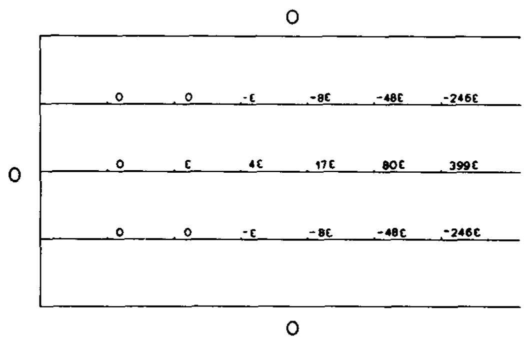

Fig. 2. The error-growth in the step-ahead process.

6. The work for solving Laplace's and Poisson's equations

As a measure for the work for solving u we take the number of multiplications and additions that must be made. We consider rectangular regions but we can get an upper bound for the work for irregular regions if s denotes the maximum number of points in a row of the region.

To form the triangular matrices we need about $rs^3/2$ multiplications and additions (cf. sec. 8), and to solve u from these matrices this number is about $2\,rs^2$. Thus we ought to turn the region so that r will be greater than s.

The difference between Laplace's equation and Poisson's equation from the numerical point of view lies in the right member of (1), and the work to solve them with triangular matrices will be about the same, i.e., $2\,rs^2$ multiplications and additions. We compare this with the number we get in computing u from the inverse matrix. For Laplace's equation we need $rs^2 + r^2s$ multiplications and additions. For Poisson's equation this number is r^2s^2, because every u value demands $r \cdot s$ multiplications and additions. The work for solving u from the triangular matrices is thus equivalent to compute two rows by the inverse matrix.[1]

However, when using the inverse matrix we can reduce the work considerably by computing two adjacent rows with the inverse matrix and computing next row by the formula

$$u_{3k} = 4\,u_{2k} - u_{2,\,k-1} - u_{2,\,k+1} - u_{1k} \qquad k = 1, 2, \ldots, s$$

[1] Poisson's eq. in a rectangular region can also be solved by using finite Fourier analysis and synthesis (cf. e. g. Charney, Fjørtoft, von Neumann, 1950). The work is about $2\,rs^2 + 2\,r^2s$ multiplications and additions.

When the third row has been computed we compute the fourth and so on. The process, however, is unstable, which theoretically has been shown by HYMAN (1952). The following ε-scheme (COLLATZ, 1951) establishes the error-growth in which we have an error ε in the middle row (see fig. 2).

Hyman calls this process step-ahead and shows that the loss after stepping 13 rows is 10 decimal figures. According to our experience the step-ahead process is favourable only when the boundary values have such a character that the solution is of exponential order in which case the relative error is constant. In other cases the error-growth very soon causes trouble and computing fresh rows with the inverse is necessary. Below will be shown that the triangular procedure is stable and an error (for example a rounding-off) will be damped. Besides, the gain with the step-ahead is none for Poisson's equation if we must compute with the inverse the u-values in more than two rows in the region.

7. The inverse matrix regarded as a function of the matrix D

For rectangular regions we can choose an alternative method (cf. section 8) for solving the u-values instead of that given by equations (11) and (12). The computation of the triangular matrices by this method has been replaced by the computation of a submatrix of the inverse matrix, and we now derive some matrix relations for the inverse matrix. By introducing allied Chebyshev polynomials for the inverse matrix we are led to expressions that are nicer than Hyman's for numerical purposes.

For Poisson's equation our expressions contain simple sums while Hyman's contain double sums corresponding directly to Green's function of the continuous case (see below equation 21). LAASONEN (1948) gives for Laplace's equation the same relations as we do.

The determinant of $\mathbf{D}$ can be put in a closed form (RUTHERFORD, 1946). Denote the determinant to the matrix $\mathbf{D}$ of the s^{th} order by D_s. Expanding D_{s+1} after the first row we have the recurrence relation

$$D_{s+1} = 4 \cdot D_s - D_{s-1} \tag{13}$$

Now we introduce a polynomial $S_n(a)$ of the n^{th} degree defined by the relations

$$S_{n+1}(a) = a \cdot S_n(a) - S_{n-1}(a); \; S_0(a) = 1; \quad S_1(a) = a \tag{14}$$

Thus,

$$S_s(4) = D_s$$

The difference equation (14) has the solution

$$S_n(a) = \sin(n+1)x/\sin x \quad 2\cos x = a \tag{15}$$

which is a polynomial allied to the Chebyshev polynomials. The eigenvalues d_k of $\mathbf{D}$ are derived immediately from (15).

$$d_k = 4 - 2 \cdot \cos\frac{\pi \cdot k}{s+1} \quad k = 1, 2 \ldots s \tag{16}$$

Because a > 2 we put (15) in the more convenient form

$$S_n(a) = \sinh(n+1)x/\sinh x \quad 2\cosh x = a \tag{17}$$

The eigenvector matrix (transformation matrix) of $\mathbf{D}$ is

$$\left\{ \sin\frac{jk\pi}{s+1} \right\}$$

We denote the normalized eigenvector matrix by $\mathbf{T}$.

$$\mathbf{T} = \sqrt{\frac{2}{s+1}} \cdot \left\{ \sin\frac{jk\pi}{s+1} \right\}$$

With the knowledge of $\mathbf{T}$ we can according to a known theorem of the matrix algebra compute an arbitrary matrix polynomial of $\mathbf{D}$ by the formula ($\mathbf{D}$ and $\mathbf{T}$ are symmetric).

$$F(\mathbf{D}) = \mathbf{T} \cdot \{ F(d_k) \} \cdot \mathbf{T}$$

where $\{ F(d_k) \}$ is a diagonal matrix formed by the values $F(d_k)$ where d_k are defined by (16).

The inverse matrix is decomposed in r^2 submatrices each of the order s.

$$\mathbf{G}^{-1} = \{ \mathbf{G}_{jk} \} \quad \mathbf{G}_{jk} = \mathbf{G}_{kj}$$

Since

$\mathbf{G} \cdot \mathbf{G}^{-1} =$ unit matrix of the order $r \cdot s$, we can for the submatrices $\mathbf{G}_{1k}$ write the relations

$$\begin{aligned}
\mathbf{D}\,\mathbf{G}_{11} - \mathbf{G}_{12} &= \mathbf{I} \\
-\mathbf{G}_{11} + \mathbf{D}\,\mathbf{G}_{12} - \mathbf{G}_{13} &= \mathbf{O} \\
-\mathbf{G}_{12} + \mathbf{D}\,\mathbf{G}_{13} - \mathbf{G}_{14} &= \mathbf{O} \\
- \; - \; - & \\
-\mathbf{G}_{1,r-1} + \mathbf{D} \cdot \mathbf{G}_{1r} &= \mathbf{O}
\end{aligned} \tag{18}$$

(18) satisfies in matrix notation the same difference equation as the polynomial $S_n(a)$.

We thus write the solution of (18) in the form

$$\mathbf{G}_{1k} = \text{constant} \cdot S_{r-k}(\mathbf{D})$$

because $\mathbf{G}_{1k}$ decreases with increasing k.

The constant is determined from the first equation of (18).

Hence,

$$\mathbf{G}_{1k} = S_{r-k}(\mathbf{D}) \cdot S_r^{-1}(\mathbf{D})$$

but since both factors are functions of $\mathbf{D}$ no ambiguity results if we write

$$\mathbf{G}_{1k} = S_{r-k}(\mathbf{D})/S_r(\mathbf{D}) \quad k = 1, 2 \ldots r \tag{19}$$

which will be used in the following.

We multiply $\mathbf{G}$ by the next column of its inverse and have the relations

$$\begin{aligned}
\mathbf{D}\,\mathbf{G}_{12} - \mathbf{G}_{22} &= \mathbf{O} \\
-\mathbf{G}_{12} + \mathbf{D}\,\mathbf{G}_{22} - \mathbf{G}_{23} &= \mathbf{I} \\
-\mathbf{G}_{22} + \mathbf{D} \cdot \mathbf{G}_{23} - \mathbf{G}_{24} &= \mathbf{O} \\
- \; - \; - & \\
-\mathbf{G}_{2,r-1} + \mathbf{D}\,\mathbf{G}_{2r} &= \mathbf{O}
\end{aligned}$$

To solve this we make the same attempt and the constant is determined by the second of the equations above. We have

$$\mathbf{G}_{2k} = S_1(\mathbf{D}) \cdot S_{r-k}(\mathbf{D})/S_r(\mathbf{D}) \quad k = 2, 3 \ldots r$$

By induction we obtain the general formula

$$\mathbf{G}_{jk} = S_{j-1}(\mathbf{D}) \cdot S_{r-k}(\mathbf{D}) \cdot / S_r(\mathbf{D}) \quad j \leq k \quad (20)$$

From (20) we easily derive formulas for computing the inverse matrix for half open regions, e.g., $r = \infty$, s finite. If we let s approach infinity in the formulas, the elements in the inverse matrix yield integrals that correspond to relations established by STÖHR (1950).

For comparison we establish the solutions of the continuous case. We divide the solution in two parts, one due to the boundary values denoted $f(\sigma)$, and an other part due to the function $J(x, y)$. Then, the solution of Poisson's equation

$$\nabla^2 u = J(x, y)$$

can be written in the form

$$u(x, y) = \int \frac{\partial K}{\partial n} \cdot f(\sigma) \cdot d\sigma -$$

$$- \iint\limits_{\text{region}} K(x, y; \xi, \eta) \cdot J(\xi, \eta) \cdot d\xi \cdot d\eta$$

where $K(x, y; \xi, \eta)$ is Green's function. The first integral is taken round the boundary. In the particular case when having a rectangle with the vertices (0, 0), $(a, 0)$, (a, b) and $(0, b)$, the normal derivative of Green's function for the side $y = 0$ and $0 \leq x \leq a$ is

$$\frac{\partial K}{\partial n} = \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sinh n \cdot \frac{\pi}{a}(b - y)}{\sinh n \cdot \frac{\pi}{a} \cdot b} \cdot$$

$$\cdot \sin n \frac{\pi x}{a} \cdot \sin n \frac{\pi \xi}{a}$$

and Green's function itself if expressed in Fourier series is

$$K = \frac{4}{\pi^2 a \cdot b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin n \frac{\pi x}{a} \cdot \sin m \frac{\pi y}{b} \cdot \sin n \frac{\pi \xi}{a} \cdot \sin m \frac{\pi \eta}{b}}{\frac{m^2}{b^2} + \frac{n^2}{a^2}} \quad (21)$$

Now we write the solution in matrix notation for the difference equation (2). Let $u = 0$ for $\sigma = 0$ and for $\sigma = s + 1$. We put

$$\mathbf{t} = \mathbf{t}' + \mathbf{t}''$$

where $\mathbf{t}'$ is due to the boundary values and $\mathbf{t}''$ is due to the function $J(x, y)$. Then,

$$\mathbf{u}_j = \mathbf{G}_{1j} \cdot \mathbf{t}_1' + \mathbf{G}_{jr} \cdot \mathbf{t}_r' + \sum_{k=1}^{r} \mathbf{G}_{jk} \cdot \mathbf{t}_k''$$

$\mathbf{G}_{jk}$ is defined by (19) and $\mathbf{t}$ is the right member of (3). If $J \equiv 0$ we only need the submatrices $\mathbf{G}_{1k}$, for $\mathbf{G}_{jr} = \mathbf{G}_{1, r-j+1}$. When comparing our matrix solution with the continuous solution we find that the normal derivative of Green's function corresponds to the "boundary matrices" $\mathbf{G}_{1k}$ and that Green's function mainly corresponds to the "kernel matrices" $\mathbf{G}_{jk}$, $jk \neq 1$.

We also consider the inverse matrix for rectangular nets. The eigenvalues of $\mathbf{D}$ will be changed but the eigenvector matrix $\mathbf{T}$ will be unchanged. If we denote the quotient between the mesh length of ϱ- and σ- directions by p, the new eigenvalues are then given by

$$d_j' = 2(1 + p^2) - 2p^2 \cdot \cos \frac{j\pi}{s+1}$$

$$i = 1, 2, \ldots, s$$

The mesh length h occurring in some equations will now mean the mesh length in the ϱ-direction.

8. A matrix method for solving u from the triangular matrices

A slight modification of the procedure indicated by equations (11) and (12) will give a convenient procedure to compute the u-values of Poisson's equation for a *rectangular* region. We split the partitioned matrix $\mathbf{G}$ into quasi-triangular matrices consisting of quadratic submatrices of the order s. Now the elements b_{jk} and c_{jk} derived from formulas (7) will become submatrices in the triangular matrices **B** and **C**. By induction we find that the **B** and **C** matrices can be written

$$\mathbf{B} = \begin{bmatrix} -S_1 & \mathbf{O} & \mathbf{O} & \mathbf{O}\ldots & & & & \\ \mathbf{I} & -\frac{S_2}{S_1} & \mathbf{O} & \mathbf{O}\ldots & & & & \\ \mathbf{O} & \mathbf{I} & -\frac{S_3}{S_2} & \mathbf{O}\ldots & & & & \\ & & & --- & & & & \\ & & & & \ldots & \mathbf{I} - \frac{S_j}{S_{j-1}} & \mathbf{O}\ldots & \mathbf{O} \\ & & & & & & --- & \\ & & & & & & & \mathbf{I} - \frac{S_r}{S_{r-1}} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -\mathbf{I} & \frac{1}{S_1} & \mathbf{O} & \mathbf{O}\ldots & & & & \\ \mathbf{O} & -\mathbf{I} & \frac{S_1}{S_2} & \mathbf{O}\ldots & & & & \\ \mathbf{O} & \mathbf{O} & -\mathbf{I} & \frac{S_2}{S_3}\ldots & & & & \\ & & & --- & & & & \\ & & & & \ldots -\mathbf{I} & \frac{S_{j-1}}{S_j} & \mathbf{O}\ldots & \\ & & & & & & --- & \\ & & & & & & & -\mathbf{I} \end{bmatrix}$$

For abbreviation the polynomial $S_n(\mathbf{D})$ is written S_n.

We introduce a matrix $\mathbf{H}_j$ defined by

$$\mathbf{H}_j = \frac{S_{j-1}}{S_j} = \sinh j\mathbf{X}/\sinh(j+1)\mathbf{X} \quad (22)$$

where $2 \cdot \cosh \mathbf{X} = \mathbf{D}$

We form an arbitrary row in **B** and obtain a difference equation for the matrix $\mathbf{H}_j$.

$$\mathbf{H}_j + \mathbf{H}_{j+1}^{-1} = \mathbf{D} \quad (23)$$

Now, equation (22) expresses the first submatrix $\mathbf{G}_{11}$ in the partitioned inverse matrix of a region of j rows according to equation (19).

To solve u from the partitioned triangular matrices **B** and **C** we first form an auxiliary function **z** that corresponds to **v** in equation (9). Thus we have

$$\begin{aligned} \mathbf{z}_1 &= \mathbf{H}_1 \cdot \mathbf{t}_1 \\ \mathbf{z}_{j+1} &= \mathbf{H}_{j+1} \cdot (\mathbf{z} + \mathbf{t}_{j+1}) \quad j = 1, 2, \ldots, r \end{aligned} \quad (24)$$

To solve u from

$$\mathbf{C} \cdot \mathbf{u} = -\mathbf{z}$$

we easily find from the partitioned matrix **C**

$$\begin{aligned} \mathbf{u}_r &= \mathbf{z}_r \\ \mathbf{u}_j &= \mathbf{z}_j + \mathbf{H}_j \cdot \mathbf{u}_{j+1} \quad j = r-1, r-2, \ldots, 2, 1. \end{aligned} \quad (25)$$

If we solve u according to the equations (24) and (25) we need $\mathbf{H}$ and $\mathbf{z}$ only. We also can apply the usual checking procedure for matrices when solving u in this way. We thus have transfered our problem to tabulate $\mathbf{G}_{11}$ for $r = 1, 2, 3, \ldots$ and these matrices can be computed free from each other. The eigenvalues of $\mathbf{H}_j$ denoted h_{jk} can be computed recurrently from (23) which gives a stable procedure.[1]

$$h_{jk} = \frac{1}{d_k - h_{j-1,\,k}} \qquad d_k = 4 - 2 \cdot \cos\frac{\pi k}{s+1}$$

The number of stored constants for solving Poisson's equation in a rectangular region is $r(s+1)$ using $\mathbf{H}_j$ matrices. For we have

$$\mathbf{H}_j = \frac{2}{s+1}\left\{\sum_{n=1}^{s} \sin\mu\frac{\pi n}{s+1}\cdot\sin\nu\frac{\pi n}{s+1}\cdot h_{jn}\right\}$$

$$= \frac{1}{s+1}\left\{\sum_{n=1}^{s}\left(\cos(\mu-\nu)\frac{\pi n}{s+1} - \right.\right.$$

$$\left.\left. - \cos(\mu+\nu)\frac{\pi n}{s+1}\right)\cdot h_{jn}\right\}$$

This implies that every $\mathbf{H}_j$ matrix demands $2s$ constants. But using congruence properties we find that $s+1$ constants are enough. Hence, having r rows we need $r(s+1)$ constants. But, as we have seen, $r(s+1)$ constants can be used to solve Poisson's equation in r regions. The first region contains 1 row, the second 2 rows and the last region contains r rows. Using inverse matrices we should need for these r regions considerably more constants. The work to compute the r $\mathbf{H}$ matrices is about rs^2.

In irregular regions it is sometimes convenient to split up the original matrix into triangular matrices that are partitioned in submatrices. See also § 11.

9. Stability

A stability investigation is now easy to perform on the equations (24) and (25), for an error will satisfy the same equations as $\mathbf{z}$ and $\mathbf{u}$ do. The error will be increased if an eigenvalue $\mathbf{H}_j$ is greater than 1, but will be decreased if it is less than 1. From (22) we immediately have the expression for h_{jk}, the eigenvalues of $\mathbf{H}_j$.

$$h_{jk} = \sinh jx / \sinh(j+1)x$$

$$\cosh x = 2 - \cos\frac{\pi k}{s+1}$$

from which equation we conclude that the eigenvalues must be less than 1.

10. Numerical considerations

To illustrate the foregoing paragraphs we give an example of triangular matrices for $s = 2$, shown in table 1. The space only allows one matrix to be written and this is written in columns. The first column should be regarded as the head diagonal of the triangular matrix $\mathbf{B}$. In this case the elements are stationary after 8 rows if rounded off to 5 decimals. MOSKOVITZ (1944) has tabulated parts of the inverse matrix for $r = 1$, 2 and 3 so these tables can be used in special cases.

For computations on BARK, the Swedish relay-computer, the triangular matrices have been formed by direct elimination because the "shorter" method has some drawbacks. At the preliminary planning of program for BESK, the Swedish electronic computer under construction, we also have used Gauss' method, and normally no difficulties arise in forming the triangular matrix because of the fixed binary point. Having rectangular regions, the forming of necessary matrices is an extremely simple procedure. Very large systems can have their matrices stored on the magnetic drum, which is favourable. In meteorological computations we can use information from preceding steps and compute corrections instead of the function itself. This will reduce the computational work.

[1] The formulas (24) and (25) can be expressed in $\mathbf{T}$ and diagonal matrices formed by h_{jk}. This means a finite Fourier analysis of the function J in the x-direction. In this case the elements h_{jk} only need to be tabulated.

Table 1. The triangular matrix B for $s = 2$

r	Head Diagonal	2	3
1	— 4.00000	—	—
	— 3.75000	1.00000	—
2	— 3.73333	0.25000	1
	— 3.42857	1.06667	1
3	— 3.70833	0.28571	1
	— 3.39185	1.08333	1
4	— 3.70518	0.29214	1
	— 3.38679	1.08612	1
5	— 3.70474	0.29314	1
	— 3.38606	1.08655	1
6	— 3.70467	0.29329	1
	— 3.38596	1.08662	1
7	— 3.70466	0.29331	1
	— 3.38594	1.08663	1
8	— 3.70466	0.29331	1
	— 3.38594	1.08663	1

11. Poisson's equation for non-rectangular regions

The method of triangular matrices can be applied, of course, to irregular regions. In these cases the submatrices in the diagonal of the partitioned matrix **G** will have their order given by the number of inner points in an actual row. Still we can add extra rows to the region without affecting the preceding elements in the triangular matrices.

12. Other elliptic problems in two dimensions

We first consider Helmholtz's equation

$$\nabla^2 u - \lambda \cdot u = J(x, y)$$

an equation which has interest in certain atmospheric models. If the λ-values are given, this equation can be solved in the same way as Poisson's equation. In a rectangular region the only difference from a numerical point of view is that the eigenvalues d_k in equation (16) will be replaced by the new expression

$$d_k = 4 + \lambda \cdot h^2 - 2 \cdot \cos \frac{\pi k}{s + 1} \qquad k = 1, 2, \ldots s$$

h denotes the mesh length.

The second boundary value problem where we have the derivative of u instead of u itself given on the boundary can also be solved by triangular matrices. The original matrix for this problem has the same simple structure as the **G** matrix.

Elliptic equations with variable coefficients give variable elements in the matrix which in general will be non-symmetric. The work for solving the function from the triangular matrices will be the same as for Poisson's equation. Self-adjoint equations give a symmetric matrix if the variable coefficients are tabulated in the half-points. Let the equation be

$$(a \cdot u_x)_x + (b \cdot u_y)_y = 0 \qquad u_x = \frac{\partial u}{\partial x} \text{ etc.}$$

where the functions a and b are functions of x and y only. Then, the corresponding difference equation is

$$- b_{j-1,k} \cdot u_{j-1,k} - a_{j,k-1} \cdot u_{j,k-1} + \\ + (b_{j-1,k} + a_{j,k-1} + a_{jk} + b_{jk}) \cdot u_{jk} - \\ - a_{jk} \cdot u_{j,k+1} - b_{jk} \cdot u_{j+1,k} = 0$$

Fig. 3 explains the index notation.

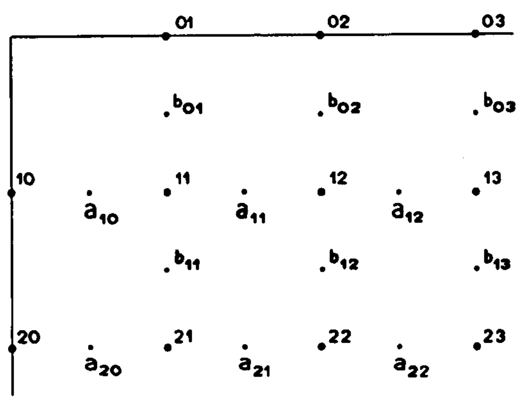

Fig. 3. Numbering for self-adjoint equations.

The biharmonic equation is an example of an equation of the fourth order that can be solved conveniently by triangular matrices. The corresponding difference equation gives $2s + 1$ non-vanishing diagonals where, as usual, s denotes the number of points per row.

13. *n*-dimensional problems

The matrix **G** will for Poisson's equation contain twice as many non-vanishing diagonals around the head diagonal as there are dimensions. For example, the matrix for a parallelepiped of cubic net is

$$\mathbf{G}_3 = \begin{bmatrix} \mathbf{G}_2 & -\mathbf{I}_2 & \mathbf{O} & \mathbf{O} \dots & \\ -\mathbf{I}_2 & \mathbf{G}_2 & -\mathbf{I}_2 & \mathbf{O} \dots & \\ \mathbf{O} & -\mathbf{I}_2 & \mathbf{G}_2 & -\mathbf{I}_2 \dots & \\ & & - & - \; - & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \dots & -\mathbf{I}_2 \; \mathbf{G}_2 \end{bmatrix}$$

where $\mathbf{I}_2$ is a unit matrix of the r^{th} order and $\mathbf{G}_2$ means the matrix

$$\mathbf{G}_2 = \begin{bmatrix} \mathbf{D}_2 & -\mathbf{I}_3 & \mathbf{O} & \mathbf{O} \dots & \\ -\mathbf{I}_3 & \mathbf{D}_2 & -\mathbf{I}_3 & \mathbf{O} \dots & \\ \mathbf{O} & -\mathbf{I}_3 & \mathbf{D}_2 & -\mathbf{I}_3 \dots & \\ & & - & - \; - & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \dots & -\mathbf{I}_3 \; \mathbf{D}_2 \end{bmatrix}$$

where $\mathbf{I}_3$ is a unit matrix of the s^{th} order and $\mathbf{D}_2$ means a matrix related to **D**.

$$\mathbf{D}_2 = \begin{bmatrix} 6 & -1 & 0 & 0 \dots & & \\ -1 & 6 & -1 & 0 \dots & & \\ 0 & -1 & 6 & -1 \dots & & \\ & & - & - \; - & & \\ 0 & 0 & 0 & 0 \dots & -1 & 6 \end{bmatrix}$$

The inverse can be derived in a similar way as we did for rectangular regions. We also can find a procedure for solving u that corresponds to equations (24) and (25).

14. Non-linear equations

For non-linear equations we often can use a linearized solution as a first approximation. Then, we add successive corrections to this approximation by solving a linearized perturbation. This equation gives matrices of the kind we have considered.

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