Theory of Development and of Thickness Patterns

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Abstract

The equations of motion, of continuity, of vorticity and of heat-transfer are transformed from coordinates (x, y, z) to coordinates (x, y, p) without the use of the hydrostatic equation in the vertical. An approximation based on the empirical fact that the isobaric surfaces are slightly inclined to the horizontal together with the use of dimensionless variables, gives the hydrostatic equation and simplifies the fundamental equations by the rejection of many of their terms. Thereafter definitions of certain types of geostrophic and nongeostrophic motions lead, respectively, to (a) Rossby's potential-vorticity equation; (b) the development and thickness-patterns theory of R. C. Sutcliffe, which is discussed in detail; and (c) J. G. Charney's (1949) treatment of the equivalent barotropic atmosphere. These models of atmospheric motions are compared and contrasted.

1. Introduction

In recent years, many investigations in dynamical meteorology have dealt with the vorticity equation, which is a formula giving the rate of change "following the motion" of the vertical component of vorticity of the air. Attention has also been directed to the part played by the rate of change "following the motion" of the pressure, and to the use of upper air charts of "thickness patterns", which exhibit the difference in height between two isobaric surfaces above some part of the Earth's surface. Amongst investigations of this kind is the theory of development and thickness patterns due to R. C. SUTCLIFFE (1947, 1950), A. G. Forsdyke and their co-workers, papers to which we shall refer hereafter as SI and SII. The first paper deals with the question of development, the main formulae being SI (10), (11) and (14); whilst the second is concerned with the use of a formula, SII (3), for the local rate of change of the thickness between two isobaric surfaces. In view of its importance in current British meteorological research, it has been thought worth while to analyze again from first principles the mathematical basis of this theory. The investigation given in the following pages seeks to discover how far the formulae referred to above depend on empirical approximations, on one hand, and on *a priori* definitions of certain types of motion, presumed to be possible in the atmosphere, on the other. The plan of the work is as follows: — In par. 3, the fundamental equations of motion, continuity, vorticity and heat-transfer are transformed from local rectangular coordinates (x, y, z) to coordinates (x, y, p), where p is the pressure, without using the assumption that the atmosphere is (vertically) in hydrostatic equilibrium. This part of the analysis is a straightforward application of the general theory of transformations given by the author (McVITTIE 1948, 1949), papers hereafter referred to as MI and MII, and therefore most of the mathematical details are omitted. In par. 4, dimensionless variables are introduced and it is shown how the hydrostatic approximation is derived from a numerical approximation depending on the fact that the isobaric surfaces are only slightly inclined to the horizontal. By the same approximation, many terms in the fundamental equations can be neglected and these equations cleared of all the parameters introduced by the conversion to dimensionless variables. The vorticity equation is dealt with in par. 5 and it is shown that, by means of *a priori* definitions, the Rossby potential-vorticity equation and the formulae SI (10), (11) and (14) can be obtained. The latter are more restrictive than the former. In par. 6, the thickness formula SII (3) is discussed and a criterion for the order of magnitude of its various terms suggested. It is pointed out that SII (3) cannot be expected, by itself, to throw much light on the dynamics of atmospheric motions. It plays a part analogous to a boundary condition and so must be combined with the formulae of par. 5 before dynamical conclusions can be reached. Finally in par. 7 we apply the theory to the equivalent-barotropic atmosphere and contrast the results with those of Sutcliffe's method.

The Coriolis parameters,

$$k = 2\omega \cos \varphi, \;\; l = 2\,\omega \sin \varphi,$$

are treated throughout as constants, which, at first sight, appears to make the treatment less general than SUTCLIFFE's (1947). But, in fact, Sutcliffe makes little use of the variation of l with latitude. If it is desired to take account of the Earth's curvature, it is better, in the present author's opinion, to abandon local rectangular coordinates and use a system appropriate to a curved Earth (MCVITTIE 1951). The theory of par. 3 onwards could easily be re-written in terms of such a coordinate system.

The author wishes to emphasize that he is concerned with mathematical theory and not with practical forecasting. The numerical values of meteorological elements, particularly in the upper air, are usually known only to a limited degree of accuracy. Now if it happens that a relation between them is hit upon which is useful in practical forecasting, this fact by itself does not establish the relation *theoretically*: its usefulness may be contingent on the inaccuracy of the data. The theoretician is then confronted by the problem: how, if at all, can the empirically useful relation be derived from the fundamental equations of hydrodynamics and thermodynamics? To argue that a solution of this problem is superfluous is, in the present author's opinion, to relinquish the hope of understanding meteorological processes.

2. Pressure as vertical coordinate. Notation

At a point O on the Earth's surface, whose N. latitude is φ, the conventional local rectangular coordinates are set up. The plane Oxy is the tangent plane to the Earth's surface at O, Ox is drawn to the East and Oy to the North, whilst Oz coincides with the vertical at O. Consider an unsteady motion of the atmosphere so that an isobaric surface has the equation

$$p\,(x, y, z, t) = C, \tag{2.1}$$

where C is a constant. If we solve this equation for z, we express this coordinate as a function of x, y, t and C. Hence for all values of C we can write

$$z = z\,(x,\; y,\; p,\; t) \tag{2.2}$$

in which each value of p is associated with one isobaric surface. We regard (2.2) as an equation which permits us to replace (x, y, z) by (x, y, p) as the coordinates describing the position at time t of an elementary mass of air. Moreover any function, f, of (x, y, z, t) can, by (2.2), also be regarded as a function of (x, y, p, t) and we need a notation for the partial derivatives of f in the two cases. When f is regarded as a function of (x, y, z, t) we shall denote its partial derivatives by $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial t}$; but when f is regarded as a function of (x, y, p, t), its partial derivatives* will be denoted by f_x, f_y, f_p, f_t. Consider now a set of five differentials δx, δy, δz, δt, δp related by

$$\delta p = \frac{\partial p}{\partial x}\,\delta x + \frac{\partial p}{\partial y}\,\delta y + \frac{\partial p}{\partial z}\,\delta z + \frac{\partial p}{\partial t}\,\delta t$$

*In SI, these partial derivatives are written

$$\left(\frac{\partial f}{\partial x}\right)_p, \;\left(\frac{\partial f}{\partial y}\right)_p, \;\frac{\partial f}{\partial p}, \;\left(\frac{\partial f}{\partial t}\right)_p$$

and by .

$$\delta z = z_x \delta x + z_y \delta y + z_p \delta p + z_t \ \delta t.$$

Substitution of either of these expressions into the other must lead to an identity in the differentials and we thus obtain

$$\left.\begin{aligned} &\frac{\partial p}{\partial x} + \frac{\partial p}{\partial z} z_x = 0, \quad \frac{\partial p}{\partial y} + \frac{\partial p}{\partial z} z_y = 0, \\ &1 - \frac{\partial p}{\partial z} z_p = 0, \quad \frac{\partial p}{\partial t} + \frac{\partial p}{\partial z} z_t = 0, \end{aligned}\right\} \tag{2.3}$$

in which, of course, p is regarded as a function of (x, y, z, t) and z as a function of (x, y, p, t). As to the operator for differentiation following the motion, we write, when $f\ (x, y, z, t)$,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}, \tag{2.4}$$

in which u, v, w are also regarded as functions of (x, y, z, t): but when $f\ (x, y, p, t)$, we write

$$\frac{Df}{Dt} = f_t + u f_x + v f_y + \tilde{\omega} f_p, \tag{2.5}$$

where, u, v, $\tilde{\omega}$ are functions of (x, y, p, t) and

$$\tilde{\omega} = \frac{Dp}{Dt}. \tag{2.6}$$

In Sutcliffe's terminology, a weather situation in which $\tilde{\omega}$ is different from zero, is said to be "developing", whilst if $\tilde{\omega}$ is zero, there is no development.

If we identify f with p and use (2.3), it is easy to show that

$$\frac{dp}{dt} = \tilde{\omega}, \tag{2.7}$$

so that $\tilde{\omega}$ is the result of transforming

$$\frac{d\,p\,(x, y, z, t)}{dt}$$

by (2.2).

In the following pages we shall be concerned with the (x, y, p) system in which w, strictly speaking, plays no part being replaced by Dz/Dt. Nevertheless is it convenient to retain w and to write

$$w\ (x, y, p, t) \equiv z_t + u z_x + v z_y + \tilde{\omega} z_p. \tag{2.8}$$

3. The kinetic metric and the fundamental equations

In order to translate the general formulae of MI and MII to our special case we shall bear in mind the following identifications

$$\left.\begin{aligned} &x^4 = t, \ x^1 = x, \ x^2 = y, \ x^3 = p, \\ &\dot{x} = u^1 = u, \ \dot{y} = u^2 = v, \ \dot{p} = u^3 = \tilde{\omega}, \end{aligned}\right\} \tag{3.1}$$

in accordance with the usual convention of the tensor calculus. If then K is the kinetic energy of unit mass of air we have (MI (4.4)), in the (x, y, z) system,

$$K = \frac{1}{2} \{ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + [k\,(z + a) - ly]\,\dot{x} + \\ + lxy - kx\dot{z} \}, \tag{3.2}$$

where a is the Earth's radius. In this expression for the kinetic energy the "centrifugal" part — proportional to w^2 — is neglected for it can be regarded, in the equations of motion, as being incorporated in g. Using (2.2), (2.8) we have, in the (x, y, p) system

$$K = \frac{1}{2} \{ u^2 + v^2 + (z_t + u z_x + v z_y + \tilde{\omega} z_p)^2 + \\ + [k\,(z + a) - ly]\,u + lxv - \\ - kx\,(z_t + u z_x + v z_y + \tilde{\omega} z_p) \}. \tag{3.3}$$

By (3.1) the corresponding kinetic metric is, (MII par. 2),

$$ds^2 = \left(1 - \frac{\gamma_{44}}{c^2}\right)(dx^4)^2 - \\ - \frac{1}{c^2} \sum_{q, r = 1}^{3} (\gamma_{qr}\, dx^q dx^r + 2\,\gamma_{q4}\, dx^q dx^4), \tag{3.4}$$

where

$$\left.\begin{aligned} &\gamma_{44} = z_t^2 - kxz_t, \\ &\gamma_{14} = \frac{1}{2}\{2\,z_t z_x + k\,(z + a) - ly - kxz_x\}, \\ &\gamma_{24} = \frac{1}{2}\{2\,z_t z_y + lx - kxz_y\}, \\ &\gamma_{34} = \frac{1}{2}\{2\,z_t z_p - kxz_p\}, \\ &\gamma_{11} = 1 + z_x^2, \ \gamma_{22} = 1 + z_y^2, \ \gamma_{33} = z_p^2, \\ &\gamma_{12} = z_x z_y, \ \gamma_{13} = z_x z_p, \ \gamma_{23} = z_y z_p. \end{aligned}\right\} \tag{3.5}$$

Since γ_{12}, γ_{13}, γ_{23} are not zero, the system (x, y, p) is not an orthogonal coordinate-system, but is "skew". We shall require the determinant, $\triangle$, of the γ_{qr} $(q, r=1, 2, 3)$ whose value by (3.5) is

$$\triangle = \begin{vmatrix} 1+z_x^2 & z_x z_y & z_x z_p \\ z_y z_x & 1+z_y^2 & z_y z_p \\ z_p z_x & z_p z_y & z_p^2 \end{vmatrix} = z_p^2. \quad (3.6)$$

It is essential that $\triangle \neq 0$, a condition satisfied in the atmosphere where z_p cannot be zero.

(i) We can now write down the expressions for the *divergence* and the *equation of continuity* which, by (3.1), (3.6) above and MII (3.4) and (5.12), are respectively

$$\Theta = \frac{1}{z_p}\frac{Dz_p}{Dt} + u_x + v_y + \tilde{\omega}_p, \quad (3.7)$$

$$\frac{1}{\varrho z_p}\frac{D(\varrho z_p)}{Dt} + u_x + v_y + \tilde{\omega}_p = 0 \quad (3.8)$$

(ii) Since the coordinate-system is skew the simplest way of finding the *equations of motion* is to use the Lagrangian method (McConnell 1931). These equations in terms of the notation (3.1) are

$$\frac{D}{Dt}\left(\frac{\partial K}{\partial u^q}\right) - \frac{\partial K}{\partial x_q} = -\frac{1}{\varrho}\frac{\partial p}{\partial x^q} + F_q,$$
$$(q = 1, 2, 3), \quad (3.9)$$

where F_q is the (generalised) force, apart from the pressure-gradient. In the (x, y, p) system we have, obviously,

$$p_x = 0, p_y = 0, p_p = 1, \quad (3.10)$$

whilst if gravity is the only external force acting, F_q is derivable from the potential gz so that

$$F_1 = -gz_x, \; F_2 = -gz_y, \; F_3 = -gz_p. \quad (3.11)$$

Using (3.3) in (3.9) and also taking account of (3.5), (3.10) and (3.11), the equations become, after some calculation,

$$\left.\begin{aligned} &\gamma_{11}\frac{Du}{Dt} + \gamma_{12}\frac{Dv}{Dt} + \gamma_{13}\frac{D\tilde{\omega}}{Dt} = \\ &-(g+b-ku)z_x - kw + lv, \\ &\gamma_{12}\frac{Du}{Dt} + \gamma_{22}\frac{Dv}{Dt} + \gamma_{23}\frac{D\tilde{\omega}}{Dt} = \\ &\quad -(g+b-ku)z_y - lu, \\ &\gamma_{13}\frac{Du}{Dt} + \gamma_{23}\frac{Dv}{Dt} + \gamma_{33}\frac{D\tilde{\omega}}{Dt} = \\ &-\left(\frac{1}{\varrho} + gz_p\right) - (b-ku)z_p, \end{aligned}\right\} \quad (3.12)$$

where

$$b = \frac{Dz_t}{Dt} + u\frac{Dz_x}{Dt} + v\frac{Dz_y}{Dt} + \tilde{\omega}\frac{Dz_p}{Dt}. \quad (3.13)$$

We can solve equations (3.12), with the help of (3.6), for the relative accelerations and thus obtain the final form of the *equations of motion* in the (x, y, p) system, viz: —

$$\frac{Du}{Dt} - lv + kw = -gz_x + \frac{z_x}{z_p}\left(\frac{1}{\varrho} + gz_p\right), \quad (3.14)$$

$$\frac{Dv}{Dt} + lu = -gz_y + \frac{z_y}{z_p}\left(\frac{1}{\varrho} + gz_p\right), \quad (3.15)$$

$$\frac{D\tilde{\omega}}{Dt} + \frac{z_x}{z_p}(lv - kw) - \frac{z_y}{z_p}lu - \frac{g}{z_p}(z_x^2 + z_y^2) + \frac{1}{z_p}(b-ku) = -\frac{1}{z_p^2}(1 + z_x^2 + z_y^2)\left(\frac{1}{\varrho} + gz_p\right), \quad (3.16)$$

where w is given by (2.8).

(iii) The *rate of change of vorticity* following the motion may be found thus: — By MII (3.13) and (3.5) above, the components of (absolute) vorticity are

$$\zeta_1 = -\frac{1}{z_p}\{v_p + (wz_y)_p - (wz_p)_y\}, \quad (3.17)$$

$$\zeta_2 = k + \frac{1}{z_p}\{u_p - (wz_p)_x + (wz_x)_p\}, \quad (3.18)$$

$$\zeta_3 = \frac{1}{z_p}\{l + v_x - u_y + (wz_y)_x - (wz_x)_y - - kz_y\}, \quad (3.19)$$

the indices 1, 2, 3 referring to the x, y and p components respectively. The rate of change of ζ_3 is then by MII (6.8),

$$\frac{D\zeta_3}{Dt} = -\Theta\zeta_3 + \zeta_1\tilde{\omega}_x + \zeta_2\tilde{\omega}_y + + \zeta_3\tilde{\omega}_p - \frac{X_3 + Y_3}{\sqrt{\triangle}},$$

where by MII (6.7) and (3.10), (3. 11) above,

$$X_3 = (F_1)_y - (F_2)_y = -g(z_{xy} - z_{yx}) = 0,$$

$$Y_3 = \left(\frac{1}{\varrho}\right)_x p_y - \left(\frac{1}{\varrho}\right)_y p_x = 0.$$

Hence by (3.7)

$$\frac{D\zeta_3}{Dt} = -\frac{\zeta_3}{z_p}\frac{Dz_p}{Dt} - \zeta_3(u_x + v_y) + + \zeta_1\tilde{\omega}_x + \zeta_2\tilde{\omega}_y. \quad (3.20)$$

Now in (3.19) write

$$\zeta = v_x - u_y, \; \zeta' = (wz_y)_x - (wz_x)_y - kz_y, \quad (3.21)$$

and then (3.20) is

$$\frac{1}{z_p}\frac{D}{Dt}(l + \zeta + \zeta') - \frac{1}{z_p^2}\frac{Dz_p}{Dt}(l + \zeta + \zeta') = = -\frac{1}{z_p^2}\frac{Dz_p}{Dt}(l + \zeta + \zeta') - - \frac{1}{z_p}(u_x + v_y)(l + \zeta + \zeta') + \zeta_1\tilde{\omega}_x + \zeta_2\tilde{\omega}_y,$$

or, finally, by (3.17), (3.18) *the rate of change of the quantity* $(l + \zeta + \zeta')$ *is given by the vorticity-equation*

$$\frac{D}{Dt}(l + \zeta + \zeta') = -(u_x + v_y)(l + \zeta + \zeta') - - \tilde{\omega}_x v_p + \tilde{\omega}_y u_p + \tilde{\omega}_y(w_p z_x - w_x z_p + kz_p) - - \tilde{\omega}_x(w_p z_y - w_y z_p), \quad (3.22)$$

where $(u_x + v_y)$ is the isobaric divergence.

(iv) Lastly the *equation of heat transfer* by bodily motion (conductivity, viscosity, radiation, etc., not explicitly analysed) is, by MII (7.5),

$$J\frac{Dq}{Dt} = Jc_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\varrho}\right), \quad (3.23)$$

where J is the mechanical equivalent of heat, q is the heat content of unit mass of air and c_v the specific heat of air at constant volume.

It should be noticed that the equations found above are quite general within the limitation imposed by the use of local rectangular coordinates. In particular, they do not depend on the assumption that the equilibrium of the atmosphere in the vertical is hydrostatic.

4. Dimensionless variables. Hydrostatic approximation

We proceed to find approximate forms for the fundamental equations of the preceding section by an appeal to observation. To do this we introduce dimensionless variables, indicated by capital letters except in the case of the time, the density and the temperature when lower-case Greek letters will be used. We write, for the dimensionless variables,

$$\left.\begin{aligned} X &= \frac{x}{r_0}, \; Y = \frac{y}{r_0}, \; Z = \frac{z}{z_0}, \; \tau = lt, \\ P &= \frac{p}{p_0}, \; \sigma = \frac{\varrho}{\varrho_0}, \; \vartheta = \frac{T}{T_0}. \end{aligned}\right\} \quad (4.1)$$

where p_0, ϱ_0, T_0 are the values of the pressure, density and temperature at the point whose coordinates in the (x, y, z) system are (r_0, r_0, z_0). We also write

$$U = \frac{DX}{D\tau} = \frac{u}{lr_0},\ V = \frac{DY}{D\tau} = \frac{v}{lr_0},$$

$$\Pi = \frac{DP}{D\tau} = \frac{\tilde{\omega}}{lp_0},\ W = \frac{DZ}{D\tau} = \frac{w}{lz_0}. \tag{4.2}$$

The conversion of some of the principal functions to the dimensionless variables is given in the following table. —

$$\left.\begin{aligned}
z_x &= \frac{z_0}{r_0} Z_X,\ z_y = \frac{z_0}{r_0} Z_Y;\\
\frac{Du}{Dt} &= l^2 r_0 \frac{DU}{D\tau},\ \frac{Dv}{Dt} = l^2 r_0 \frac{DV}{D\tau},\\
\frac{D\tilde{\omega}}{Dt} &= l^2 p_0 \frac{D\Pi}{D\tau};\\
b &= l^2 z_0 B = l^2 z_0 \left(\frac{DZ_\tau}{D\tau} + \right.\\
&\left. + U \frac{DZ_X}{D\tau} + V \frac{DZ_Y}{D\tau} + \Pi \frac{DZ_P}{D\tau}\right);\\
\zeta &= l(V_X - U_Y) = l\Xi;\\
\zeta' &= \frac{lz_0^2}{r_0^2} \{(WZ_Y)_X - (WZ_X)_Y\} -\\
&\quad - k \frac{z_0}{r_0} Z_Y;\\
(wz_y)_p &- (wz_p)_y =\\
&= \frac{lz_0^2}{p_0 r_0} \{(WZ_Y)_P - (WZ_P)_Y\};\\
(wz_x)_p &- (wz_p)_x =\\
&= \frac{lz_0^2}{p_0 r_0} \{(WZ_X)_P - (WZ_P)_X\}.
\end{aligned}\right\} \tag{4.3}$$

It is a fact of observation that the isobaric surfaces in the atmosphere are only slightly inclined to the horizontal, a gradient of 200 ft. in 50 miles being large and 200 ft. in 100 miles being appreciable. Thus we may say that z_x, z_y have values of about 7.6×10^{-4} to 3.8×10^{-4}. If then we take z_x, z_y to be equal to 5×10^{-4} and assume that the corresponding values of Z_X, Z_Y are of order unity, we have by (4.3).

$$\frac{z_0}{r_0} = 5 \times 10^{-4}. \tag{4.4}$$

In temperate latitudes (say $\varphi = 60°$) we also have

$$\frac{g}{l^2} = 6 \times 10^{10} \text{ cm}. \tag{4.5}$$

Consider the equation of motion (3.16) converted to dimensionless units by (4.3) so that it reads, after multiplication by z_0/g,

$$\begin{aligned}
&\frac{z_0 l^2}{g}\left\{\frac{D\Pi}{D\tau} + \frac{Z_X}{Z_P}\left(V - \frac{k}{l}\frac{z_0}{r_0} W\right) - \right.\\
&\left. - \frac{Z_Y}{Z_P} U + \frac{B}{Z_P}\right\} - \frac{z_0^2}{r_0^2}\frac{Z_X^2 + Z_Y^2}{Z_P} -\\
&- \frac{klr_0}{g}\frac{U}{Z_P} = -\frac{1}{Z_P^2}\left\{1 + \frac{z_0^2}{r_0^2}(Z_X^2 + Z_Y^2)\right\}\cdot\\
&\cdot\left\{\frac{p_0}{g\varrho_0 z_0}\frac{1}{\sigma} + Z_P\right\}
\end{aligned} \tag{4.6}$$

In order to find z_0 and r_0, let us assume that

$$\frac{z_0 l^2}{g} = \frac{z_0^2}{r_0^2} = 2.5 \times 10^{-7}, \tag{4.7}$$

whence

$$\left.\begin{aligned}
&r_0^2 = \frac{gz_0}{l^2},\quad \frac{klr_0}{g} = \frac{k}{l}\frac{z_0}{r_0},\\
&z_0 = 1.5 \times 10^4 \text{ cm},\ r_0 = 3 \times 10^7 \text{ cm}.
\end{aligned}\right\} \tag{4.8}$$

The location of the "standard" point (r_0, r_0, z_0) is therefore at some 300 km from O and near the plane Oxy. Thus this point is not far from the stratosphere and the temperature there (T_0) is therefore low. However it is not necessary to know T_0 at all accurately: its value must certainly lie between 200°A and 300° A. In this range the coefficient of $1/\sigma$ in (4.6) is, using (4.8) also,

$$\frac{p_0}{g\varrho_0 z_0} = \frac{RT_0}{gz_0} = 49 \pm 9. \tag{4.9}$$

Thus, if in (4.6) we neglect all terms whose coefficients are of order 10^{-4} or 10^{-7} and retain those with coefficients greater than or equal to unity, we find

$$\frac{p_0}{g\varrho_0 z_0}\frac{1}{\sigma} + Z_P = 0,$$

or

$$\frac{1}{\varrho} + g z_p = 0. \qquad (4.10)$$

Reference to (2.3) shows that this equation in the (x, y, z) system is

$$\frac{\partial p}{\partial z} = -g\varrho,$$

the familiar hydrostatic equation. Thus the hydrostatic approximation may be deduced from the observed fact that the isobaric surfaces in the atmosphere are, very roughly, parallel to the Earth's surface.

The equation of motion (3.14) may be written, using (4.3), (4.7) and (4.10),

$$V = Z_X + \frac{DU}{D\tau} + \frac{k}{l}\frac{z_0}{r_0} W.$$

Neglecting the term in this equation whose coefficient is of order 10^{-4}, the *equations of motion* (3.14), (3.15) may be written in dimensionless units as

$$V = Z_X + \frac{DU}{D\tau}, \qquad (4.11)$$

$$U = -Z_Y - \frac{DV}{D\tau}. \qquad (4.12)$$

The *equation of continuity* (3.8) reduces, by (4.3) and (4.10), to

$$U_X + V_Y + \Pi_P = 0, \qquad (4.13)$$

a form due to Godart (quoted in SI p. 371). The *vorticity equation* (3.22) becomes, using (4.3) and again neglecting all terms with coefficients of orders 10^{-4} and 10^{-7},

$$\frac{D}{D\tau}(1 + \Xi) = -(U_X + V_Y)(1 + \Xi) - $$
$$-\Pi_X V_P + \Pi_Y U_P. \qquad (4.14)$$

The equation of heat-transfer we shall deal with in par. 6.

Equations (4.11) to (4.14) are therefore the simplified forms of the fundamental equations which result from an approximation equivalent to the hydrostatic one. In the next section we shall consider further simplifications arising from certain *definitions* but we note one such definition now. The *geostrophic wind* is defined to be a wind whose relative acceleration $(DU/D\tau,\ DV/D\tau)$ is zero. Hence by (4.11), (4.12)

$$U = -Z_Y,\ V = Z_X.$$

If the wind is not geostrophic, the so-called "ageostrophic components" are, by (4.11), (4.12), $(-DV/D\tau,\ DU/D\tau)$. Thus, apart from sign, the "ageostrophic wind" is merely another name for the relative acceleration of the true wind. For brevity we write (4.12), (4.11) as

$$U = -Z_Y - \dot{V}, \qquad (4.15)$$

$$V = Z_X + \dot{U}. \qquad (4.16)$$

5. The vorticity equation

In the previous section we have seen how we can reject certain terms in the fundamental equations by means of an empirical approximation. In the course of doing this we have also cleared the equations of the parameters g, l, r_0, z_0, p_0, ϱ_0 and T_0 so that we are now no longer able to perform further approximations of this kind. We are therefore thrown back onto definitions of certain types of motion which may, or may not, be realised in nature. An analogy may make this point clearer: in elementary hydrodynamics we begin with the study of the irrotational motion of an incompressible fluid, i.e. motions in which, by definition, a velocity potential Φ exists such that

$$u = \frac{\partial \Phi}{\partial x},\ v = \frac{\partial \Phi}{\partial y},\ w = \frac{\partial \Phi}{\partial z}.$$

But in writing down these three "restrictive conditions", as we shall call them, we do not imply either that all fluid motions are irrotational or even that they are approximately irrotational: we are merely defining the simplest case that arises.

Turning now to the vorticity equation (4.14), the first restrictive condition we consider is expressed by the equation

$$\Pi_X \; V_P - \Pi_Y \, U_P = 0. \qquad (5.1)$$

It would be satisfied if one or other of the three following situations arose in an atmospheric motion: —

(a) U_P, V_P were negligibly small or zero, i.e. if there were no change of wind-speed or direction on passing from one isobaric surface to another, keeping (X, Y) fixed;

(b) Π_X, Π_Y were negligibly small or zero, i.e. if there were no change of Π on passing from one point to another in an isobaric surface;

(c) Π_X, Π_Y, U_P, V_P were large but happened to satisfy (5.1) throughout the region of the atmosphere considered.

Whether plausible or not, (5.1) is usually asserted in discussing the vorticity equation which then becomes, using (4.13) also,

$$\frac{D}{D_\tau}(1 + \Xi) = -(U_X + V_Y)(1 + \Xi) = $$
$$= \Pi_P (1 + \Xi). \qquad (5.2)$$

The second restrictive condition we consider concerns the "ageostrophic" wind $(-\dot{V}, \dot{U})$. We have two mutually exclusive possibilities, viz: —

(A) The ageostrophic wind is comparable in magnitude to Z_X, Z_Y;

(B) The ageostrophic wind is negligibly small compared to Z_X, Z_Y.

(i) Consider first case (A) in which, by (4.15), (4.16), (4.13),

$$\left.\begin{aligned} \Xi &= V_X - U_Y = Z_{XX} + Z_{YY} + \dot{U}_X + \dot{V}_Y \\ &= \nabla^2 Z + \dot{U}_X + \dot{V}_Y, \\ \Pi_P &= -(U_X + V_Y) = \dot{V}_X - \dot{U}_Y. \end{aligned}\right\} \qquad (5.3)$$

Hence, in this case, $\Xi \neq \nabla^2 Z$ and $\Pi_P \neq 0$, and we may write (5.2) as

$$\frac{D}{D\tau}\{\log(1 + \Xi)\} = \Pi_P.$$

This equation may be integrated with respect to P from an isobaric surface where $P = P_1$ to another surface where $P = P_2$ $(< P_1)$, and the result divided by $P_2 - P_1$. We obtain

$$\frac{1}{P_2 - P_1}\int_{P_1}^{P_2} \frac{D}{D\tau}\{\log(1 + \Xi)\}\, dP =$$
$$= \frac{\Pi_2 - \Pi_1}{P_2 - P_1} = \frac{D}{D\tau}\{\log(P_2 - P_1)\}. \qquad (5.4)$$

If we define a pressure-average, $\bar{\Xi}$, of Ξ by

$$\frac{1}{P_2 - P_1}\int_{P_1}^{P_2} \frac{D}{D\tau}\{\log(1 + \Xi)\}\, dP =$$
$$= \frac{D}{D\tau}\{\log(1 + \bar{\Xi})\},$$

we obtain from (5.4)

$$\frac{D}{D\tau}\left\{\frac{1 + \bar{\Xi}}{P_2 - P_1}\right\} = 0,$$

or, returning to our original variables, and writing $\bar{\zeta} = l\bar{\Xi}$,

$$\frac{D}{Dt}\left\{\frac{l + \bar{\zeta}}{p_2 - p_1}\right\} = 0, \qquad (5.5)$$

which is a form of the Rossby potential-vorticity equation for the case of ageostrophic motion.

(ii) Consider next case (B) in which $\dot{U} = 0$, $\dot{V} = 0$, so that, by (4.15), (4.16), (5.3),

$$U = -Z_Y,\; V = Z_X,\; \Xi = \nabla^2 Z,\; \Pi_P = 0. \qquad (5.6)$$

Using these expressions in (5.2), we have

$$\nabla^2 Z_\tau - Z_Y \nabla^2 Z_X + Z_X \nabla^2 Z_Y = -$$
$$- \Pi \nabla^2 Z_P. \qquad (5.7)$$

From this equation we can obtain a partial differential equation of the third order in Z by satisfying (5.1) in either of the following two ways: —

(1) $U_P = 0$, $V_P = 0$, which by (5.6) lead to $Z_{YP} = 0$, $Z_{XP} = 0$, and so to $\nabla^2 Z_P = 0$. This restrictive condition is therefore equivalent to satisfying (5.1) by means of (a);

(2) $\Pi = 0$, so that $\Pi_X = 0$, $\Pi_Y = 0$ and (5.1) is satisfied in virtue of (b).

In either case (5.7) is

$$\nabla^2 Z_\tau - Z_Y \nabla^2 Z_X + Z_X \nabla^2 Z_Y = 0 \tag{5.8}$$

which may also be written by (5.6) as

$$\nabla^2 Z_\tau + U(1 + \Xi)_X + V(1 + \Xi)_Y = 0. \tag{5.8a}$$

If we convert this equation to our original variables, use (4.7) to eliminate r_0^2, and write $h = gz$, we obtain

$$u\,(l + \zeta)_x + v\,(l + \zeta)_y + \frac{1}{l}\,(h_{txx} + h_{tyy}) = 0, \tag{5.9}$$

which, in our notation, is Sutcliffe's equation SI (11) for the case he describes as being that of "non-development" $dp/dt = 0$ (i.e. $\Pi = 0$). The equation however is valid also under the conditions $U_P = 0$, $V_P = 0$, $\Pi_P = 0$, which imply that U, V, Π are functions of (X, Y, τ) but not of P. Equation (5.9) can thus be established for a geostrophic wind provided that (5.1) is satisfied in virtue of (1) and (2) above.

(iii) To obtain the remaining formulae of Sutcliffe's theory we return to case (A) in which, by (5.3), $\Xi \neq \nabla^2 Z$ and $\Pi_P \neq 0$. Equation (5.2) may be written as

$$\nabla^2 Z_\tau + \dot{U}_{X\tau} + \dot{V}_{Y\tau} + U\Xi_X + V\Xi_Y + \Pi\,\Xi_P = \Pi_P + \Pi_P\,\Xi. \tag{5.10}$$

If we now introduce the restrictive condition expressed by the equation

$$\dot{U}_{X\tau} + \dot{V}_{Y\tau} = \Pi_P \Xi - \Pi\,\Xi_P, \tag{5.11}$$

we find that (5.10) becomes, using (4.13) again,

$$\nabla^2 Z_\tau + U\Xi_X + V\Xi_Y = \Pi_P = -(U_X + V_Y). \tag{5.12}$$

On returning to our original variables, with $h = gz$, we have

$$-\frac{1}{l}\,(h_{txx} + h_{tyy}) - \{\,u\,(l+\zeta)_x + v\,(l+\zeta)_y\,\} = l\,(u_x + v_y), \tag{5.13}$$

which, in our notation, is Sutcliffe's equation SI (10) for the case of ageostrophic motion. It is therefore valid whenever it is known *a priori* that (5.11) is satisfied in the atmospheric motion considered.

Sutcliffe however proceeds further by means of a second restrictive condition. Consider the equation (5.12) at two isobaric surfaces $P = P_1$ and $P = P_2$ so that

$$(\nabla^2 Z_\tau)_2 + U_2\,(\Xi_X)_2 + V_2\,(\Xi_Y)_2 = (\Pi_P)_2,$$
$$(\nabla^2 Z_\tau)_1 + U_1\,(\Xi_X)_1 + V_1\,(\Xi_Y)_1 = (\Pi_P)_1.$$

Hence by subtraction

$$(\nabla^2 Z_\tau)_2 - (\nabla^2 Z_\tau)_1 + U_2(\Xi_X)_2 - U_1(\Xi_X)_1 + V_2\,(\Xi_Y)_2 - V_1\,(\Xi_Y)_1 = (\Pi_P)_2 - (\Pi_P)_1. \tag{5.14}$$

If we now impose the restrictive condition expressed by the equation

$$(\nabla^2 Z_\tau)_1 + U_2(\Xi_X)_1 + V_2(\Xi_Y)_1 = (\nabla^2 Z_\tau)_2 + U_1(\Xi_X)_2 + V_1\,(\Xi_Y)_2, \tag{5.15}$$

and use it to eliminate $(\nabla^2 Z_\tau)_2 - (\nabla^2 Z_\tau)_1$ from (5.14), we can write the result in the form

$$(U_2 - U_1)\,(1 + \Xi_1 + \Xi_2)_X + (V_2 - V_1)\,(1 + \Xi_1 + \Xi_2)_Y = (\Pi_P)_2 - (\Pi_P)_1 = -(U_X + V_Y)_2 + (U_X + V_Y)_1 \tag{5.16}$$

Returning to our original variables, this equation is

$$-(u_2-u_1)(l+\zeta_1+\zeta_2)_x-$$
$$-(v_2-v_1)(l+\zeta_1+\zeta_2)_y=$$
$$=l\{(u_x+v_y)_2-(u_x+v_y)_1\}, \quad (5.17)$$

which is equivalent to Sutcliffe's final equation, SI (14), as may be seen by writing

$$\mathbf{V}'=(u_2-u_1)\,\mathbf{i}+(v_2-v_1)\,\mathbf{j},\ \zeta_0=\zeta_1, \zeta=\zeta_2,$$
$$(\mathbf{V}'.\nabla_p)(l+\zeta+\zeta_0)=(u_2-u_1)(l+\zeta_1+\zeta_2)_x+$$
$$+(v_2-v_1)(l+\zeta_1+\zeta_2)_y,$$
$$\mathrm{div}_p\mathbf{V}=(u_x+v_y)_2,\ \mathrm{div}_p\,\mathbf{V}_0=(u_x+v_y)_1.$$

Thus equation (5.17) is valid only if it is known *a priori* that both the restrictive conditions (5.11) and (5.15) are satisfied in the atmospheric motion under investigation. In terms of our original variables, with $h=gz$, these conditions are

$$\left(\frac{Du}{Dt}\right)_{xt}+\left(\frac{Dv}{Dt}\right)_{yt}=l(\tilde{\omega}_p\zeta-\tilde{\omega}\zeta_p), \quad (5.11\,\mathrm{a})$$

$$\frac{1}{l}(h_{txx}+h_{tyy})_1+u_2(\zeta_x)_1+v_2(\zeta_y)_1=$$
$$=\frac{1}{l}(h_{txx}+h_{tyy})_2+u_1(\zeta_x)_2+v_1(\zeta_y)_2. \quad (5.15\,\mathrm{a})$$

Hence Sutcliffe's theory imposes more conditions on the motion than that which led to the potential-vorticity equation (5.5). It is true that each side of the equation (5.15 a) bears a superficial resemblance to the left-hand side of (5.9), which is valid for *geostrophic* motion. But Sutcliffe is not considering geostrophic motion, for the purpose of (5.17) is to deal with cases when (u_x+v_y), i.e. his $\mathrm{div}_p\mathbf{V}$, is not zero. It also does not seem possible to deduce one of the equations (5.11a), (5.15a) from the other. Before, therefore, any synoptic conclusions are drawn from (5.13) or (5.17) it would seem necessary to investigate empirically to what extent, and in what synoptic situations, (5.11a) and (5.15a) hold good.

6. The equation of heat-transfer

The continued recurrence of $\nabla^2 Z_\tau$ in the formulae of the preceding section suggests that further light would be thrown on the problem if an independent method for calculating Z_τ as a function of (X, Y, P, τ) could be found. To take a simple example: in the geostrophic case, Z is a solution of the partial differential equation (5.8) and Z would therefore involve a number of arbitrary "functions of integration". Hence an independently calculated value of Z_τ would serve as a kind of "boundary condition" and fix some of these unknown functions. Sutcliffe and Forsdyke in SII therefore seek such a determination of Z_τ by means of the hydrostatic equation (4.10), the gas equation $p=R\varrho T$ and the equation of heat-transfer (3.23). We have in fact by (4.10)

$$gz_p=-R\frac{T}{p}, \quad (6.1)$$

and we also have the identity $\tilde{\omega}=p_t+up_x+$ $+vp_y+\tilde{\omega}p_p$ which, by (3.10), gives $p_t=0$. Thus differentiating (6.1) partially with respect to t, we obtain

$$gz_{pt}=-R\frac{T_t}{p}. \quad (6.2)$$

Integrating with respect to p from the isobaric surface $p=p_1$ to the surface $p=p_2$, we have

$$h_t=g\{(z_t)_2-(z_t)_1\}=R\int_{p_2}^{p_1}T_t d\log p. \quad (6.3)$$

The equation of heat-transfer (3.23) and the gas-equation yield

$$J\frac{Dq}{Dt}=Jc_p\frac{DT}{Dt}-\frac{RT}{p}\tilde{\omega},$$

whence, if Γ is the adiabatic lapse-rate, we obtain

$$T_t=-(uT_x+vT_y)+\tilde{\omega}\left(\frac{\Gamma}{g}\frac{1}{\varrho}-T_p\right)+$$
$$+\frac{1}{c_p}\frac{Dq}{Dt}.$$

If this is introduced into (6.3) we have

$$h_t = g\{(z_t)_2 - (z_t)_1\} =$$

$$= R\int_{p_2}^{p_1}\left\{-(uT_x + vT_y) + \tilde{\omega}\left(\frac{\Gamma}{g}\frac{1}{\varrho} - T_p\right) + \right.$$

$$\left. + \frac{1}{c_p}\frac{Dq}{Dt}\right\} d\log p, \qquad (6.4)$$

which is, in our notation, the equation SII (3). In order to find the magnitude of its various terms, we convert to dimensionless units by (4,1), (4.3) and introduce also the dimensionless heat-unit $Q = q/q_0$. We find

$$(Z_\tau)_2 - (Z_\tau)_1 = \frac{RT_0}{gz_0}\int_{P_2}^{P_1}\left\{-(U\vartheta_X + V\vartheta_Y) + \right.$$

$$\left. + \Pi\left(\frac{\Gamma R}{g}\frac{1}{\sigma} - \vartheta_P\right) + \frac{q_0}{c_p T_0}\frac{DQ}{D\tau}\right\} d\log P. \qquad (6.5)$$

Since

$$\frac{\Gamma R}{g} = \frac{2.868 \times 10^6}{0.2405 \times 4.1863 \times 10^7} \simeq \frac{3}{10},$$

it follows that all the terms in the integrand of (6.5) are of the same order of magnitude provided that the heat-content, q_0, and the temperature, T_0, of unit mass of air at our "standard" point (r_0, r_0, z_0) satisfy the condition $q_0 \simeq c_p T_0$. If therefore the right-hand side of (6.5) can be evaluated directly, we shall have a value of Z_τ at an arbitrary level for use in the equations of par. 5.

We note however that a study of equation (6.4) by means of the data of observation does not, by itself, throw light on the dynamics of an atmospheric motion. For this equation depends only on the hydrostatic equation and the equation of heat-transfer. It is only when (6.4) is used in conjunction with equations derived from the two "horizontal" equations of motion and the equation of continuity that dynamical consistency can be hoped for.

7. The equivalent-barotropic atmosphere

The methods of this paper can be applied to the analysis of a form of the vorticity-equation which occurs in the work of J. G. Charney (1949). This author employs the (x, y, z) system in his work but, in order to permit comparison with our results, we shall convert his formulae to the (x, y, p) system. We deal again with case (A) of par. 5 and once more assert (5.1). Following Charney, the first restrictive condition on the motion is expressed by

$$U = A(P)\,U'(X, Y, \tau), \quad V = A(P)\,V'(X, Y, \tau), \qquad (7.1)$$

where A is an unknown function of P. This condition means that, for fixed (X, Y, τ), there is no change of horizontal wind-direction, but a possible change of wind-speed, as we pass from one isobaric surface to another. Let $P = P_1$ now correspond to the Earth's surface whilst $P = P_2 = 0$ corresponds to the "top" of the atmosphere where we also assume that $\Pi = 0$. We define $P = \overline{P}$ to be an isobaric surface on which $A(P)$ is equal to its pressure average, i.e.

$$A(\overline{P}) \equiv \overline{A} = \int_0^{P_1} A(P)\,dP. \qquad (7.2)$$

Let

$$\overline{U} = A(\overline{P})\,U', \quad \overline{V} = A(\overline{P})\,V',$$

so that, if U, V refer to some isobaric surface other than $P = \overline{P}$, we have

$$U = \frac{A}{\overline{A}}\,\overline{U}, \quad V = \frac{A}{\overline{A}}\,\overline{V}. \qquad (7.3)$$

We define

$$\overline{\Xi} = \overline{V}_X - \overline{U}_Y,$$

so that

$$\Xi = V_X - U_Y = \frac{A}{\overline{A}}\,\overline{\Xi}(X, Y, \tau) \qquad (7.4)$$

Hence the vorticity equation (5.2) becomes

$$\frac{A}{\overline{A}}\,\overline{\Xi}_\tau + \left(\frac{A}{\overline{A}}\right)^2 (\overline{U}\,\overline{\Xi}_X + \overline{V}\,\overline{\Xi}_Y) + \Pi\,\frac{A_P}{\overline{A}}\,\overline{\Xi} =$$

$$= \Pi_P\left(1 + \frac{A}{\overline{A}}\,\overline{\Xi}\right), \qquad (7.5)$$

since $\bar{\Xi}_P = 0$. We next define the dimensionless number

$$K = \{ \int_0^{P_1} A^2(P)dP \} / [\int_0^{P_1} A(P)dP]^2, \quad (7.6)$$

which, apart from a reversal of limits in the first integral that is probably due to a misprint in his paper, plays the same part as Charney's K. If we then integrate (7.5) with respect to P from 0 to P_1 and use (7.2) and (7.6), we find

$$\bar{\Xi}_\tau + K(\bar{U}\bar{\Xi}_X + \bar{V}\bar{\Xi}_Y) + \frac{\bar{\Xi}}{\bar{A}} \int_0^{P_1} \Pi A_P dP =$$
$$= \int_0^{P_1} \Pi_P \left(1 + \frac{A}{\bar{A}} \bar{\Xi}\right) dP. \quad (7.7)$$

We now impose a second restrictive condition expressed by

$$\int_0^{P_1} \Pi A_P dP = 0, \quad (7.8)$$

which corresponds to Charney's neglect of terms in the vertical velocity-component w. Considering also the equation of continuity (4.13) we obtain, by (7.2),

$$\int_0^{P_1} \Pi_P dP =$$
$$= -(\bar{U}_X + \bar{V}_Y) \int_0^{P_1} \frac{A}{\bar{A}} dP =$$
$$= -(\bar{U}_X + \bar{V}_Y)$$

or, if $\Pi = \Pi_1$ at the Earth's surface

$$\Pi_1 = -(\bar{U}_X + \bar{V}_Y).$$

Hence integrating (7.8) by parts, using the equation of continuity and this formula for Π_1 ($\neq 0$), we find

$$A(P_1) - KA(\bar{P}) = 0, \quad (7.9)$$

which shows that the restrictive condition (7.8) is essentially a limitation on the function A. Again

$$\int_0^{P_1} \Pi_P \left(1 + \frac{A}{\bar{A}} \bar{\Xi}\right) dP = -$$
$$-(\bar{U}_X + \bar{V}_Y) \int_0^{P_1} \left\{ \frac{A}{\bar{A}} + \left(\frac{A}{\bar{A}}\right)^2 \bar{\Xi} \right\} dP =$$
$$= \Pi_1 (1 + K\bar{\Xi}).$$

Thus finally (7.7) may be written in either of the equivalent forms

$$\bar{\Xi}_\tau + K(\bar{U}\bar{\Xi}_X + \bar{V}\bar{\Xi}_Y) +$$
$$+ (1 + K\bar{\Xi})(\bar{U}_X + \bar{V}_Y) = 0, \quad (7.10)$$

or

$$\bar{\Xi}_\tau + K(\bar{U}\bar{\Xi}_X + \bar{V}\bar{\Xi}_Y) - \Pi_1(1 + K\bar{\Xi}) = 0. \quad (7.11)$$

Returning to our original variables by (4.3) we have

$$p_0 A(\bar{p}/p_0) = \int_0^{p_1} A(p/p_0) dp,$$

$$K = \{ p_0 \int_0^{p_1} A^2 dp \} / (\int_0^{p_1} A dp)^2,$$

and (7.10), (7.11) become, respectively,

$$\bar{\zeta}_t + K(\bar{u}\bar{\zeta}_x + \bar{v}\bar{\zeta}_y) + (l + K\bar{\zeta})(\bar{u}_x + \bar{v}_y) = 0, \quad (7.12)$$

$$\frac{1}{l + K\bar{\zeta}} \{ \bar{\zeta}_t + K(\bar{u}\bar{\zeta}_x + \bar{v}\bar{\zeta}_y) \} - \frac{\omega_1}{p_0} = 0, \quad (7.13)$$

where

$$\bar{u} = l r_0 \bar{U},$$
$$\bar{v} = l r_0 \bar{V},$$
$$\bar{\zeta} = \bar{v}_x - \bar{u}_y = l\bar{\Xi}.$$

The equation (7.12) corresponds, in the (x,y,p) system and for l = constant, to an equation given by Charney (1949, foot of page 383). However, (7.13) differs from his next equation (1949, top of page 384) in the following respects:—By (2.7) and the hydrostatic equation we have, for the last term,

$$-\frac{\tilde{\omega}_1}{p_0} = -\frac{1}{p_0}\left\{\frac{\partial p}{\partial t} - g\varrho w + u\frac{\partial p}{\partial x} + \right.$$
$$\left. + v\,\frac{\partial p}{\partial y}\right\}_1,$$

whilst Charney has — in our notation —

$$-\frac{\tilde{\omega}_1}{p_0} = +\frac{1}{p_1}\left\{\frac{\partial p}{\partial t} - g\varrho w\right\}_1.$$

The discrepancy in sign is explained by an error due to a reversal of the limits of integration in Charney's manipulation of the tendency equation at the foot of page 383. The substitution of the surface pressure p_1 for our p_0 is a consequence of the use of local rectangular coordinates. The pressure p_0 is that at our standard point (r_0, r_0, z_0) which is near the plane Oxy. In the "flat earth" theory which results from the use of local rectangular coordinates, p_0 could be identified with p_1, though we have preferred not to do so in par. 4. Apart from these two points, our formula agrees with Charney's in the case of geostrophic motion when $u\,\frac{\partial p}{\partial x} + v\,\frac{\partial p}{\partial y} = 0$.

We return to the consideration of the geostrophic case, and write (5.8 a) as

$$(1 + \Xi)_T + U\,(1 + \Xi)_X + V\,(1 + \Xi)_Y = 0,$$

or, returning to our original variables by (4.3), we have

$$(l + \zeta)_t + u\,(l + \zeta)_x + v\,(l + \zeta)_y = 0. \tag{7.14}$$

This equation corresponds, in the (x, y, p) system, to the equation [1949, (42)] taken by Charney as governing the motion of the equivalent barotropic atmosphere. Since, by (7.1), U_P, V_P are not zero, the condition (1) of par. 5 (ii) is not satisfied and therefore condition (2) must apply. This means that (7.14) is valid when $\tilde{\omega} = 0$ (Sutcliffe's non-development) throughout the atmospheric motion, a restriction which replaces Charney's condition that $\partial p/\partial t$ shall be zero at the Earth's surface. Thus in the geostrophic case, Sutcliffe's equation (5.9) is identical with Charney's (7.14) and both are equivalent forms of the third-order partial differential equation (5.8) for Z.

If the foregoing analysis is accepted, it appears that Rossby, Sutcliffe and Charney are each using models of atmospheric motions. All three models use the restrictive condition (5.1) and, in Rossby's case, no further restrictions are needed in order to obtain the potential-vorticity equation (5.5). This appears to be the most general case for ageostrophic motion. At the other extreme is Charney's equivalent barotropic atmosphere, with the additional restrictive conditions of geostrophic motion and (7.1), (7.9). Whether these additional restrictions are accepted or not, they are certainly easy to interpret in physical terms. Intermediately comes Sutcliffe's model, in which the additional restrictive conditions are expressed by equations (5.11 a) and (5.15a). Whilst Sutcliffe has thus escaped from the case of geostrophic motion it is not apparent that his additional restrictions have any obvious physical interpretation.

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