Comment on a Note on the Kinetic Energy Balance of Zonal Wind Systems

By J. van MIEGHEM

University of Brussels[1]

March 5th, 1952

Since Kuo's equation of balance of the kinetic energy of zonal motion in the atmosphere is of basic importance for the study of the general circulation (KUO 1951), I wish to point out that this equation may be simplified if account is taken of the equation of continuity for the mean motion of the atmosphere. It turns out that the constancy of the specific mass ϱ along latitude circles and with time appears to be an unnecessary hypothesis. Moreover the terms related to the meridional circulation are more easely interpreted.

In orthogonal and axial symmetrical coordinates ($x^1 \equiv \lambda$, x^2, x^3), the equation of motion along a latitude circle assumes the form of a balance:

$$\frac{\partial}{\partial t}\left[\varrho(\gamma_{11}\,\omega + v_1)\right] + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}\left\{\sqrt{\gamma}\left[(\gamma_{11}\,\omega + v_1)\,\varrho v^i + \varepsilon_1^i\, p\right]\right\} = 0,$$
$$(i = 1,\ 2,\ 3)$$

from which the equation of the mean motion may easely be deduced:

$$\bar{\varrho}\,\frac{d\hat{v}_1}{dt} + \omega\,\frac{\partial\gamma_{11}}{\partial x^\alpha}\,\overline{\varrho v^\alpha} + \frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\overline{\varrho v_1'' v''^\alpha}\right) = 0,\ (\alpha = 2,\ 3) \tag{1}$$

where $\overline{X}$ designates the zonal mean of X and $\hat{X}$ the corresponding weighted mean ($\bar{\varrho}\hat{X} = \overline{\varrho X}$), $X'' \equiv X - \hat{X}$ the fluctuation of X with respect to the weigthed mean, γ_{ij} the coefficients of the metric form ($\gamma_{ii} \neq 0$, $\gamma_{ij} \equiv 0$ when $i \neq j$; $i, j = 1,\ 2,\ 3$), $v^i \equiv \dot{x}^i$ the contravariant and $v_i \equiv \gamma_{ij} v^j$ the covariant components of the air velocity $\mathbf{v}$ relative to the Earth, γ the determinant $\|\gamma_{ij}\|$; where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \hat{v}^\alpha\,\frac{\partial}{\partial x^\alpha}$ and $\varepsilon_1^i = 1$ or 0, resp. when $i \equiv 1$ or $\neq 1$.

After multiplication of (1) by $\hat{v}^1$ and substitution of the equation of continuity $\frac{\partial\bar{\varrho}}{\partial t} + \frac{1}{\sqrt{\gamma}}\,\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\overline{\varrho v^\alpha}\right) = 0$, one obtains the equation of balance

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,\bar{\varrho}\,\hat{v}^1\hat{v}_1\right) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left[\sqrt{\gamma}\left(\frac{1}{2}\,\bar{\varrho}\,\hat{v}_1\hat{v}^1\hat{v}^\alpha + \overline{\varrho v_1'' v''^\alpha}\,\hat{v}^1\right)\right] + \frac{1}{2}\,\frac{\partial\gamma_{11}}{\partial x^\alpha}\,\hat{v}^1(\hat{v}^1 + 2\,\omega)\,\overline{\varrho v^\alpha} - \overline{\varrho v_1'' v''^\alpha}\,\frac{\partial\hat{v}^1}{\partial x^\alpha} = 0, \tag{2}$$

or, explicitly in spherical coordinates ($\gamma_{11} \equiv r^2\cos^2\varphi$, $\gamma_{22} \equiv r^2$, $\gamma_{33} \equiv 1$; $\frac{\partial}{\partial y} \equiv \frac{1}{r}\,\frac{\partial}{\partial\varphi}$, $\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial r}$), using the classical zonal, meridional and vertical components (u, v, w) of the velocity,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\,\bar{\varrho}\,(\hat{u})^2\right] + \frac{1}{\cos\varphi}\,\frac{\partial}{\partial y}\,(Y\cos\varphi) + \frac{1}{r^2}\,\frac{\partial}{\partial z}\,(Zr^2) = \sigma$$

where

$$Y \equiv \frac{1}{2}\,(\hat{u})^2\,\overline{\varrho v} + \overline{\varrho u'' v''}\,\hat{u} \quad \text{and}$$

$$Z \equiv \frac{1}{2}\,(\hat{u})^2\,\overline{\varrho w} + \overline{\varrho u'' w''}\,\hat{u}$$

[1] At present: Institute of Meteorology of the University of Stockholm.

are the meridional and vertical components of the flux of the "mean zonal kinetic energy" $\left(\frac{1}{2}\,(\hat{u})^2\right)$ and

$$\sigma \equiv \hat{u} \left\{ \left[2\,\omega \sin\varphi + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi \right] \overline{\varrho v} - \left[2\omega\cos\varphi + \frac{\hat{u}}{r} \right] \overline{\varrho w} \right\} + \overline{\varrho u'' v''} \cdot \left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi \right) + \overline{\varrho u'' w''} \left(\frac{\partial \hat{u}}{\partial z} - \frac{\hat{u}}{r} \right)$$

is the corresponding rate of production.

The flux $(Y,\,Z)$ of "mean zonal kinetic energy" in the meridional plane includes a convective flux and a flux related to the flux of linear zonal momentum; this non convective flux is directed to the North in the belt of westerlies (VAN MIEGHEM, 1949).

The production of "mean zonal kinetic energy" results from: a) the mean meridional circulation $(\overline{\varrho v},\,\overline{\varrho w})$ and the vorticity created by the Earth's rotation and by the curvature of the zonal flow; b) the eddy stresses $\overline{\varrho u'' v''}$ and $\overline{\varrho u'' w''}$ and the horizontal and vertical shears of the mean zonal wind $\hat{u}$.

It should also be noted that the "mean zonal kinetic energy" $\left(\frac{1}{2}\,(\hat{u})^2 \equiv \frac{1}{2}\,\hat{v}_1\,\hat{v}^1\right)$ is not a scalar quantity, (but disregarding the factor $\frac{1}{2}$) one of the components of the tensor $\hat{v}_i\,\hat{v}^j$, $(i,\,j = 1,\,2,\,3)$.

Finally Kuo's conclusive statement "*the eddy process in the atmosphere is a self consistent and self sufficient mechanism in maintaining the zonal wind systems*" may be misleading and instigate unnecessary controverses. In order to avoid misunderstandings, it should be mentioned that the rates of production (σ) of the potential (Φ) and kinetic (K_m) energy of the mean three-dimensional motion, of the kinetic energy (K_e) of the large-scale eddies and of the thermal energy (E_t) (sensible and latent heat) assume the form

$$\begin{cases} \sigma(K_m) \equiv \bar{p}\,\mathrm{div}\,\hat{\mathbf{v}} + \triangle & -\sigma(\Phi), \\ \sigma(K_e) \equiv \quad -\triangle + (-\overline{\nabla p \cdot \mathbf{v}''}) & , \\ \sigma(E_t) \equiv -\bar{p}\,\mathrm{div}\,\hat{\mathbf{v}} \quad -(-\overline{\nabla p \cdot \mathbf{v}''}) & , \end{cases}$$

where

$$\triangle \equiv \overline{\varrho u'' u''} \left(-\frac{\hat{v}}{r}\,\mathrm{tg}\,\varphi + \frac{\hat{w}}{r} \right) + \overline{\varrho v'' v''} \left(\frac{\partial \hat{v}}{\partial y} + \frac{\hat{w}}{r} \right) + \overline{\varrho w'' w''}\,\frac{\partial \hat{w}}{\partial z} + \overline{\varrho v'' w''} \left(\frac{\partial \hat{v}}{\partial z} + \frac{\partial \hat{w}}{\partial y} - \frac{\hat{v}}{r} \right) + \overline{\varrho u'' w''} \left(\frac{\partial \hat{u}}{\partial z} - \frac{\hat{u}}{r} \right) + \overline{\varrho u'' v''} \left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi \right).$$

For simplicity the viscosity and the small-scale turbulence have been neglected here. The above formulas show that:

1) $\bar{p}$ div$\hat{\mathbf{v}}$ and $-\overline{\nabla p \cdot \mathbf{v}''}$ express the rate of conversion of thermal energy resp. into kinetic energy of the mean motion and into kinetic energy of the large-scale eddies;

2) $\triangle$ expresses the rate of conversion of kinetic energy of the large-scale eddies into kinetic energy of mean motion, or, more precisely, a positive term of $\triangle$, such as $\overline{\varrho u'' v''} \left(\frac{\partial \hat{u}}{\partial y} + \frac{\hat{u}}{r}\,\mathrm{tg}\,\varphi \right) > 0$, gives a rate of conversion of K_e into K_m and, a negative term, such as $\overline{\varrho u'' w''} \left(\frac{\partial \hat{u}}{\partial z} - \frac{\hat{u}}{r} \right) < 0$, a rate of conversion of K_m into K_e.

Conversion of thermal energy into kinetic energy of the large scale eddies occurs when the work performed by the pressure force along the path of the large-scale eddying motion is positive; when this work is negative (stable large-scale eddying motion) the conversion of energy takes place in the opposite direction.

In a three-dimensional diverging flow, thermal energy is transformed into kinetic energy K_m; the reversed transformation is taking place in a three-dimensional converging flow.

From the point of view of large scale energy transformations in the atmosphere, it should be emphazised that all possible conversions between the mechanical energy $K_m + \Phi$ of the mean motion, the thermal energie E_t and the kinetic energy K_e of the large-scale eddies may occur and in fact, do occur.

Stockholm, February, 1952.

REFERENCES

KUO, H. L., 1951: A Note on the Kinetic Energy Balance of the Zonal Wind Systems. *Tellus*, 3, 205—207.

VAN MIEGHEM, J., 1949: Production et redistribution de la quantité de mouvement et de l'énergie cinétique dans l'atmosphère. Application à la circulation atmosphérique générale. *J. Sc. de la Météor.*, I, 55—67.

Reply

By H. L. KUO.

Massachusetts Institute of Technology

March 1952

Prof. Van Mieghem's introduction of the mean value weighted by the density distribution is interesting. Through this definition of the mean, the restriction on the density put forward in my short note becomes unnecessary. As for the sentence "the eddy process in the atmosphere is a self consistent and self sufficient mechanism in maintaining the zonal wind systems" quoted by Prof. Van Mieghem, I have only intended to point out that this eddy process is important both for the angular momentum balance and for the kinetic energy balance of the mean zonal winds and that its effects are of the right order of magnitude and direction, so that it may account for the major portions of these balances. It is hoped that it does not give the impression that no energy supply is needed by these eddies, nor that it excludes the effects of other physical processes. The note is primarily concerned with the kinetic energy balance of the mean zonal wind systems, in which the work done by the pressure forces does not occur, although it is a very important factor for the production of kinetic energy of almost all other kinds of motions. This is one advantage of my equation. The energy exchange between eddies of different scales discussed by Van Mieghem is surely an important process in the atmosphere, but not directly connected with the maintenance of the mean zonal wind systems.