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Simplified Dynamic Models of the Atmosphere, designed for the Purpose of Numerical Weather Prediction

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Abstract

The properties of several atmospheric models, which have been proposed for the purpose of numerical computation of prognostic maps, are discussed, and a new version of the "$2^1/_2$-dimensional model" is given. It is pointed out that the "barotropic" model is unable to explain adequately the growth of wave perturbations, and that the advective model gives too strong instability for short waves, whereas the two-layer model and the continuous $2^1/_2$-dimensional model seem to describe almost correctly the behaviour of waves in a baroclinic current.

1. Introduction

Looking back at the results of the research on numerical weather forecasting carried out after the last war, we may say that a first stage of the research program is now finished, since it is definitely established that the two-dimensional, non-divergent model is able to explain a considerable part of the changes that we observe on the 500 mb maps. We have thus arrived at a serviceable first approximation, which enables us to carry on the research work with more confidence than before. The next stage, the advance from this bridgehead towards more accurate methods for numerical weather forecasting, has already begun. The goal is to find a system of prognostic equations which is realistic enough to give forecasts of practical value, and at the same time simple enough to permit computations of such forecasts with a reasonable amount of labour. Since this is a compromise, there may not be just one solution, suitable for all purposes. We may perhaps need several methods, of various degrees of accuracy and mathematical complexity; which one to use will depend on the nature of the forecast in question, as well as on the capacity of the available calculating equipment.

The derivation of suitable prognostic equations, which fulfil the conditions mentioned above, is one of the two main problems of numerical weather forecasting. We may hope to arrive at such equations by testing a variety of intelligent proposals. A discussion of some of the models that have been proposed, together with the author's version of the "2 ½-dimensional model", is given below. The second main problem of numerical forecasting, to overcome the mathematical difficulties involved in the process of numerical integration, will not be discussed here.

2. Quasi-geostrophic equations in three dimensions

So far, most of the prognostic equations that have been considered have been based on the quasi-static and the quasi-geostrophic approximations. The quasi-static approximation is valid for motions whose horizontal extent is large in comparison with their vertical extent, and is generally believed to be legitimate for all large-scale motions in the atmosphere. More questionable is the legitimacy of the quasi-geostrophic approximation, which has been discussed by CHARNEY (1948). If we study solutions of linearized equations, we find that this approximation, together with the quasi-static approximation, eliminates all sound waves and ordinary gravitational-inertial waves whose perturbation velocities are completely non-geostrophic. On the other hand, the same equations describe almost correctly the behaviour of waves with nearly geostrophic perturbation velocities, such as the gravitational-inertial waves in a baroclinic current discussed by CHARNEY (1947) and EADY (1949), as well as Rossby-waves. Since most large-scale disturbances connected with weather seem to belong to the group with almost geostrophic perturbation velocities, we have reason to hope that the quasi-geostrophic approximation will serve our purpose. On the other hand, we know from synoptic experience that large non-geostrophic winds sometimes exist in the upper troposphere; in particular, we must expect severe difficulties when applying the geostrophic approximation to sharp upper ridges, whose importance for the dynamics of the flow has been stressed by J. BJERKNES (1951). We must therefore expect the quasi-geostrophic approximation to cause considerable errors in some cases.

The quasi-static approximation enables us to use pressure (p) instead of height (z) as a vertical coordinate, which is convenient for several reasons. This means that partial differentiation with respect to horizontal coordinates (x, y) and time (t) will be performed at constant pressure, such that $\partial/\partial x$ and $\partial/\partial y$ denote differentiation along an isobaric surface, and $\partial/\partial t$ means rate of change as measured by an observer which follows an isobaric surface in its vertical motion.

It is convenient to use vector notation $\mathbf{v}$ for the horizontal wind velocity (with components u and v), and to introduce a horizontal del-operator ∇ (with components $\partial/\partial x$ and $\partial/\partial y$) to express differentiation with respect to horizontal coordinates (at constant p). Let further $\mathbf{k}$ denote a vertical unit vector, pointing upward; ω individual rate of change of pressure, α specific volume, Θ potential temperature, f Coriolis parameter and g acceleration of gravity.

The hydrostatic equation may then be written

$$(1)\qquad \frac{\partial z}{\partial p} = -\frac{\alpha}{g}.$$

The equation of continuity assumes the simple form

$$(2)\qquad \nabla\cdot\mathbf{v} + \frac{\partial\omega}{\partial p} = 0.$$

Neglecting frictional forces, we derive from the horizontal equation of motion the vorticity equation

$$(3)\qquad \frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\nabla(f+\zeta) + \omega\frac{\partial\zeta}{\partial p} + \\ + \mathbf{k}\cdot\nabla\omega\times\frac{\partial\mathbf{v}}{\partial p} + (f+\zeta)\nabla\cdot\mathbf{v} = 0,$$

where

$$(4)\qquad \zeta = \mathbf{k}\cdot\nabla\times\mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the "isobaric" vorticity, which differs from the vertical vorticity component in the differentiations being taken along isobaric instead of level surfaces.

Elimination of $\nabla\cdot\mathbf{v}$ between (2) and (3) gives the vorticity equation in the form

$$(5)\qquad \frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\nabla(f+\zeta) + \omega\frac{\partial\zeta}{\partial p} + \\ + \mathbf{k}\cdot\nabla\omega\times\frac{\partial\mathbf{v}}{\partial p} - (f+\zeta)\frac{\partial\omega}{\partial p} = 0.$$

Since the amount of heating and cooling is not known, we are at the present time unable to take this effect into account, and will have to assume that the potential temperature is conserved. The equation expressing this assumption is

$$\text{(6)}\qquad \frac{1}{\alpha}\frac{\partial \alpha}{\partial t}+\frac{1}{\alpha}\mathbf{v}\cdot\nabla\alpha+\frac{\omega}{\Theta}\frac{\partial\Theta}{\partial p}=0,$$

since differentiation at constant pressure gives $\partial\ln\Theta/\partial t=\partial\ln\alpha/\partial t$ and $\nabla\ln\Theta=\nabla\ln\alpha$. Making use of the hydrostatic equation (1) in the first term, this equation may also be written

$$\text{(7)}\qquad \frac{\partial}{\partial p}\frac{\partial z}{\partial t}-\frac{1}{g}\mathbf{v}\cdot\nabla\alpha-\frac{\alpha\omega}{g\Theta}\frac{\partial\Theta}{\partial p}=0.$$

The quasi-geostrophic method consists in substituting the geostrophic wind for the true wind in the terms which are not critically affected by this substitution. Thus the geostrophic wind approximation cannot be used in the divergence term, or in the expression for the work of the horizontal pressure force (if the energy equations are considered). The simplest and most logical way to introduce the geostrophic approximation is given by CHARNEY (1948). His method consists in substituting the geostrophic wind for the true wind everywhere in the thermodynamic energy equation and in the vorticity equation, after having eliminated the horizontal divergence by means of the equation of continuity.

From the geostrophic wind formula

$$\text{(8)}\qquad \mathbf{v}_g=\frac{g}{f}\mathbf{k}\times\nabla z$$

we find the following approximate formula for the geostrophic vorticity

$$\text{(9)}\qquad \zeta_g=\frac{g}{f}\nabla^2 z=\frac{g}{f}\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right).$$

From these equations and the hydrostatic equation (1) we find

$$\text{(10)}\qquad \frac{\partial \mathbf{v}_g}{\partial p}=-\frac{1}{f}\mathbf{k}\times\nabla\alpha,$$

$$\text{(11)}\qquad \frac{\partial \zeta_g}{\partial p}=-\frac{1}{f}\nabla^2\alpha.$$

Using Charney's method, we now introduce the geostrophic wind approximation into equations (5) and (7), and obtain

$$\text{(12)}\qquad \nabla^2\frac{\partial z}{\partial t}-\frac{f(f+\zeta_g)}{g}\frac{\partial\omega}{\partial p}-\frac{1}{g}\nabla\cdot(\omega\nabla\alpha)= =-\mathbf{k}\cdot\nabla z\times\nabla(f+\zeta_g),$$

$$\text{(13)}\qquad \frac{\partial}{\partial p}\frac{\partial z}{\partial t}-\frac{\alpha}{g\Theta}\frac{\partial\Theta}{\partial p}\omega=\frac{1}{f}\mathbf{k}\cdot\nabla z\times\nabla\alpha.$$

This may be considered as a system of linear differential equations for the two unknown functions $\partial z/\partial t$ and ω, whose coefficients can be determined if the field $z\ (x, y, p)$ of the contour height is known at the instant considered. When the absolute vorticity and the static stability are both positive, which corresponds to normal conditions in the atmosphere, the system is of the elliptic type. Together with the boundary conditions at the top and the bottom of the atmosphere

$$\text{(14)}\qquad \begin{cases} \omega=0 & \text{at } p=0,\\ \dfrac{\partial z}{\partial t}-\dfrac{1}{g}\alpha\omega=0 & \text{at the ground, assumed to be level}\\ & (p=p_0\approx 1000\text{ mb})\end{cases}$$

and suitable conditions for the lateral boundaries, the system (12, 13) determines $\partial z/\partial t$ and ω throughout the region considered. The solution may be worked out numerically, and may be used to extrapolate the field $z(x, y, p)$ over a small interval of time. Iterating this procedure, we obtain an integration in small steps of time. This method, based on essentially the same system of equations, has been discussed previously by the author (ELIASSEN, 1949).

In order to solve the system (12, 13) numerically, we may eliminate one of the unknown functions, whence we obtain a Poisson-type differential equation in three dimensions for the other, which may be solved e. g. by relaxation. Such an equation for the height (or pressure) tendency has been discussed by CHARNEY (1949).

It is interesting to note that if the third term on the left of (12) is neglected (which is often done), then the system (12), (13) is equivalent to the equilibrium equations for a loaded elastic system consisting of a number of horizontal elastic membranes exposed to weights of variable magnitude, and connected with vertical springs. Interpreting $\partial z/\partial t$ as the vertical displacement of the membranes and ω as the tension in the vertical springs, we see that the equation expressing equilibrium between the vertical forces has the form (12), whereas (13) expresses a linear stress-strain relationship for the vertical springs. The boundary conditions (14) also have obvious interpretations in this elastic model. Since we have an immediate feeling of how this model will behave under the influence of a given distribution of weights, and for given values of the constant term in Hooke's law (right-hand side of (13)), it may serve to give us an idea of what the solutions of our meteorological problem will look like.

Computations of forecasts based on the three-dimensional, quasi-geostrophic equations involve a large amount of computational work. To the author's knowledge, no such forecasts have yet been computed. So far, the tests have been based on simplified equations, which reduce the problem to one in two dimensions in space. This is attained by making certain assumptions, based on empirical knowledge, of the variation with height of the field of motion.

3. The barotropic model

The simplest model of this kind is the so-called barotropic model. The suggestion that the large-scale motion of the atmosphere may be considered as two-dimensional and non-divergent was first put forward by ROSSBY (1939). CHARNEY (1949) has given a more precise formulation of the physical basis for this hypothesis. He uses the vorticity equation (5) in the simplified form

$$(15)\qquad \frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla (f + \zeta) - f \frac{\partial \omega}{\partial p} = 0$$

with the simplified boundary conditions

$$(16)\qquad \omega = 0 \text{ at } p = 0 \text{ and at } p = p_0.$$

Furthermore, he makes use of the observed fact that as far as the large-scale motion is concerned, the magnitude of the wind velocity tends to change with height in a typical fashion, whereas the streamline pattern stays about the same from level to level. This assumption makes it possible to carry out an integration of (15) with respect to p from 0 to p_0, thus eliminating ω. The resulting equation expresses that at a certain intermediate level, which is found to be in the vicinity of 500 millibars, vorticity changes are due to advection only:

$$(17)\qquad \frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla (f + \zeta) \quad \text{at } p \approx 500 \text{ millibars.}$$

The motion at this level may therefore be considered as horizontal and non-divergent. When the geostrophic assumption is introduced, a two-dimensional Poisson equation for the height tendency is obtained, which may be used as a prognostic equation[1].

Several numerical tests have been carried out by comparing the changes computed from this equation with the observed ones (CHARNEY, ELIASSEN, 1949; CHARNEY, FJØRTOFT, VON NEUMANN, 1950; BOLIN, CHARNEY, 1951; STAFF MEMBERS, UNIVERSITY OF STOCKHOLM, 1952). The results are very promising indeed; the discrepancies, although important, are nothing more than we would have to expect from such a crude model.

The properties of horizontal, two-dimensional flow, which is governed by equation (17), has been subject to many theoretical studies. ROSSBY (1939) investigated horizontal, non-divergent wave perturbations on a uniform zonal current, and found that these waves move with the speed

$$(18)\qquad c = U - \frac{\beta}{k^2}$$

where U is the velocity of the current, $\beta = df/dy$, and k the wave number. These waves are stable for all wave lengths. CRAIG (1945) and HØILAND (1950) have found solutions representing two-dimensional non-divergent waves with finite amplitude on a spherical earth. These theoretical motions have a great resemblance with the observed wave patterns on the 500 mb maps.

The most apparent defect of the barotropic model seems to be connected with the growth

[1] Concerning the details, the reader is referred to CHARNEY's paper (1949) and also to CHARNEY and ELIASSEN (1949), CHARNEY, FJØRTOFT, VON NEUMANN (1950).

of wave perturbations. Unstable waves do exist in the barotropic model, but the only source of energy for the growing perturbations is the kinetic energy of the basic flow. Investigations made by CHARNEY (1947), EADY (1949) and FJØRTOFT (1950), of wave perturbations in a baroclinic, zonal current, based on the three-dimensional equations, have shown the existence of unstable waves that grow at the expense of the potential energy of the system; and there is rather strong evidence that most waves in the westerlies develop as a result of instability of this kind. Such unstable waves do not exist in the barotropic model, since transformations of potential into kinetic energy are excluded a priori in this model. We must conclude that the barotropic model is not capable of an adequate explanation of the growth of wave perturbations, and we must therefore look for more realistic models.

4. The advective model

It is possible to take into account important effects related to transformation of potential into kinetic energy, i. e. to the baroclinicity, without using the complicated three-dimensional equations. So called 2 ½-dimensional models have been designed, which in a crude manner take into account the three-dimensional structure of the atmosphere and the vertical motion, although they are still two-dimensional, mathematically speaking. In these models, two horizontal fields (e. g. one contour field and one temperature field) are being used to define the state and motion of the atmosphere. A better name would therefore perhaps be "two-parametric models".

The simplest model of this type is *the advective model*. It is based on the assumption that temperature changes are caused by horizontal advection only. Moreover, one assumes that the thermal wind has a similar variation in magnitude with height in all places, whereas its direction is independent of height. In other words, one assumes that the wind field can be represented by the formula

$$(19)\quad \mathbf{v}(x, y, p, t) = \overline{\mathbf{v}}(x, y, t) + A(p)\,\mathbf{v}_T(x, y, t)$$

where the bar denotes the mean value with respect to pressure, $A(p)$ represents the variation of thermal wind with height, and $\mathbf{v}_T$ is the thermal wind measured between two selected isobaric levels. Introducing the thickness h between these isobaric levels, and the mean contour height $\bar{z}$, the geostrophic assumption gives

$$(20)\quad \overline{\mathbf{v}} = \frac{g}{f}\,\mathbf{k} \times \nabla \bar{z}, \quad \mathbf{v}_T = \frac{g}{f}\,\mathbf{k} x \nabla h,$$

and the corresponding vorticities

$$(21)\quad \bar{\zeta} = \frac{g}{f}\nabla^2 \bar{z}, \quad \zeta_T = \frac{g}{f}\nabla^2 h.$$

When these expressions are introduced into the approximate vorticity equation (15), and the mean value with respect to pressure is taken, one obtains

$$(22)\quad \frac{\partial \bar{\zeta}}{\partial t} = -\overline{\mathbf{v}} \cdot \nabla (f + \bar{\zeta}) - \overline{A^2}\,\mathbf{v}_T \cdot \nabla \zeta_T,$$

where the last term gives the contribution of the thermal wind field to the mean vorticity change. The advective hypothesis gives finally

$$(23)\quad \frac{\partial h}{\partial t} = -\overline{\mathbf{v}} \cdot \nabla h.$$

The two last equations are seen to contain the two horizontal fields $\bar{z}$ and h only, and may be integrated numerically in steps of time to give the change of these fields.

In a form suitable for numerical integration, the advective model is described in a paper by CHARNEY, FJØRTOFT and VON NEUMANN (1950), (see also FJØRTOFT, 1951, and CHARNEY, 1951). The assumption that temperature changes are due to horizontal advection only has actually been used earlier by ROSSBY (1942) and by J. BJERKNES and HOLMBOE (1944) in the study of long waves in a baroclinic zonal current, although without the derivation of the corresponding formula for the wave velocity. If we linearize (22) and (23) for the case of small perturbations on a baroclinic zonal current without horizontal shear, we find for sine waves of wave number k and of infinite lateral extent the velocity of propagation

$$(24)\quad c = \overline{U} - \frac{\beta}{2k^2} \pm \sqrt{\left(\frac{\beta}{2k^2}\right)^2 - \overline{A^2}\,U_T^2}$$

where $\overline{U}$ is the mean wind speed and U_T the thermal wind of the current. This formula has been derived by FJØRTOFT (1950) and by SUTCLIFFE (1951). A similar formula is given by BERSON (1950).

Equation (24) gives stable or unstable waves, according as the wave length is greater or smaller than a certain critical wave length, the magnitude of which increases with the square root of the thermal wind. This is in agreement with results derived by CHARNEY (1947) from the three-dimensional equations. Thus the advective model seems to be more realistic than the barotropic model, which in this case gives stable waves only. For wave lengths considerably larger than the critical wave length, one of the values (24) of the wave speed is in approximate agreement with the Rossby formula (18), whereas the other is considerably larger. This second value, however, corresponds to waves with a nodal surface in the middle of the atmosphere, roughly at the level where the barotropic model applies. These waves are therefore eliminated a priori in the barotropic model. We may conclude that the barotropic model is in agreement with the advective model as far as waves considerably longer than the critical wave length are concerned. For waves comparable with, or shorter than the critical wave length, the results derived from the two models differ appreciably (compare SUTCLIFFE 1951).

The unstable waves derived from the advective model have the property that their growth-rate increases indefinitely as the wave length approaches zero. This "ultra-violet catastrophe" has the unagreeable consequence that we are unable to solve the linearized initial value problem by means of Fourier analysis; most likely, this problem has no meaning. We may find a solution by dropping the shortest wave components, which is automatically done if the continuous functions are replaced by their values in a grid of points. However, such a procedure will not eliminate the difficulty, because the solution will depend critically on the wavelength of the shortest waves kept in the computation, i. e. on the grid size. We must expect to find the same difficulty also if we apply the equations of the advective model in non-linear form.

The extreme instability for very short waves, obtained from the advective model, is not supported by the theory based on the three-dimensional equations. This instability is just an error, introduced by the advective assumption, which is not legitimate for short waves. As we shall see, the error is quite unnecessary, and the advective model is therefore not a particularly good model to use for the purpose of numerical forecasting.

We shall now proceed to discuss some two-parametric models where also the vertical transport of potential temperature is taken into account.

5. Phillips' two-layer model

PHILLIPS (1951) has designed a model consisting of two homogeneous and incompressible fluid layers separated by a quasi-horizontal interface and bounded by two rigid, horizontal planes. The density of the upper layer is somewhat smaller than that of the lower, so that the system is statically stable. The system is baroclinic if the isobaric surfaces intersect the interface; and since each layer is autobarotropic, the baroclinicity is concentrated in the interface. The interface is permitted to slope and to move up or down, corresponding to vertical motion and horizontal divergence in each layer.

From the hydrostatic relation and the dynamical boundary condition at the interface one derives a linear relationship between the contour heights $z(x, y, t)$ and $z'(x, y, t)$ of the two layers and the height $h(x, y, t)$ of the interface. The state of the system is thus defined by two two-dimensional fields, e. g. the fields of z and h. Application of the geostrophic potential vorticity equation to the two layers gives for these two fields differential equations of the form

$$\nabla^2 \frac{\partial h}{\partial t} - a(x, y) \frac{\partial h}{\partial t} = q(x, y) \tag{25}$$

$$\nabla^2 \frac{\partial z}{\partial t} = r(x, y) \tag{26}$$

where the coefficient functions a and q can be computed from knowledge of the fields h and z, and r can be computed from knowledge of h, z and $\partial h/\partial t$.[1] Together with

[1] For further details see PHILLIP'S paper (1951).

suitable lateral boundary conditions, these equations may be used to determine $\partial h/\partial t$ and $\partial z/\partial t$ when z and h are known at the instant considered. The tendencies thus computed may be used to extrapolate z and h through a finite interval of time; and iteration of this procedure gives a numerical integration in steps of time. It is important to note that although the motion of the model is three-dimensional, the forecast problem is mathematically two-dimensional in space, since the vertical coordinate (p) does not occur as an independent variable.

Phillips has applied his model to study the behaviour of perturbations on a uniform zonal current with different wind speeds in the two layers. For long waves, his results are in agreement with Charney's results and with the advective model: waves longer than a critical wave length are stable, and shorter waves unstable. However, as the wave length decreases, the growth rate of the unstable waves does not increase indefinitely, as in the advective model, but instead increases to a maximum and then falls down to zero at another critical wave length. Still shorter waves are stable in Phillips' model. This part of the frequency curve is in agreement with the results obtained by EADY (1949) from the three-dimensional perturbation equations for small wave perturbations on a baroclinic current. The whole frequency curve for the two-layer model corresponds closely to the frequency curve for a continuous medium derived by FJØRTOFT (1950) from energy considerations. This is a strong indication that the two-layer model in many respects will behave like the three-dimensional atmosphere.

Phillips has also applied his model to an actual weather situation and computed tendencies and vertical velocities which look very promising.

A disadvantage of the two-layer model is its geometrical dissimilarity with the real atmosphere, which makes it difficult to define a proper correspondence between the states of the model and the states of the atmosphere. Phillips establishes such a correspondence by requiring the frequency curve, derived from the model for waves on a zonal current, to be quantitatively correct in its main properties. But even with the best possible choice of correspondence, one would expect the model to reproduce atmospheric motion with a certain amount of distortion. For the purpose of numerical forecasting, a more life-like model would therefore probably give better results.

6. Continuous two-parametric model with vertical stability

The continuous two-parametric model with static stability is a natural generalization of the barotropic and advective models, and it is therefore not surprising that several research workers seem to have arrived at this model more or less independently. Thus Eady has proposed a very simple and elegant version of this model which he presents in an article in this issue of Tellus (cf pp 157—167).

In the author's version of the 2 ½-dimensional model, described below, it has been attempted to attain the closest possible correspondence with atmospheric conditions, in order to minimize errors due to geometrical dissimilarity between the atmosphere and the model. The author's model is somewhat more complicated than Eady's, but does not differ much from Eady's model in its main properties.

In the two-parametric model, the temperature distribution along the vertical is represented by one parameter. In other words, the possible ascent curves of our model are restricted to a one-dimensional mannifold of curves. Moreover, it will be convenient to have the temperature (T) as a linear function of the parameter defining the particular ascent curve. We will therefore make the following assumption for the vertical temperature distribution:

$$T(p) = T_n(p) + \tau(p)(h - h_n). \tag{27}$$

Here h is the parameter which defines the temperature curve, and h_n a constant included for the sake of convenience. The function $T_n(p)$ is chosen as a "normal" ascent curve (fig. 1), and $\tau(p)$ is chosen positive in the troposphere and negative in the stratosphere, as shown in fig. 2. Formula (27) then gives, for different values of h, a family of ascent curves as shown in fig. 3. These curves may be thought of as a set of homologes for different air masses; they are in accordance with real conditions in so far as a warm troposphere

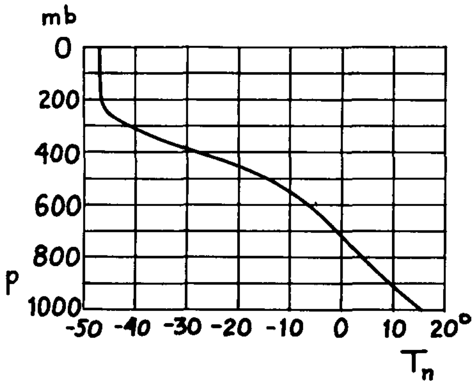

Fig. 1. "Normal" ascent curve T_n.

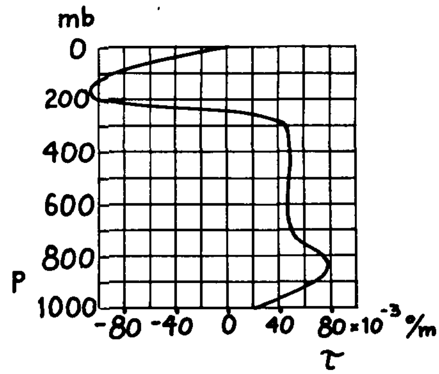

Fig. 2. Temperature deviation τ per unit thickness.

goes together with a cold stratosphere and vice versa. Since the result is unchanged if τ is reduced and at the same time h and h_n increased by the same factor, it follows that we may always choose h to represent the thickness of a prescribed isobaric layer, e. g. from 1,000 to 500 mb; h_n will then be the corresponding thickness for the "normal" temperature curve T_n. Assuming h to be a function of x, y and t, formula (27) will represent a three-dimensional temperature field.

The corresponding field of specific volume is

$$(28) \qquad \alpha(x, y, p, t) = R\frac{T_n(p)}{p} + R\frac{\tau(p)}{p}[h(x, y, t) - h_n],$$

(R, gas constant), and the contour field

$$(29) \quad z(x, y, p, t) = \bar{z}(x, y, t) + H(p) + A(p)[h(x, y, t) - h_n].$$

Here the bar denotes as before the mean value with respect to pressure, and the functions $H(p)$ and $A(p)$ are defined by

$$(30) \qquad H' = -\frac{RT_n(p)}{gp}, \quad \overline{H} = 0,$$

$$(31) \qquad A' = -\frac{R\tau(p)}{gp}, \quad \overline{A} = 0.$$

The three-dimensional field $z(x, y, p, t)$ thus depends on the two horizontal fields of $\bar{z}$ and h.

From (29) we find the field of geostrophic wind

$$(31) \quad \mathbf{v}(x, y, p, t) = \bar{\mathbf{v}}(x, y, t) + A(p)\mathbf{v}_T(x, y, t),$$

and the corresponding geostrophic vorticity

$$(32) \quad \zeta(x, y, p, t) = \bar{\zeta}(x, y, t) + A(p)\zeta_T(x, y, t),$$

where $\bar{\mathbf{v}}$, $\mathbf{v}_T$, $\bar{\zeta}$, and ζ_T are defined by (20) and (21). Fig. 4 shows the function $A(p)$.

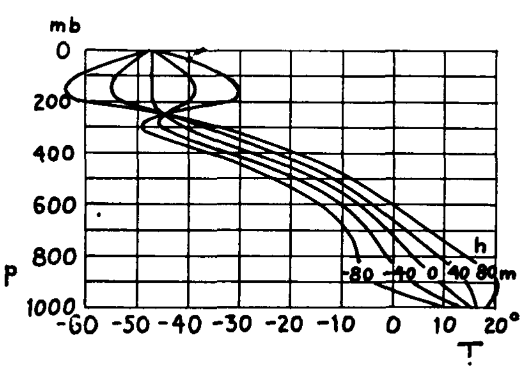

Fig. 3. Ascent curves for different values of h.

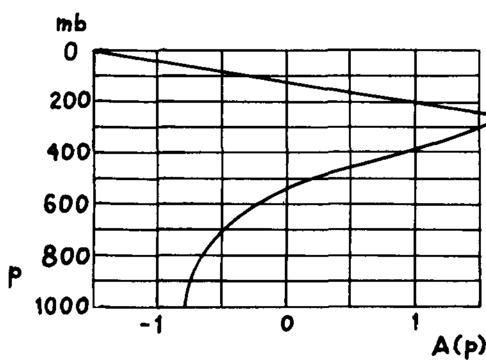

Fig. 4. The function $A(p)$.

We will also have to make a certain assumption about the distribution with height of vertical motion, and put

$$(33)\quad \omega\ (x,\ y,\ p,\ t) = B\ (p)\cdot w\ (x,\ y,\ t)$$

where $B\ (p)$ is chosen as a function of the type shown in fig. 5. The approximate boundary conditions (16) are automatically fulfilled when B is chosen so that $B\ (0) = B\ (p_0) = 0$.

We now insert the formulae (28), (31) and (33) into the thermodynamic equation (6). Approximating α, when undifferentiated, as well as the static stability factor, by their normal values, we obtain

$$(34)\quad \frac{\tau}{T_n}\left(\frac{\partial h}{\partial t} + \bar{\mathbf{v}}\cdot\nabla h\right) + \frac{B}{\Theta_n}\frac{d\Theta_n}{dp}\,w = 0,$$

where Θ_n is the potential temperature corresponding to the "normal" temperature T_n. As a rule, we cannot attain this equation to be exactly fulfilled at all levels; a reasonable requirement is, however, that the mean value of its left hand side shall vanish. This gives

$$(35)\qquad w = -\gamma\left(\frac{\partial h}{\partial t} + \bar{\mathbf{v}}\cdot\nabla h\right),$$

where we have put

$$(36)\quad \gamma = \overline{\left(\frac{\tau}{T_n}\right)}\cdot\left[\overline{\left(\frac{B}{\Theta_n}\frac{d\Theta_n}{dp}\right)}\right]^{-1}.$$

The constant γ can be evaluated when the functions T_n, τ and B have been chosen.

The vorticity equation shall be used in the form

$$(37)\quad \frac{\partial\zeta}{\partial t} + \mathbf{v}\cdot\nabla(f+\zeta) + \omega\frac{\partial\zeta}{\partial p} - (f+\zeta)\frac{\partial\omega}{\partial p} = 0,$$

where only the term involving $\nabla\omega$ of the "exact" equation (5) has been dropped. It is even possible to take this term into account, but that would complicate the computations considerably. Introduction of (31), (32) and (33) into (37) gives

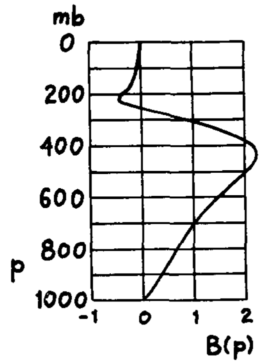

Fig. 5. The function $B(p)$.

$$(38)\quad \frac{\partial\bar\zeta}{\partial t} + A\frac{\partial\zeta_T}{\partial t} + \bar{\mathbf{v}}\cdot\nabla(f+\bar\zeta) + A[\bar{\mathbf{v}}\cdot\nabla\zeta_T + \mathbf{v}_T\cdot\nabla(f+\bar\zeta)] + A^2\,\mathbf{v}_T\cdot\nabla\zeta_T - B'\,(f+\bar\zeta)\,w + (A'B - AB')\,\zeta_T\,w = 0.$$

As in the case of the thermodynamic equation (34), we cannot hope to have the vorticity equation exactly satisfied at all levels. Our model is thus inconsistent, a fact which, however, will not disturb a theoretical meteorologist. We have the freedom to impose two conditions on the left-hand side of the vorticity equation, because we need two prognostic equations for the two horizontal fields $\bar{z}$ and h. What we have to do is to choose these conditions so as to minimize the error in the vorticity equation.

For fixed values of x, y and t, the left-hand side of (38) is a function of p, which will be denoted by $M(p)$. To this function we define a linear regression function $L(p)$, such that $\overline{(L-M)^2}$ = a minimum. We now require $L(p)\equiv 0$ for all values of p. This gives us the two wanted conditions

$$(39)\quad \overline{M} = 0\ \text{and}\ \overline{pM} - \bar{p}\cdot\overline{M} = 0.$$

When we here insert for M the left-hand side of (38), carry out the details of the calculation and eliminate w by means of (35), we obtain the two prognostic equations

$$(40)\quad \frac{\partial\bar\zeta}{\partial t} + \bar{\mathbf{v}}\cdot\nabla(f+\bar\zeta) + a\,\mathbf{v}_T\cdot\nabla\zeta_T - q\,\zeta_T\left(\frac{\partial h}{\partial t} + \bar{\mathbf{v}}\cdot\nabla h\right) = 0,$$

$$(41)\quad \frac{\partial \zeta_T}{\partial t} + \mathbf{v}_T \cdot \nabla (f + \bar{\zeta}) + \bar{\mathbf{v}} \cdot \nabla \zeta_T + \\ + b\,\mathbf{v}_T \cdot \nabla \zeta_T - r\,(f + \bar{\zeta} + A^{\star} \zeta_T) \\ \left(\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h\right) = 0,$$

where

$$(42)\begin{cases} a = \overline{A^2} \\ b = \dfrac{\overline{pA^2} - \bar{p}\,\overline{A^2}}{\overline{\overline{pA}}} \\ q = 2\gamma\,\overline{A'B} \\ r = \dfrac{\gamma \bar{B}}{\overline{\overline{pA}}} \\ A^{\star} = \dfrac{1}{\bar{B}}\left(\overline{AB} + 2\,\overline{pA'B} - 2\bar{p}\,\overline{A'B}\right). \end{cases}$$

Equations (40) and (41) represent, when the geostrophic formulae (20) and (21) are inserted, two linear differential equations for the tendencies $\partial\bar{z}/\partial t$ and $\partial h/\partial t$, of the same type as the prognostic equations (25), (26) obtained for the two-layer model. The system (40), (41) may be integrated numerically in steps of time. At each step, we will first have to determine $\partial h/\partial t$ from (41); having done that, $\partial\bar{z}/\partial t$ may be obtained from (40). Both equations may be solved by relaxation. If desired, the contour height at any particular level (e. g. 500 mb) may be used instead of the mean height $\bar{z}$, the relation between them being given by (29). This has the advantage that the two dependent variables h and z (500) can then be compared directly with the synoptic maps.

Each step will involve the evaluation of the fields of five advection terms, which have the form of Jacobian determinants. Their number may, however, be reduced to three. Let

$$(43)\quad J_1 = (\bar{\mathbf{v}} + A_1 \mathbf{v}_T) \cdot \nabla (f + \bar{\zeta} + A_1 \zeta_T), \\ J_2 = (\bar{\mathbf{v}} + A_2 \mathbf{v}_T) \cdot \nabla (f + \bar{\zeta} + A_2 \zeta_T)$$

denote the vorticity advection at two levels p_1 and p_2, which are chosen such that $A_1 = A(p_1)$ and $A_2 = A(p_2)$ are the roots of the equation

$$(44)\quad A^2 - bA - a = 0.$$

Then we have

$$(45)\quad \bar{\mathbf{v}} \cdot \nabla (f + \bar{\zeta}) + a\,\mathbf{v}_T \cdot \nabla \zeta_T = \frac{A_2 J_1 - A_1 J_2}{A_2 - A_1},$$

$$(46)\quad \mathbf{v}_T \cdot \nabla (f + \bar{\zeta}) + \bar{\mathbf{v}} \cdot \nabla \zeta_T + \\ + b\,\mathbf{v}_T \cdot \nabla \zeta_T = \frac{J_1 - J_2}{A_1 - A_2},$$

which we may introduce into (40) and (41).

The coefficients of the prognostic equations (40), (41) may be computed from (42) and (36), when the functions T_n, A and B have been chosen. If we adopt the values given in figs. 1, 4 and 5, which are based on estimates made by the author's colleague, Mr. Smebye, we find

$$(47)\quad a = 0.65, \qquad b = 0.74, \\ q = 0.89 \cdot 10^{-2}\, m^{-1}, \quad r = 0.76 \cdot 10^{-2}\, m^{-1}, \\ A^{\star} = 0.85.$$

However, there is no unique way of defining the functions A and B, and the above values of the coefficients may therefore not be the best possible ones.

To the author's knowledge, all two-parametric models designed so far give the same prognostic equations (40), (41), only with different numerical values of the coefficients. This applies to the model proposed by Eady, as well as to Phillips' two-layer model, when suitably interpreted. For the purpose of numerical forecasting, it is therefore not necessary to pay too much attention to the details of the structure of the model; the best approach is perhaps to determine the coefficients of (40), (41) empirically so as to get the least error in the computed forecast.

7. Waves in a baroclinic zonal current.

We shall finally investigate on the basis of the continuous two-parametric model the speed of propagation and growth-rate of wave perturbations in a baroclinic zonal current. Let the wind velocity of the basic current at any level be given by $\bar{U} + A(p)\, U_T$, where $\bar{U}$ and U_T are constants. For simplicity, we shall let our coordinate system move with the mean wind, so that we may assume $\bar{U} = 0$. Assuming the perturbations $\bar{z}$

and h of the mean height and thickness to be functions of x and t only, we find from (40) and (41) the perturbation equations:

$$\frac{\partial^3 \bar{z}}{\partial x^2 \partial t} + \beta \frac{\partial \bar{z}}{\partial x} + a U_T \frac{\partial^3 h}{\partial x^3} = 0, \tag{48}$$

$$U_T \frac{\partial^3 \bar{z}}{\partial x^3} + k_0^2 U_T \frac{\partial \bar{z}}{\partial x} + \frac{\partial^3 h}{\partial x^2 \partial t} - \tag{49}$$

$$- k_0^2 \frac{\partial h}{\partial t} + \beta \frac{\partial h}{\partial x} + b U_T \frac{\partial^3 h}{\partial x^3} = 0$$

where

$$k_0^2 = \frac{r f^2}{g}. \tag{50}$$

These equations may be satisfied by a trigonometric wave solution of the form: $\bar{z}, h \sim \exp i\,(kx + \nu t)$, provided

$$\nu = \frac{\beta}{k} - \nu_b \pm \sqrt{\nu_b^2 - \nu_a^2} \tag{51}$$

where

$$\nu_b = \frac{k_0^2 \beta + b U_T k^4}{2k\,(k_0^2 + k^2)}, \tag{52}$$

$$\nu_a^2 = a U_T^2 k^2 \frac{k_0^2 - k^2}{k_0^2 + k^2}. \tag{53}$$

When $\nu_b^2 > \nu_a^2$ we obtain stable waves with the speed of propagation $c = -\nu/k$. When $\nu_b^2 < \nu_a^2$, the waves are unstable; their speed of propagation is $c = -\beta/k^2 + \nu_b/k$, and their growth-rate $\nu_i = \sqrt{\nu_a^2 - \nu_b^2}$. For the values of a and b given in (47), the speed of propagation and the growth-rate are shown in fig. 6 as functions of the wave length L and the thermal wind U_T. The quantities L, ν_i, c and U_T are in the diagrams represented as the non-dimensional quantities $k_0 L/2\pi$, ν_i/ν_0, c/U_0, and U_T/U_0, respectively, where

$$\nu_0 = \frac{\beta}{k_0}, \qquad U_0 = \frac{\beta}{k_0{}^2}. \tag{54}$$

This has the advantage that the diagrams are valid for all latitudes and for all values of the coefficient r in (50), which involves the static stability. In latitude 45°, and with the value of r given in (47), we find

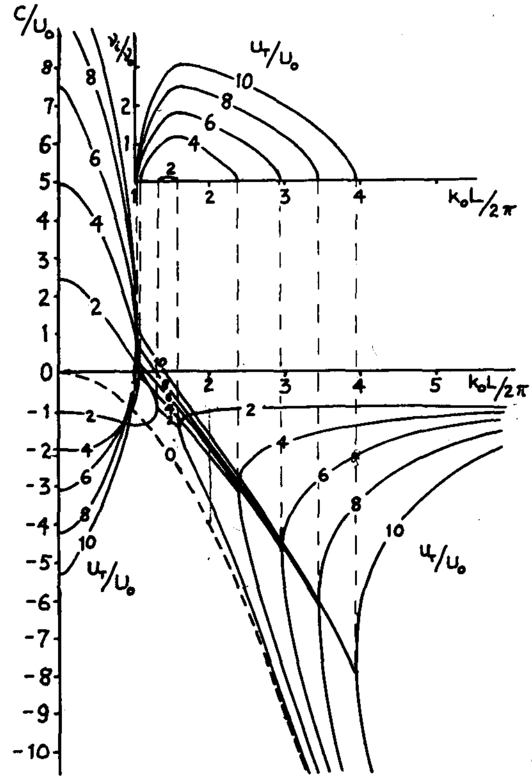

Fig. 6. Upper part: Growth rate ν_i/ν_0 of unstable waves as a function of wave-length $K_0 L/2\pi$, for different values of thermal wind U_T/U_0.
Lower part: Speed of propagation c/U_0 of stable and unstable waves as a function of wave-length $K_0 L/2\pi$, for different values of thermal wind U_T/U_0.
Dashed curve: speed of propagation of Rossby-waves ($U_T = 0$).

$$\frac{2\pi}{k_0} = 2{,}200 \text{ km}, \qquad \nu_0 = (2 \text{ days})^{-1},$$

$$U_0 = 2.0 \text{ m/sec}.$$

We notice that the waves are unstable in an intermediate band of wave lengths, in agreement with the findings of FJØRTOFT (1950) and PHILLIPS (1951). The growth-rate is a maximum when $k_0 L/2\pi \approx 1.5$, which with the numerical values given above corresponds to a wave length of about 3,300 km. The wave length of maximum instability is seen to increase in proportion to the square root of the static stability, and will also vary in inverse proportion to the Coriolis parameter; this is in good agreement with synoptic experience. Thus, for less stable air, or in higher latitudes, we will obtain considerably smaller values of the wave length of maximum instability. The values of the growth-rate and the speed of propagation obtained from the diagrams seem to be reasonable.

Thus the results obtained from the two-parametric model are plausible, as far as the behaviour of wave perturbations are concerned. This gives us some reason to hope that this model will give good results also when the equations (40), (41) are applied in non-linear form. Should these equations prove insufficient, there is still the possibility of increasing the number of parameters in the model.

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