On a Numerical Method of Integrating the Barotropic Vorticity Equation

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Abstract

A method is given by means of which quasi-permanent "velocity" fields may be constructed in which absolute vorticity is conserved if the atmosphere is treated as barotropic. For vorticity displacements these fields may be considered as constant for time intervals of the order of magnitude 24 hrs. Some simple approximate integration formulae are given for the Poisson- and Helmholz-equations. A graphical procedure is introduced to derive the isolines of vorticity and to perform the integration work.

1. Introduction

In numerical weather prognosis one works with a certain system of partial differential equations. These equations may differ considerably in form in accordance with how the problem has been formulated physically. Apart from this essentially *physical* aspect of the forecasting problem there is one *mathematical* and one *practical* aspect. As to the first one a number of mathematical problems arise in connection with an attempt of solving the equations. Certainly one is forced to solve the equations numerically, i. e. to replace in some way differentials with finite differences. How this has to be done must to some extent depend upon the physical assumptions which are made. However, even for a given system of equations different ways of numerical procedures may be tried, and it is not altogether certain which one should be preferred from a mathematical point of view. The partial differential equations in question are namely so complicated that a pure mathematical discussion of how well a numerical solution approximates the true one will have a limited value only. From a practical point of view, however, this may not present any great difficulty. The *relative* goodness of different numerical approaches may namely be tested from a comparison between the different solutions with respect to how well they approximate the actual development (integration) having taken place in the atmosphere.

The practical aspect of the numerical weather forecasting problem is related to the time necessary for carrying out a numerical solution. For it is quite clear that a forecast, how good it else may be, is completely useless in daily weather forecasting if it cannot be completed ahead of the weather. From that reason one may prefer one numerical method for another even on the cost of accuracy if the computation time is reduced essentially. This will especially be the case if the additional errors introduced thereby are well within the limits of errors present for other reasons.

In the following it is dealt with a particular numerical approach for solving the so-called barotropic vorticity equation. The chief achievements of this method may be summarized as follows:

1. For an area approximately as large as the one covered by the map in fig. 2 a forecast for 24 hrs. can be computed in the course of 2—3 hrs., reckoned from the conclusion of the analysis of the 500 mb map.

2. The work can be made entirely by one man and exclusively with the aid of standard means present at any weather forecasting section.

Recalling the integration time for the same problem on high-speed electronic computers, especially if the necessary introductory readings of data from the map are included, one would not expect particularly high integration accuracy from such a method. However, from a total of about 45 forecasts, including the four Princeton-forecasts (CHARNEY, FJØRTOFT, VON NEUMANN, 1950) it has turned out as a very pleasant fact that the integration accuracy appearingly is of the same order of magnitude as the one obtained on the electronic computer used for the Princeton-forecasts. It will be a subject for further work to make a systematic investigation of the integration accuracy. Such an investigation will be much easified after the completion of more forecasts based upon different numerical approaches. In spite of the fact that no systematic investigation of the integration accuracy has yet been carried through the author will dare to write down as a third achievement of the particular method in question:

3. In spite of the short integration time and in spite of the simple means by which the integration is carried through, the solutions obtained seem to be good approximations to the solutions of the barotropic vorticity equation.

In this article weight will be put on explaining the particular features of the method in question which are responsible for the surprisingly short integration time.

It is hardly worth while mentioning that it is the author's hope that weather forecasting centers will start using the method. This is not because of an opinion that forecasts based upon a barotropic model always will give useful results. Certainly it will be necessary later to include effects from the changes in the mass-field of the atmosphere. However, provided the numerical solutions may be considered as useful approximations to barotropic developments the meteorologists will themselves be able to encircle all the cases where barotropic forecasts fail and thereby gain valuable experience with regard to the importance of additional physical factors and how they possibly could be incorporated into the whole complex of computation and arguments building up a forecast. In this connection it should be mentioned that the numerical approach presented in this paper also will be valuable when applied to more extended physical models. This will be seen already in this paper from some arguments concerning the solution of the so-called Helmholz-equation $\nabla^2\alpha - a^2\alpha = h$ which appears in more extended models. Otherwise, this problem of using the particular numerical approach which is discussed in this article for less specialized models will be discussed more throughly in an article to appear later.

2. The barotropic problem

In the barotropic model it is assumed that the motion at levels close to the 500 mb level is horizontal and non-divergent thereby conserving the vertical component of absolute vorticity by the additional assumption of conservative forces. To reveal most clearly the essential features of the particular numerical method which is discussed in this article it will at first be assumed that the motion is a plane one. The fundamental property of periodicity around the earth for any quantity α derived from some atmospheric variable may in the plane representation be expressed by writing the quantity as a Fourier series. Since we are always dealing with numerical approaches to the solutions we will approximate the quantity in question by a finite Fourier series

$$(1) \qquad \alpha = \sum^{(l_N)} \alpha_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m}+\frac{y}{\omega_n}\right)} + c$$

where l_N now indicates the shortest wavelength occuring in the series and c is the constant in the series.

The law of individual conservation of vorticity may be written

$$(2) \qquad \frac{\partial \hat{\eta}}{\partial t} = -\mathbf{v} \cdot \nabla \hat{\eta},$$

$\hat{\eta}$ symbolizing vorticity, $\mathbf{v}$ velocity and ∇ the del-operator in the plane of motion. Because of the assumption of non-divergence for the velocity field, velocity may be expressed by a stream function ψ, $\mathbf{v} = -\nabla\psi \times \mathbf{k}$, $\mathbf{k}$ denoting a unit vector perpendicular to the plane of motion. Vorticity is then given by $\nabla^2\psi = \hat{\eta}$. The problem is now to solve (2) under the condition of periodicity expressed in (1). To get numerical solutions one will have to replace $\nabla^2\psi$ by an approximation, the corresponding finite difference expression. One may write

$$(3) \qquad \hat{\eta} = \nabla^2\psi \approx -\frac{4}{d^2}(\psi - \overline{\psi})$$

where $\overline{\psi}$ is defined from

$$(4) \quad \overline{\psi} = \frac{1}{4}\Big\{\psi(x+d, y) + \psi(x-d, y) + \psi(x, y+d) + \psi(x, y-d)\Big\}.$$

One may remark that even if $\nabla^2\psi$ is computed from the approximation (3) we do not necessarily require the solutions for gridpoints only. Even if a numerical solution cannot from evident reasons yield a solution everywhere we will to begin with in theory require a solution valid for any x and y. If now $\hat{\eta} = -\frac{4}{d^2}(\psi - \overline{\psi})$ is substituted into (2) and the factor $-\frac{4}{d^2}$ divided out of the equation one obtains as a new continuous problem to be solved under condition (1):

$$(5) \qquad \frac{\partial \eta}{\partial t} = -\mathbf{v} \cdot \nabla \eta$$

where

$$\eta \equiv \psi - \overline{\psi}.$$

The numerical problem of solving (5) for a sufficiently short time $\triangle t$ may be divided into the two independent steps:

a) To displace properly the initial isolines η = const over a time interval $\triangle t$ to find the new position of these lines and by subtraction the increment $\triangle\eta$ up to time $\triangle t$.

b) To solve thereafter $\triangle\psi - \overline{\triangle\psi} = \triangle\eta$ to find the corresponding increment in the stream-function of the velocity field.

These two steps will be discussed separately. At first, however, we shall in the next section discuss more closely the importance of the operation defined in (4) and some consequences which may be drawn from this operation.

3. Some consequences of the operation $\overline{}$

Performing the operation $\overline{}$ L times on each term of the Fourier representation (1) one obtains

$$(6) \quad \overline{\alpha}^{\}L} = \sum^{(l_N)} A^L_{m,n}\, \alpha_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m} + \frac{y}{\omega_n}\right)} + c$$

with

$$(7) \quad A_{m,n} = \frac{1}{2}\left(\cos\frac{4d}{\lambda_m}\frac{\pi}{2} + \cos\frac{4d}{\omega_n}\frac{\pi}{2}\right).$$

From this last formula it may be concluded that for all components for which $4\,d/\lambda_m$ and $4\,d/\omega_n$ are not both equal to an even number one must have

$$|A_{m,n}| < 1.$$

The operation $\overline{}$ means therefore particularly a damping of all components with wavelengths $\lambda_m, \omega_n > 2\,d$. The damping is complete for $\lambda_m, \omega_n = 4\,d$, and is from here on decreasing asymptotically to no damping for $\lambda_m, \omega_n = 2\,d$, and to a reduced damping when λ_m, ω_n approach their highest values λ_1 and ω_1, respectively. By assuming $\lambda_m, \omega_n > 2\,d$ one obtains therefore by repeated $\overline{}$ operations in the limit:

$$(8) \quad \overline{\alpha}^{\}L} \underset{L \to \infty}{\longrightarrow} c, \text{ provided } \quad \lambda_m, \omega_n > 2\,d. \qquad l_N > 2\,d.$$

The expression for $\alpha - \overline{\alpha}$ will in a finite Fourier series be given by

$$(9) \qquad \alpha - \overline{\alpha} \equiv \sum^{(l_N)} B_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m} + \frac{y}{\omega_n}\right)}$$

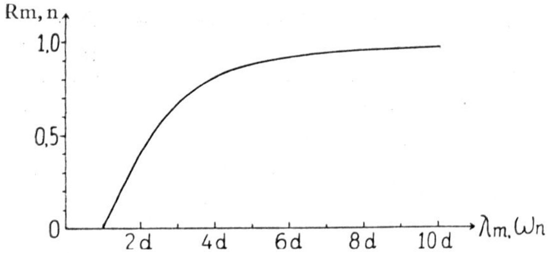

Fig. 1. The curve illustrates the goodness of the approximation (3) for different wavelengths.

where $B_{m,n}$ is given from

$$(10)\quad B_{m,n} = \left(\sin^2 \frac{4\,d}{\lambda_m}\frac{\pi}{4} + \sin^2 \frac{4\,d}{\omega_n}\frac{\pi}{4}\right) \alpha_{m,n}.$$

On the other hand $-\frac{d^2}{4} \nabla^2 \alpha$ will as obtained from differentiation of (1) be given by

$$(11)\quad -\frac{d^2}{4} \nabla^2 \alpha = d^2 \pi^2 \sum^{(l_N)} \left(\frac{1}{\lambda_m^2} + \frac{1}{\omega_n^2}\right) \alpha_{m,n} e^{2\pi i \left(\frac{x}{\lambda m} + \frac{y}{\omega n}\right)}$$

The ratio $R_{m,n}$ between corresponding coefficients in (9) and (11) gives us then a measure for how well the approximation $\nabla^2 \alpha = -\frac{4}{d^2}(\alpha - \overline{\alpha})$ is valid for the different components. We obtain $R_{m,n}$ from

$$(12)\quad R_{m,n} = \frac{\sin^2 \frac{4\,d}{\lambda_m}\frac{\pi}{4} + \sin^2 \frac{4\,d}{\omega_n}\frac{\pi}{4}}{d^2\pi^2/\lambda_m^2 + d^2\pi^2/\omega_n^2}$$

A curve is drawn in fig. 1 to illustrate the variation of $R_{m,n}$ for similar variations of λ_m and ω_n. As would have been expected $R_{m,n} < 1$, and is the closer to zero the smaller are the wave-lenghts. It is worth while mentioning, however, that it is the relative variation of $R_{m,n}$ and not the value of $R_{m,n}$ itself that introduces errors in the vorticity equation

$$\frac{\partial \hat{\eta}}{\partial t} = -\mathbf{v} \cdot \nabla \hat{\eta}$$

when $\hat{\eta}$ there is replaced by $-\frac{4}{d^2}(\psi - \overline{\psi})$ to give

$$\frac{\partial (\psi - \overline{\psi})}{\partial t} = -\mathbf{v} \cdot \nabla (\psi - \overline{\psi}).$$

The last equation may namely always be brought to give complete identity between the two equations for a single arbitrary component by multiplication with a suitable constant. The relative variation in $R_{m,n}$ is seen to be slight in the range of wavelengths $\lambda_m, \omega_n > 3\,d$, but some approximation to $\nabla^2 \alpha$ is obtained even down to $\lambda_m, \omega_n = 2\,d$. In the following we shall fix the range of wavelengths in which we may expect useful solutions of (2) from the solutions of (5) to the range

$$\lambda_m, \omega_n > 3\,d.$$

4. The problem of time-integration

In Lagrangian terms the time-integration of $\frac{\partial \eta}{\partial t} = -\mathbf{v} \cdot \nabla \eta$ may in theory be accomplished by repeated displacements of the isolines in velocity fields which are considered as constant for a sufficiently small time increment $\triangle t$. This Lagrangian formulation of the time-integration problem will make sure that the fundamental assumption of conservation of vorticity is realized during the computation in the sence that no isolines η = const then will be present at any later time which did not already exist in the beginning, nor will any isolines be lost.[1]

The assumption of constant velocity fields during the partial displacements of the isolines of vorticity introduces necessarily in the general case an error in the distribution of η at a later time t. One may, however, keep this error within arbitrarily small limits by choosing a sufficiently small value for $\triangle t$. Decisive for how rapid a forecast can be computed for a period T is the ratio $\frac{\triangle t}{T}$. In the atmosphere the actual velocity fields are usually changing so rapidly that one is forced to keep $\triangle t$ less

[1] In this connection it should be recalled that in the Princeton-forecasts referred to earlier the vorticity was not always very well conserved in this respect.

than 3—4 hrs. It is possible, however, to increase this time considerably if use is made of the following considerations:
We write

$$\psi \equiv \psi^\star + \psi^{\star\star}$$

and correspondingly

$$\mathbf{v} = \mathbf{v}^\star + \mathbf{v}^{\star\star}$$

where

$$(13) \qquad \psi^\star = \psi - c\,(\psi - \overline{\psi}), \;\; c = \text{const}$$

$$\psi^{\star\star} = c\,(\psi - \overline{\psi}).$$

Substituting $\mathbf{v} = \mathbf{v}^\star + \mathbf{v}^{\star\star}$ into the vorticity equation

$$\frac{\partial\,(\psi - \overline{\psi})}{\partial t} = -\mathbf{v} \cdot \nabla\,(\psi - \overline{\psi})$$

one gets the equivalent equation

$$(14) \qquad \frac{\partial\,(\psi - \overline{\psi})}{\partial t} = -\mathbf{v}^\star \cdot \nabla\,(\psi - \overline{\psi}).$$

The term $-\mathbf{v}^{\star\star} \cdot \nabla \eta$ drops namely out of the equation because $\mathbf{v}^{\star\star}$ is parallel to the isolines $\eta \equiv \psi - \overline{\psi} = \text{const}$ and consequently ineffective in changing the distribution of vorticity. One may therefore as well displace the equiscalar lines of vorticity in the fictitious $\mathbf{v}^\star$-fields as in the actual velocity fields. This will now imply a considerable advantage if this fictitious velocity field $\mathbf{v}^\star$ is considerably less varying in time than the actual velocity field itself. It is quite clear that if the $\psi^\star$-field shall possibly possess this property, the constant c in (13) cannot be taken too small, nor can it be taken too large. If particularly c is put equal to 1 one obtains

$$(15) \qquad \psi^\star = \overline{\psi}.$$

Provided a useful displacement field exists it would imply a considerable advantage from a practical point of view if it could be derived from (15) instead of (13) because initially $\overline{\psi}$ already exists when $\eta_{t=0}$ is computed, and because at later time-steps the solution of $\triangle\psi - \overline{\triangle\psi} = \triangle\eta$ gives us immediately the change in the $\overline{\psi}$-field. The question then arises what size should be taken for the distance d involved in the definition of ψ to make $\overline{\psi}$ most useful as a displacement field. Obviously, since $\overline{\psi} \to \psi$ when $d \to 0$ only slight advantage could possibly be gained if d were taken too small. Recalling the consequences of the operation $\overline{}$ it is clear that in order to make $\overline{\psi}$ most suitable as a displacement field d should be taken equal to the order of magnitude of the wavelengths of the most rapidly changing components of the ψ-field. Applied to the atmosphere the value

$$4\,d \approx 2{,}500 \text{ km}$$

suggests itself as a natural value in this connection. In the forecasts referred to in the introduction d was put equal to the length of 12 degrees long. at 60 degrees N. It then turned out to be possible to take $\triangle t = 24$ hrs. Appearingly this only introduced negligible errors in most cases in the forecasted position of the equiscalar lines of vorticity. The map in fig. 2 illustrates a case of computed changes in $\triangle\eta \sim \triangle(z - \overline{z})$, where z is the height of the 500 mb surface. The isolines $\triangle(z - \overline{z}) = \text{constant}$ are drawn for every 40 m. The map in fig. 3 illustrates the corresponding observed changes. As it may be seen the observed and computed changes generally compare well. It should be remarked that the presented case is not one which is particularly favourable to the method of displacing described in this article because it was a case with relatively strong changes in the mean field itself. It should also be remarked that only relative vorticities were displaced, and also that discrepancies between observed and computed *vorticity* changes when they are of a disorderly character always have a tendency to disappear when the integration of $\triangle\psi - \overline{\triangle\psi} = \triangle\eta$ have transformed the vorticity changes into corresponding changes in the velocity field.

It should now be noted that having found a distance d involved in the definition of $\overline{\psi}$ which makes $\overline{\psi}$ most suitable as a displacement field, this distance may be too large for a sufficiently accurate determination of the vorticities from the approximation $\nabla^2\psi = -\frac{4}{d^2}(\psi - \overline{\psi})$. Obviously if the distance d in this connection has to be taken much smaller, for instance equal to a $d_0 \ll d$, we have to return to the expression in (13) to get if possible

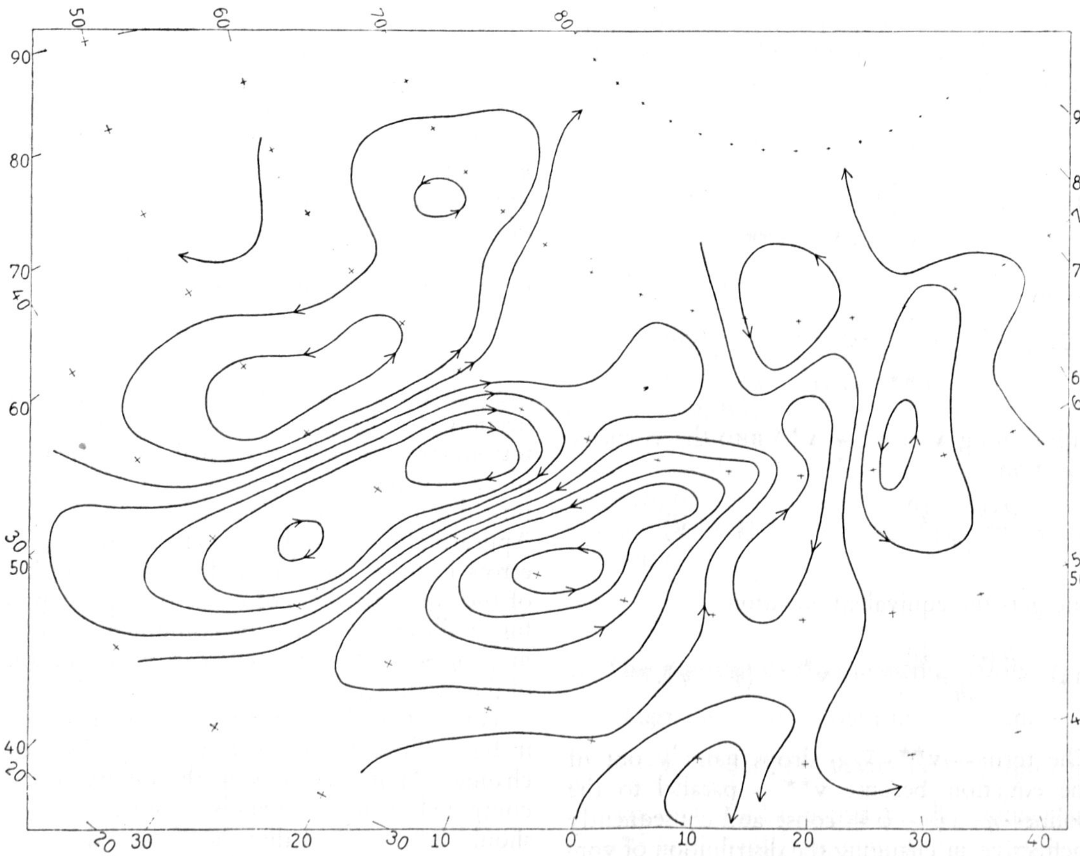

Fig. 2. November 25, 1951, 0300 GMT: The lines represent equiscalar lines for the computed 24-hr changes in $z - \bar{z}$, $\triangle (z - \bar{z})_{comp.}$, and are drawn for every 40 m. The arrows point in the directions $- \nabla \triangle (z - \bar{z})_{comp.} \times \mathbf{k}$.

a useful displacement field. We shall abstain from a discussion of what value should then at best be taken for the constant c in (13). Recalling namely what was pointed out at the end of the preceding section the approximation (3) will only be applicable for components with wavelengths down to approximately $3\,d$, which with the chosen value of $4\,d \approx 2{,}400$ km means down to wavelengths of the order of magnitude 1,800 km. Phenomena on a considerably smaller scale than this cannot easily be observed in the free atmosphere with any appreciable accuracy. It might also be that such phenomena, at least for relative short periods, do not have any noteable influence at all for the development of the more large-scale components of the motion. In that case there would also be a physical reason for abstaining from an attempt to include components with wavelengths considerably less than approximately 1,800 km.

5. The numerical solution of the Helmholz-equation $\nabla^2 \alpha - a^2 \alpha = - \hat{h}$.

When the changes $\triangle\eta$ in vorticity are found from the displacement of the isolines $\eta = \text{const}$, one is left with the problem of solving

$$\triangle\psi - \overline{\triangle\psi} = \triangle\eta .$$

This equation may be considered as a special case of

$$\alpha - M\bar{\alpha} = h \tag{16}$$

which, with M and h determined from

$$M = \frac{1}{1 + \frac{a^2 d^2}{4}} \tag{17}$$

and

$$h = - \frac{d^2 \hat{h}}{4M},$$

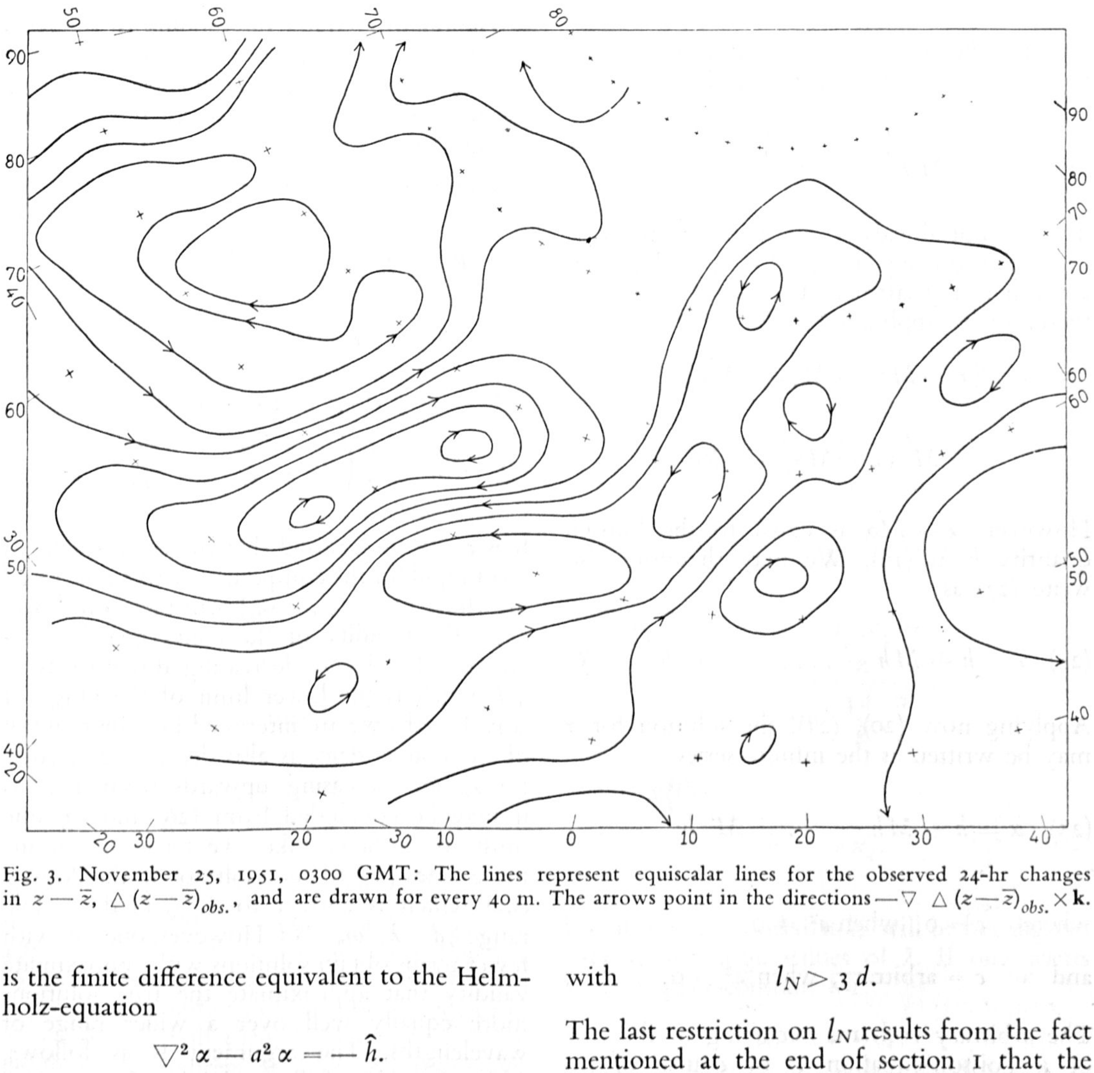

Fig. 3. November 25, 1951, 0300 GMT: The lines represent equiscalar lines for the observed 24-hr changes in $z - \bar{z}$, $\triangle(z - \bar{z})_{obs.}$, and are drawn for every 40 m. The arrows point in the directions $-\nabla \triangle (z-\bar{z})_{obs.} \times \mathbf{k}$.

is the finite difference equivalent to the Helmholz-equation

$$\nabla^2 \alpha - a^2 \alpha = -\hat{h}.$$

The following identity exists:

$$(18)\quad \alpha \equiv (\alpha - M\bar{\alpha}) + M(\bar{\alpha} - M\bar{\bar{\alpha}}) + \ldots + M^L\left(\overline{\vdots}\Big\}^{L}\alpha - M\,\overline{\vdots}\Big\}^{L+1}\alpha\right) + R_L.$$

where

$$(19)\quad R_L = M^{L+1}\,\overline{\vdots}\Big\}^{L+1}\alpha$$

We are looking for a solution of (16) where the required quantity α is expressed as a finite Fourier series

$$\alpha = \sum^{(l_N)} \alpha_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m} + \frac{y}{\omega_n}\right)} + c$$

with $\qquad l_N > 3\,d.$

The last restriction on l_N results from the fact mentioned at the end of section 1 that the approximation $\nabla^2\alpha \approx -\frac{4}{d^2}(\alpha - \bar{\alpha})$ does not apply to the components with wavelengths considerably smaller than $3\,d$. We may now apply the result of (8) in section 1 in (19). We then obtain:

$$(20)\quad R_L \underset{L \to \infty}{\longrightarrow} 0, \qquad a^2 \neq 0; \quad M < 1.$$

and

$$(21)\quad R_L \underset{L \to \infty}{\longrightarrow} c, \qquad a^2 = 0; \quad M = 1.$$

We make further the assumption that the relative variation of $M = \dfrac{1}{1 + \dfrac{a^2 d^2}{4}}$ as compared with the one of $\overline{\alpha}\}^{p}$ is sufficiently small to secure the approximate validity of

$$(22) \qquad \overline{M\overline{\alpha}\}^{p}} = M\overline{\alpha}\}^{p+1}$$

if a^2 is not already a constant. In the case of a Poisson equation, $a^2 = 0$, and (22) is authomatically satisfied. The identity (18) may therefore by applying (22) be written

$$(23) \quad \alpha \equiv (\alpha - M\overline{\alpha}) + M(\overline{\alpha - M\overline{\alpha}}) + \ldots + \\ + M^{L}\,\overline{(\alpha - M\overline{\alpha})}\}^{L} + R_L$$

However, $\alpha - M\overline{\alpha}$ is equal to the known quantity h in (16). We may therefore also write (23) as

$$(24) \quad \alpha = h + M\overline{h} + \ldots\ldots + M^{L}\overline{h}\}^{L} + R_L$$

Applying now (20), (21), the solution for α may be written as the infinite series

$$(25) \quad \alpha = h + M\overline{h} + \ldots + M^{L}\overline{h}\}^{L} + \ldots \\ + c$$

where $\quad c = 0$, when $a^2 \neq 0$

and $\quad c =$ arbitrary, when $a^2 = 0$.

The arbitrary constant occurring in the case of a Poisson-equation is of course of no physical importance, as we, for instance, may understand when α is equal to the stream function. In the case of a Poisson-equation $\alpha - \overline{\alpha} = h$ it is immediately understood that the function h cannot contain any constant in the Fourier-development. This may, however, be the case when we have a Helmholz-equation $\alpha - M\overline{\alpha} = h$. The solution for α will in this case contain a constant determined from the solution (24). By the application of the Helmholz-equation in the more extended computational models of the atmosphere α is the temperature-change. From physical reasons, now, it may easily be understood why temperature in the mean may increase or decrease in a horizontal level extending all around earth.

The question about the rapidity of the convergence for each Fourier component of the infinite series (24) may be discussed easily by means of the expression (19) for the remainer R_L in (24).

Using (6) one obtains

$$(26) \quad R_L \equiv \sum^{(l_n)} (R_L)_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m} + \frac{y}{\omega_n}\right)} \\ = \sum^{(l_n)} \alpha_{m,n}\,(A_{m,n}\cdot M)^{L+1}\, e^{2\pi i\left(\frac{x}{\lambda_m} + \frac{y}{\omega_n}\right)}$$

where now $A_{m,n}$ is given from

$$(7) \quad A_{m,n} = \frac{1}{2}\left(\cos\frac{4d}{\lambda_m}\frac{\pi}{2} + \cos\frac{4d}{\omega_n}\frac{\pi}{2}\right).$$

It is easily understood that the convergence is most rapid for the component with $\lambda_m, \omega_n = 4d$ in which case $A_{m,n} = 0$. From there on, however, the rapidity of the convergence is decreasing for λ_m, ω_n decreasing downwards to $3d$ which is the lower limit of the range of wavelengths we are interested in. The rapidity of the convergence is also decreasing steadily for λ_m, w_n increasing upwards from $4d$. As it may be concluded from (26) and (7) one must for instance take five terms in the infinite series (25) to get a solution in the Poisson case which is correct to 25 % in the whole range $4d < \lambda_m, \omega_n \leq 8d$. However, one can with fewer terms obtain solutions with approximate validity that approximate the true solutions more equally well over a wider range of wavelengths. The argument is as follows, taking for sake of illustration the Poisson case with $M = 1$:

Let now $\overline{}^{q}$ denote any of the operations $\overline{}^{1}, \overline{}^{2}, \ldots, \overline{}^{L}$ defined from

$$\overline{\beta}^{q} = \frac{1}{4}\Big\{\beta(x + d_q, y) + \beta(x - d_q, y) + \\ + \beta(x, y + d_q) + \beta(x, y - d_q)\Big\}$$

where

$$d_q < d_{q+1}, \quad q = 1, 2, \ldots, L.$$

Let us particularly define $\overline{\beta}^{0}$ from

$$\overline{\beta}^{0} = \beta.$$

We may then put down the following identity:

$$(27)\qquad \alpha \equiv \sum_{q=0}^{L}\left\{\left(\overset{\substack{-q\\ \vdots\\ -1\\ -0}}{\alpha}\right) - \overline{\left(\overset{\substack{-q\\ \vdots\\ -1\\ -0}}{\alpha}\right)}^{\,q+1}\right\} + r_L$$

where

$$(28)\qquad r_L = \overset{\substack{-L+1\\ \vdots\\ -1\\ -0}}{\alpha}$$

Corresponding to the result (6), (7), in section 3 one must have

$$(29)\qquad r_L = \sum^{(l_N)}\left(\prod_{q=1}^{L+1} A^{(q)}_{m,n}\right)\alpha_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m}+\frac{y}{\omega_n}\right)}$$

where

$$(30)\qquad A^{(q)}_{m,n} = \frac{1}{2}\left(\cos\frac{4\,d_q}{\lambda_m}\frac{\pi}{2} + \cos\frac{4\,d_q}{\omega_n}\frac{\pi}{2}\right).$$

For comparison the remainer R_L in (26) for the Poisson case with $M = 1$ is:

$$R_L = \sum^{(l_N)}\left(A_{m,n}\right)^{L+1}\alpha_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m}+\frac{y}{\omega_n}\right)}$$

While R_L is zero only for $\lambda_m, \omega_n = 4\,d$, r_L is seen to be zero for $\lambda_m, \omega_n = 4\,d_1, 4\,d_2, \ldots,$ $4\,d_{L+1}$. It would therefore so far be an advantage to use (27) instead of (18) as a base for obtaining approximate solutions of the Poisson equation $\alpha - \overline{\alpha} = 0$. It will then, however, further be necessary to express at least approximately the general term in the identity (27), after the remainder has been left out, in terms of the known function h. Denoting this general term with

$$P_q \equiv \left(\overset{\substack{-q\\ \vdots\\ -1\\ -0}}{\alpha}\right) - \overline{\left(\overset{\substack{-q\\ \vdots\\ -1\\ -0}}{\alpha}\right)}^{\,q+1},\quad q = 0, 1, \ldots L$$

and letting further Q_q be defined from

$$Q_q \equiv \left(\overset{\substack{-q\\ \vdots\\ -1\\ -0}}{\alpha}\right) - \overline{\left(\overset{\substack{-q\\ \vdots\\ -1\\ -0}}{\alpha}\right)}^{\,q},\quad q = 1, 2, \ldots, L$$

one obtains easily for the ratio between corresponding Fourier components of P_q and Q_q:

$$(31)\qquad \frac{\{P_q\}_{m,n}}{\{Q_q\}_{m,n}} = \frac{1 - A^{(q+1)}_{m,n}}{1 - A^{(q)}_{m,n}} \equiv k_{m,n}.$$

Hence by substitution for $A^{(q)}_{m,n}$ and $A^{(q+1)}_{m,n}$ from (30), one obtains

$$k_{m,n} = \frac{\sin^2\frac{4\,d_{q+1}}{\lambda_m}\frac{\pi}{4} + \sin^2\frac{4\,d_{q+1}}{\omega_n}\frac{\pi}{4}}{\sin^2\frac{4\,d_q}{\lambda_m}\frac{\pi}{4} + \sin^2\frac{4\,d_q}{\omega_n}\frac{\pi}{4}}.$$

With the notation

$$\varkappa_q \equiv \frac{d_{q+1}}{d_q},$$

$k_{m,n}$ will for similar variations of λ_m, ω_n, say $\lambda_m, \omega_n = \lambda$, vary as

$$k_\lambda = \frac{\sin^2\frac{\varkappa\,4\,d_q}{\lambda}\frac{\pi}{4}}{\sin^2\frac{4\,d_q}{\lambda}\frac{\pi}{4}}.$$

It is seen that

$$k_\lambda \underset{\lambda \to \infty}{\longrightarrow} \varkappa_q^2.$$

It is also understood that k_λ will be but slightly varying for large values of λ. If one wants k_λ as a quasi-constant for

$$\lambda_m, \omega_n > 4\,d_q,$$

one must take

$$\varkappa \leqq 1.5, \text{ approximately.}$$

In the following we shall take

$$(32)\qquad \varkappa = 1.5.$$

We may then replace k_λ with the constant 2 which is approximately the mean value of $k_{4\,d_q}$ and k_∞:

$$k_{m,n} \approx 2,\ \lambda_m, \omega_n \geqq 4\,d_q.$$

Consequently we may write (31) as:

$$(33)\qquad P_q \approx 2\,Q_q \equiv P_q^\star \equiv \sum^{(l_N)}\{P_q^\star\}_{m,n}\, e^{2\pi i\left(\frac{x}{\lambda_m}+\frac{y}{\omega_n}\right)}$$

provided

$$(34) \qquad \lambda_m, \omega_n \gtreqless 4\, d_q.$$

We note from the definitions of Q_q and P_q that

$$(35) \qquad Q_q = \overline{P_{q-1}}^{\,q}$$

Combining (33) and (35) one therefore gets:

$$P_q^\star = 2\, P_{q-1}, \quad \lambda_m, \omega_n \gtreqless 4\, d_q.$$

By repeated use of this relation and condition (34) one finally obtains in connection with (33):

$$(36) \qquad P_q^\star = 2^q\, \overline{P_0}^{\,0 \cdots q} \approx P_q$$

provided

$$\lambda_m, \omega_n \gtreqless 4\, d_q.$$

Using

$$\left\{\overline{P_0}^{\,0 \cdots q}\right\}_{m,n} = \left(\prod_{r=1}^{q} A_{m,n}^{(r)}\right) \{P_0\}_{m,n}$$

in connection with (36), we may write:

$$\{P_q^\star\}_{m,n} = 2^q \left(\prod_{r=1}^{q} A_{m,n}^{(r)}\right) \{P_0\}_{m,n},$$

or, when similar variations of λ_m, ω_n are considered:

$$\{P_q^\star\}_\lambda = 2^q \cos \frac{4\, d_1}{\lambda} \frac{\pi}{2} \cdot \cos \frac{4\, d_2}{\lambda} \frac{\pi}{2} \ldots \cos \frac{4\, d_q}{\lambda} \frac{\pi}{2} \times$$
$$\times \{P_0\}_{m,n}.$$

Consequently $P_q^\star$ is zero for $\lambda_m, \omega_n = 4\, d_1, 4\, d_2, \ldots, 4\, d_q$, and is also small for λ_m, ω_n having values between the values of $4\, d_1, 4\, d_2, \ldots, 4\, d_q$, if q is not taken to large. This is because the ratios $\frac{d_r}{d_{r-1}}$, $r = 2, \ldots, q$ have been chosen so small.

Making use of this last fact one may by substitution from (36) into the identity (27) write this approximately as:

$$(37) \quad \alpha \equiv P_0 + 2\, \overline{P_0}^{\,1} + \ldots + 2^L\, \overline{P_0}^{\,1 \cdots L} + r_L$$

for *all*

$$4\, d_1 \leq \lambda_m, \omega_n$$

if L is not taken too large. The relation (36) exists namely in parts of the range of wave-lengths which are considered, and in the remaining parts where it does not exist the terms in consideration are negligible anyway in comparison with other more dominating terms. The remainder r_L given from (29), (30) will be small in the whole interval

$$4\, d_1 \leq \lambda_m, \omega_n \leq 4\, d_{L+1}.$$

By disregarding the remainder in (37) above and by substituting

$$P_0 = \alpha - \overline{\alpha}^{\,1} = h$$

one should therefore get an approximate solution α of the Poisson-equation $\alpha - \overline{\alpha}^{\,1} = h$ as:

$$(38) \quad \alpha^\star = h + 2\, \overline{h}^{\,1} + \ldots + 2^L\, \overline{h}^{\,1 \cdots L}$$

provided

$$4\, d_1 \leq \lambda_m, \omega_n \leq 4\, d_{L+1}.$$

The accuracy of the solution may be examined for the different Fourier components in the following way:

We have

$$h_{m,n} = \{\alpha - \overline{\alpha}^{\,1}\}_{m,n} = \left(1 - A_{m,n}^{(1)}\right) \alpha_{m,n}.$$

If this is substituted into (38) one obtains for each Fourier component:

$$(39) \quad \frac{\alpha_{m,n}^\star}{\alpha_{m,n}} = \left(1 - A_{m,n}^{(1)}\right) \cdot \left(1 + 2\, A_{m,n}^{(1)} + \ldots + 2^L\, A_{m,n}^{(1)} \ldots A_{m,n}^{(L)}\right).$$

When this ratio is $+1$ the solution is correct for the corresponding component. In the tabulation below this ratio is given as a function of $\lambda_m, \omega_n = \lambda$ measured in the unit $4\, d_1$, and for L assuming successively the values 1, 2, 3.

Table I.

	$4\,d_1$	$5\,d_1$	$6\,d_1$	$7\,d_1$	$8\,d_1$	$9\,d_1$	$10\,d_1$	$11\,d_1$	$12\,d_1$	$16\,d_1$
$\alpha^{\star}_{m,n}/\alpha_{m,n}$, $L = 1$	1.00	1.12	1.00	.85	.82	.71	.62	.54	.36	—
$\alpha^{\star}_{m,n}/\alpha_{m,n}$, $L = 2$	1.00	.82	.97	1.04	1.00	.92	.85	.77	.67	—
$a^{\star}_{m,n}/\alpha_{m,n}$, $L = 3$	1.00	1.40	1.02	.85	.90	.89	.89	.88	.87	.76

It is seen that as approximate solutions of $\alpha - \overline{\alpha}^{1} = h$ we may use the following ones:

$$(40) \qquad \alpha^{\star}_{\mathrm{I}} = h + 2\,\overline{h}^{1}, \qquad 4\,d_1 \leq \lambda_m, \omega_n \leq 8\,d_1$$

$$(41) \qquad \alpha^{\star}_{\mathrm{II}} = h + 2\,\overline{h}^{1} + 4\,\overline{\overline{h}^{1}}^{2}, \qquad 4\,d_1 \leq \lambda_m, \omega_n \leq 10\,d_1$$

$$(42) \qquad \alpha^{\star}_{\mathrm{III}} = h + 2\,\overline{h}^{1} + 4\,\overline{\overline{h}^{1}}^{2} + 8\,\overline{\overline{\overline{h}^{1}}^{2}}^{3}, \qquad 4\,d_1 \leq \lambda_m, \omega_n \leq 16\,d_1.$$

If one wishes a solution with only two terms as in (40) which is better for wavelengths around $8\,d_1$ one could use a solution:

$$(43) \qquad \alpha^{\star\star} = h + 3\,\overline{h}^{1}.$$

The goodness of this approximate solution is measured by

$$\frac{\alpha^{\star\star}_{m,n}}{\alpha_{m,n}} = \left(1 - A^{(1)}_{m,n}\right) \cdot \left(1 + 3\,A^{(1)}_{m,n}\right)$$

which is tabulated below:

Table II.

	$4\,d_1$	$5\,d_1$	$6\,d_1$	$7\,d_1$	$8\,d_1$	$9\,d_1$	$10\,d_1$
$\alpha^{\star\star}_{m,n}/\alpha_{m,n}$ ...	1	1.35	1.25	1.14	.94	.83	.69

To demonstrate the usefulness already of the approximation (43) a map for the 24 hr. *observed* changes $(\triangle z)_{obs}$ in the height of the 500 mb surface, fig. 4, was used to compute corresponding "Laplacians" $(\triangle z - \overline{\triangle z}) \equiv h$ which was then in turn substituted into formula (42) to give a set of integrated values $(\triangle z)^{\star\star}$, fig. 5. As is seen the correspondence is surprisingly good in spite of the fact that the case was not a particularly favourable one to the integration method.

It is worth while mentioning that in the general case of a Helmholz-equation $\alpha - M\overline{\alpha} = h$, the convergence for larger wavelengths may be speeded up for the series (24) by proceeding in a similar way as was done for the Poisson case. But in the case of a Helmholz equation this is not as necessary as in the case with a Poisson equation because the convergence for larger wavelengths have a tendency to be rapid already because of the multiplication with M^{L+1} in the remainer (26).

6. Modifications due to the curvature of the earth. — Geostrophic approximation

The curvature of the earth modifies the results of the preceding section physically and geometrically. Let now ∇_s denote the spherical del-operator and $\mathbf{k}$ a vertical unit vector. Relative velocity $\mathbf{v}$, and relative vorticity $\hat{\eta}$ will then be expressed by a stream function ψ as follows:

$$\mathbf{v} = -\nabla_s \psi \times \mathbf{k}$$

and

$$\hat{\eta} = \nabla^2_s \psi.$$

If

$$f = 2\,\Omega \sin \varphi$$

where Ω is the angular velocity of the earth and φ is the latitude, the vertical component of absolute voticity $\hat{\eta}_{abs}$ will be given by:

$$\hat{\eta}_{abs} = \nabla^2_s \psi + f.$$

The conservation of absolute vorticity may now be expressed by:

$$(44) \qquad \frac{\partial \hat{\eta}_{abs}}{\partial t} = -\mathbf{v} \cdot \nabla_s \hat{\eta}_{abs}.$$

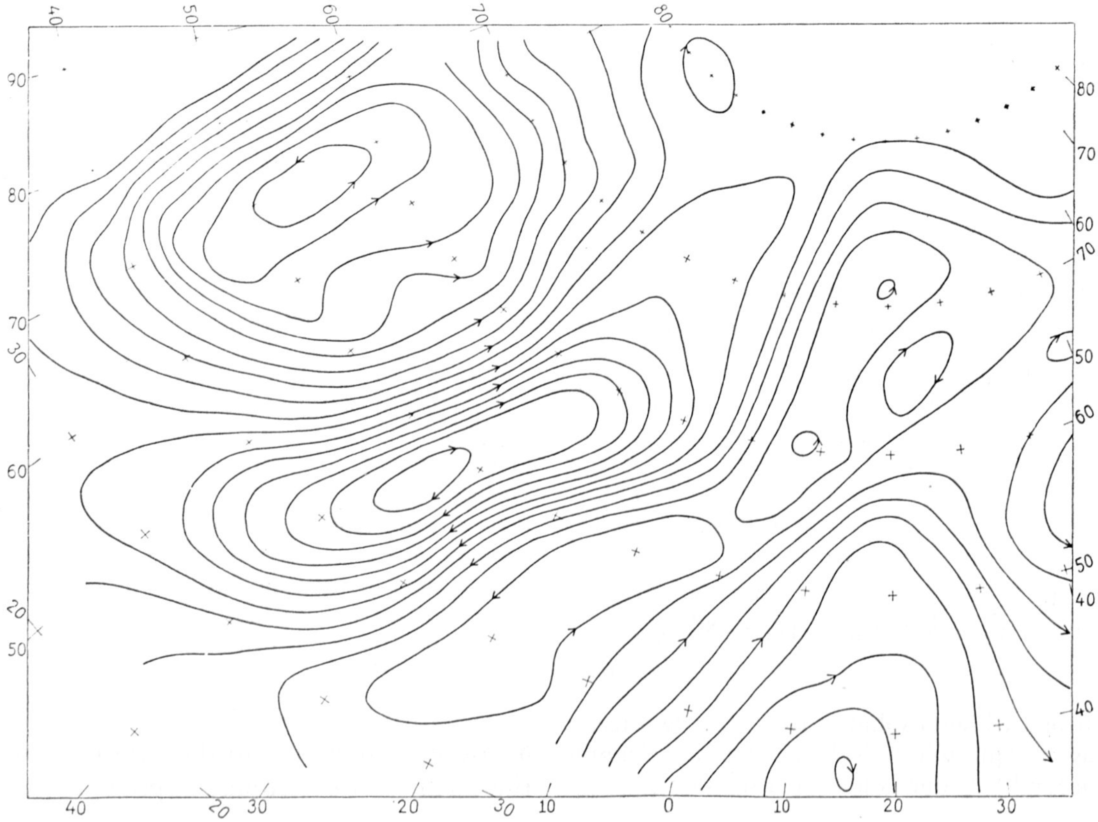

Fig. 4. November 25, 1951, 0300 GMT: The lines represent equiscalar lines for the observed 24-hr changes in z, $\triangle z_{obs.}$, and are drawn for every 40 m. The arrows point in the directions $-\nabla \triangle z_{obs.} \times \mathbf{k}$.

To find a useful displacement field one may proceed in exactly the same way as in section 4. We then write

$$\psi \equiv \psi^{\star} + \psi^{\star\star}$$

and correspondingly

$$\mathbf{v} = \mathbf{v}^{\star} + \mathbf{v}^{\star\star}$$

where

$$\psi^{\star} = \psi - c\,(\nabla_s^2 \psi + f), \quad c = \text{const}$$

and

$$\psi^{\star\star} = c\,(\nabla_s^2 \psi + f).$$

Substituting $\mathbf{v} = \mathbf{v}^{\star} + \mathbf{v}^{\star\star}$ into (44) one obtains

$$\frac{\partial \hat{\eta}_{abs}}{\partial t} = -\mathbf{v}^{\star} \cdot \nabla_s \hat{\eta}_{abs} \tag{45}$$

since the term $\mathbf{v}^{\star\star} \cdot \nabla_s \hat{\eta}_{ab}$ drops out because $\mathbf{v}^{\star\star}$ is parallel to the isolines of absolute vorticity and therefore ineffective in changing the vorticity distribution. Eq. (45) above therefore expresses the conservation of absolute vorticity in the field

$$\mathbf{v}^{\star} = -\nabla_s \left\{\psi - c\,(\nabla_s^2 \psi + f)\right\} \times \mathbf{k}.$$

We now introduce the geostrophic approximation:

$$\mathbf{v} = -\frac{g\,\nabla_s z \times \mathbf{k}}{f}.$$

Hence relative vorticity will be given as

$$\nabla_s^2 \psi = \frac{g}{f}\,\nabla_s^2 z + \nabla_s \frac{1}{f} \times f\mathbf{v} \cdot \mathbf{k}$$

or approximately as

$$\nabla_s^2 \psi \approx \frac{g}{\ }\,\nabla_s^2 z \tag{46}$$

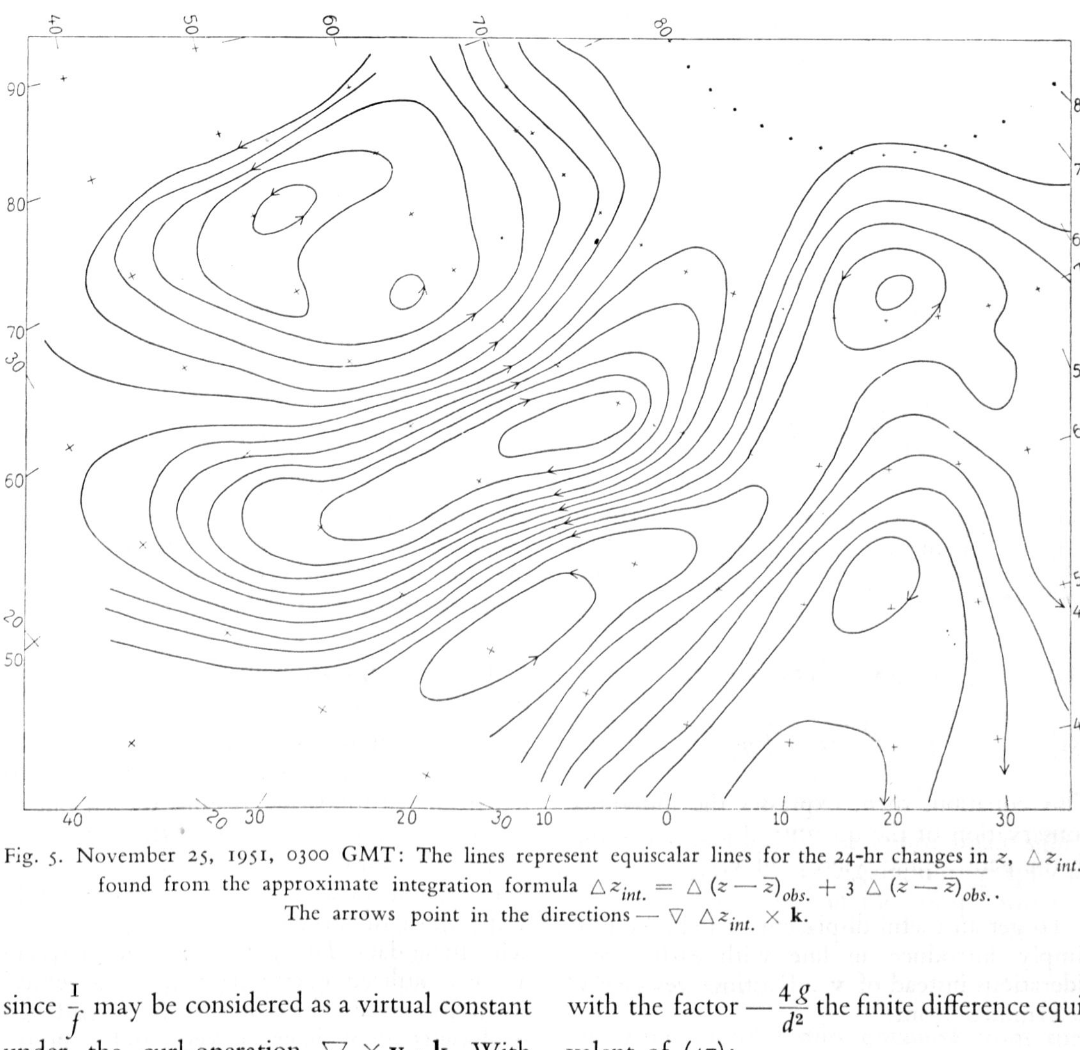

Fig. 5. November 25, 1951, 0300 GMT: The lines represent equiscalar lines for the 24-hr changes in z, $\triangle z_{int.}$, found from the approximate integration formula $\triangle z_{int.} = \triangle (z - \bar{z})_{obs.} + 3\, \overline{\triangle (z - \bar{z})}_{obs.}$. The arrows point in the directions $-\nabla\, \triangle z_{int.} \times \mathbf{k}$.

since $\frac{1}{f}$ may be considered as a virtual constant under the curl-operation $\nabla_s \times \mathbf{v} \cdot \mathbf{k}$. With (46) the vorticity-equation (44) can be written

$$(47) \quad \frac{g}{f} \frac{\partial \nabla_s^2 z}{\partial t} = -\mathbf{v}_s \cdot \nabla_s \left(\frac{g}{f} \nabla_s^2 z + f \right).$$

Working in a conformal map projection with the magnification factor m, one obtains

$$\nabla_s^2 z = m^2 \nabla^2 z.$$

Here ∇ denotes the plane del-operator. Substituting the finite difference approximation

$$\nabla^2 z = -\frac{4}{d^2} (z - \bar{z}), \quad d = \text{const.}$$

in the last equation one obtains after division with the factor $-\frac{4g}{d^2}$ the finite difference equivalent of (47):

$$\frac{m^2}{f} \frac{\partial (z - \bar{z})}{\partial t} = -\mathbf{v} \cdot \nabla_s \left\{ \frac{m^2 (z - \bar{z})}{f} - \frac{d^2 f}{g\, 4} \right\}.$$

It may be supposed that the relative variation in $\frac{m^2}{f}$ is small compared with the relative variation in the quantity $z - \bar{z}$. The last equation may therefore also be written with sufficient accuracy:

$$(48) \quad \frac{m^2}{f} \frac{\partial (z - \bar{z})}{\partial t} = -\frac{m^2}{f} \mathbf{v} \cdot \nabla_s (z - \bar{z}) + $$

$$+ \frac{m^2}{f} \mathbf{v} \cdot \frac{f d^2}{m^2\, 4g} \nabla_s f.$$

Let us suppose that m is a function of latitude only. Then the last term in (48) may be written as

$$\frac{m^2}{f} \mathbf{v} \cdot \frac{4 \Omega^2 d^2 \sin \varphi \cos \varphi}{m^2 (\varphi) \, 4 g} \nabla_s \varphi$$

or, with the notation $J(\varphi)$ defined from

$$(49) \quad J(\varphi) \equiv \int \frac{4 \Omega^2 d^2 \sin \varphi \cos \varphi \, d\varphi}{m^2 (\varphi) \, 4 g}$$

as

$$\frac{m^2}{f} \mathbf{v} \cdot \nabla_s J(\varphi).$$

Substituting now this for the last term in (48) one obtains after dividing out the factor $\frac{m^2}{f}$:

$$(50) \quad \frac{\partial \xi}{\partial t} = - \mathbf{v} \cdot \nabla_s \xi, \quad \text{where}$$

$$(51) \quad \xi \equiv z - \bar{z} - J(\varphi).$$

This equation, then, expresses the individual conservation of the quantity $\xi = z - \bar{z} - J(\varphi)$ in the geostrophic wind field $\mathbf{v}$.

To get an useful displacement field we now simply introduce in line with earlier considerations instead of $\mathbf{v}$ a fictitious velocity $\mathbf{v}^\star$ determined from:

$$\mathbf{v}^\star = - \frac{g}{f} \nabla_s z^\star \times \mathbf{k}$$

where

$$z^\star = z - c \left\{ z - \bar{z} - J(\varphi) \right\}, \quad c = \text{const.}$$

Choosing the constant equal to unity one gets:

$$z^\star = \bar{z} + J(\varphi)$$

and correspondingly

$$\mathbf{v}^\star = \bar{\mathbf{v}} + \mathbf{c}_d$$

with

$$(52) \quad \mathbf{c}_d \equiv - \frac{g}{f} \nabla_s J(\varphi) \times \mathbf{k}.$$

Instead of (50) one obtains now the equivalent equation

$$(53) \quad \frac{\partial \xi}{\partial t} = - (\bar{\mathbf{v}} + \mathbf{c}_d) \cdot \nabla \xi.$$

This equation now expresses the conservation of the quantity ξ in the field $\bar{\mathbf{v}} + \mathbf{c}_d$. $\mathbf{c}_d$ as obtained from (52) and (49) is seen to represent a velocity directed westwards and is in fact the velocity of propagation of a so-called Rossby-wave with wavelengths approximately equal to $4\,d$. Since d has been taken to be only equal to the length of 12 degr. long. at 60 degr. N this wavespeed will be a small one and in most cases only imply a small correction to the field $\bar{\mathbf{v}}$. The main contribution from the variability in the Coriolis parameter will result from the term $+ \bar{\mathbf{v}} \cdot \nabla_s J(\varphi)$ when the smoothed fields $\bar{\mathbf{v}}$ have noteable components in meridional direction.

7. Practical operation in a weather forecasting section

To accomplish a forecast in accordance with the principles outlined in this article it has from the following reasons turned out to be very convenient to make extended use of graphical additions and subtractions, rather than reading data in selected gridpoints only and furnish the necessary computational work with these data: *Firstly*, the analyzed map is in a sence utilized everywhere when graphical methods are used, whereas in a grid the gridpoints may have a very unfavourable position for a sufficiently accurate determination of the required quantities. *Secondly*, the graphical method is much faster than the grid method. For instance the whole time for the computation of a map for the isolines $z - \bar{z} =$ const reckoned from the conclusion of the analysis of the 500 mb map amounts to only 30—45 min, the area of the map being approximately as large as the ones covered by the maps presented in this article. *Thirdly*, one gets in the graphical method the isolines for the required quantities directly.

At a weather forecasting centre, the actual operational procedure then has to be as follows, or approximately as follows:

I: Construction of the initial field $\xi \equiv z - \bar{z} - J(\varphi) =$ const.

II: Displacement of the isolines $\xi =$ const in the displacement field $\bar{\mathbf{v}} + \mathbf{c}_d$, to find the new position of the lines after 24 hrs.

III: Solving the equation $\triangle z - \overline{\triangle z} = \triangle \xi$ to find the changes in the height z of the 500 mb map after 24 hrs.

I.

The 500 mb. map, here denoted as the z-map is copied over on a blank map displaced in an arbitrary direction x the distance $d =$ equal to the length of 12 degr. long. at 60 degr. N. Only every second line is copied. This gives us a z_{x+d}-map for the isolines $z_{x+d} \equiv {}^1/_2 z(x+d, y) =$ const. Thereafter is the z-map copied over once more on to the z_{x+d}-map, but now displaced a distance $-d$ in the x-direction. Again only every second isoline is copied. This gives us a map for the isolines $z_{x-d} =$ const. By graphical addition of the z_{x+d}-map and the z_{x-d}-map one obtains when only every second line is drawn, a map representing the field

$$z^{\mathrm{I}} \equiv \frac{1}{4}\left\{z_{x+d} + z_{x-d}\right\} = \text{const.}$$

By repeating this for the direction perpendicular to the chosen x-direction one obtains similarly a representation for the isolines

$$z^{\mathrm{II}} \equiv \frac{1}{4}\left\{z_{y+d} + z_{y-d}\right\} = \text{const.}$$

Then by the graphical addition of the z^{I}-map and the z^{II}-map above one obtains

$$z^{\mathrm{I}} + z^{\mathrm{II}} = \bar{z}.$$

Then finally from a graphical subtraction of the Z-field from the field $z - J(\varphi)$ one obtains the initial distribution of ξ given as isolines

$$\xi = z - \bar{z} - J(\varphi) = \text{const.}$$

It will obviously be most practical to have the isolines $J(\varphi) =$ const printed on blank maps once for all.

II.

The isolines $\xi =$ const are displaced in the field $\bar{\mathbf{v}}$ and then afterwards corrected for the relatively small westwards displacement due to the velocity $\mathbf{c}_d$. A wind-ruler of the ordinary hyperbolic type may be used. Instead of beginning with the displacement of the isolines $\xi =$ const, one may at first draw a suitable line crossing all the isolines $z =$ const and then compute the displacement of this line for $L = \pm\triangle t,\ 2\triangle t, \ldots,$ until the whole map is sufficiently densely covered of such lines. The displacement of the isolines $\xi =$ const will then afterwards be relatively easy.

III.

After having got the new position of the ξ-lines one obtains $\triangle \xi$ from the graphical subtraction

$$\xi_{24} - \xi_0 = \triangle \xi.$$

For integration of the equation

$$\triangle z - \overline{\triangle z} = \triangle \xi$$

one may use either of the approximate solutions (40), (43) or (41). Again, only graphical additions are involved in the computations.

8. Final remarks

In some future results obtained from the integration method in this article will be presented more systematically, also its extension to more extended models where absolute vorticity is not conserved. When the 45 forecasts referred to in the introduction were computed the integration formula (40) was used. It has later turned out that an use of the integration formula (41) instead, or even of (43), would have improved the integration results considerably in many cases. Also in most of the forecasts the influence from the variation of the Coriolis parameter was completely disregarded. It would therefore not be fair to the method to publish actual results at the present time. It will particularly be interesting to see how the forecasts compare with the ones obtained with electronic computers.

The time necessary for carrying out the three steps above amounts to between 2 and 3 hrs. After considerable experience one should expect that it will be possible to bring the time down to 2 hrs., particularly if one could use carbonpaper for the copying of the maps involved in the integration work. Having brought the time down to 2 hrs one could of course take the time and use systematically the more accurate integration formula (41). One could also possibly forecast for 48 hrs. if the barotropic model applies to such time intervals.

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