

The momentum imbalance paradox

By THIERRY PICHEVIN^{1,2†} and DORON NOF^{1*}, ¹*Department of Oceanography 3048 and the Geophysical Fluid Dynamics Institute, The Florida State University, Tallahassee, Florida 32306-3048, USA;* ²*LODYC, Boite 100, Université Pierre et Marie Curie, 4 Place Jussieu, 75005 Paris, France*

(Manuscript received 25 September 1995; in final form 19 June 1996)

ABSTRACT

The classical question of what happens when anomalous water enters an ocean via a meridional northward channel is addressed analytically using a reduced-gravity nonlinear model. The channel corresponds to either a conduit connecting 2 otherwise separated basins (e.g., the Yucatan Channel) or a conduit carrying water from an independent source. The traditional view is that, due to the Coriolis force, such an anomalous northward flowing current turns to the right (looking offshore) and forms a zonal boundary current that flows eastward. In this scenario, a front (corresponding to a surfacing interface) separates the oceanic and the anomalous water. Integration of the steady inviscid momentum equation along the boundary gives the long-shore flow-force and shows that such a scenario leads to a paradox. Specifically, such a flow corresponds to an unbalanced flow-force and, therefore, cannot exist. To balance the integrated momentum and resolve the paradox the inflow constantly sheds anticyclones which propagate to the left due to β . Under such conditions, the momentum of the eddies moving to the left balances the momentum of the current flowing to the right. This new eddy shedding mechanism may explain why the Loop Current produces loops and why other inflows produce anticyclones. A nonlinear analytical solution to the problem is constructed with the aid of a new and powerful theoretical approach which is based on the idea that, after each eddy generation process, the system returns to its original state. This implies that *nonlinear periodic flows* can be integrated over a control volume in a similar manner to the integration used in steady flows. This novel method enables us to extract the details of the resulting features (i.e., their size, speed, periodicity and depth of the shedded rings) without solving for the details of the incredibly complicated three-dimensional and time-dependent generation process. It turns out that the problem involves a *new eddy length scale* $R_d/\varepsilon^{1/6}$ (where R_d is the parent current Rossby radius and $\varepsilon = \beta R_d/f_0$) which is somewhat *greater* than that of most eddies. Calculations were made for both zero and finite potential vorticity flows; they show that, for currents such as the Loop Current which transports about 20 Sv, eddies are shed approximately once every 300 days. Quantitative numerical experiments using the Bleck and Boudra model show that, indeed, an inflow along a straight coastline produces eddies next to the source.

1. Introduction

The question of how and why eddies are generated in the ocean is important for our understanding of the transfer of mass, salt and heat.

Traditionally, eddy formation processes have been associated with zonal current instabilities. Such instabilities generate meanders that grow, close upon themselves and then pinch off. Although this is probably the correct formation mechanism for many eddies and rings in the ocean (e.g., Gulf Stream and Kuroshio rings), *some eddies and rings* (e.g., Loop Current eddies in the Gulf of Mexico) *are always generated in the same location suggesting some other, geographically controlled,*

* Corresponding author.
nof@ocean.fsu.edu.

† On leave from SHOM/CMO, 13 rue du Chatellier, BP 426, 29275 Brest Cedex, France.

formation mechanism. This paper focuses on a new eddy generation process that is associated with such a geographical control. Specifically, we focus on currents turning away from their original direction due to a change in the coastline geometry. We shall see that a turn of this kind leads to a paradox because it produces an unbalanced flow-force. This force can be compensated for by shedding eddies on the other side and we argue that the actual situation in the ocean involves such a compensation.

The essence of this article is a new physical process and a new mathematical technique that we have recently found; its principles can be qualitatively described as follows. Consider a warm northward flowing jet which, at some latitude, turns eastward due to a change in the coastline shape. In an analogy to a rocket and a rotating sprinkler, the turned jet exerts a force toward the west and this force must somehow be balanced. It turns out that, in the ocean, such a force can be balanced by shedding anticyclonic eddies which would move westward due to β . The westward movement of these eddies provides the necessary compensating force.

The essence of the new mathematical method which has not been used before in either oceanography or meteorology is to integrate the nonlinear shallow water equations over one period of eddy generation. We have developed this technique by taking advantage of the fact that, after each eddy generation period, the field returns to its original structure. We can then analytically compute the eddy size, speed and generation period without solving for the very complicated time-dependent (and three-dimensional) generation process.

The details of the new physical process are described in Section 2 where the problem is formulated. The new mathematical technique is described in Sections 3 and 4, and the solution for zero and uniform potential vorticity flows is given in Section 5. We also performed a set of quantitative numerical simulations that show the shedding process in detail (Section 6). Application of our theory to the generation of loop-current eddies in the Gulf of Mexico is discussed in Section 7.

2. Formulation

This section describes the physics of the problem and the mathematical approach.

2.1. The paradox

We suggest that many eddies and rings are generated by a new and intriguing process which is a result of the "momentum imbalance paradox." The principle of this imbalance is best illustrated using the following example.

Consider a northward meridional channel carrying relatively light water (with density ρ) emptying into an otherwise stagnant ocean (with density $\rho + \Delta\rho$) and assume that the streamlines remain parallel to the channel wall until the opening (i.e., the channel mouth) is reached (Fig. 1). In reality one would expect the streamlines to be somewhat curved a distance of up to $O(W)$ (where W is the channel width) upstream from the mouth but the assumption of parallel streamlines is reasonable and has been used before in many similar problems (see e.g., Nof, 1978a,b; 1981b; 1996). The reader who is concerned about the validity of this assumption is referred to Nof (1988, Fig. 4 and pages 188–191) where it is shown that in the limit of $W \rightarrow 0$ the flow becomes symmetrical with respect to the x axis so that (in conventional notation)

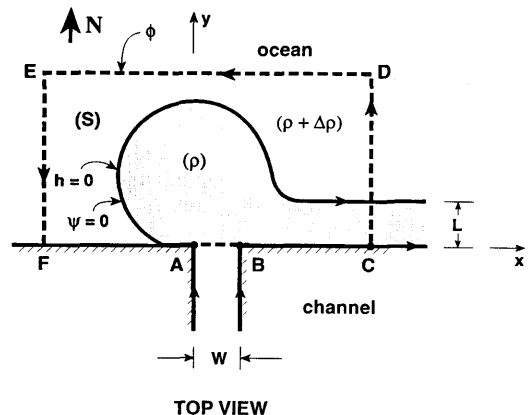


Fig. 1. An example of "the momentum imbalance paradox." A northward meridional channel carrying water with density (ρ) empties into an otherwise stagnant ocean with density ($\rho + \Delta\rho$). Without loss of generality we may choose $\psi = 0$ along the front ($h = 0$). The streamlines in the channel are assumed to remain parallel to the channel walls until the coastline is reached (i.e., section AB). Assuming that there is an inviscid steady state corresponding to the current hugging the coastline on the right-hand side, one finds that the momentum imparted on the region bounded by $ABCDEFA$ by the water exiting through CD is not balanced.

$\int_A^B h u v \, dx \rightarrow 0$. We shall see later that this implies that, at least for the cases where the channel is not very broad, the inflow does not contribute much to the momentum flux in the long-shore direction so that the assumption of parallel streamlines in the channel is adequate. We shall see later that this aspect is clearly supported by our numerical simulations (Subsection 7b).

The region immediately to the right of corner B [which is a point of infinite speed (see e.g., Batchelor, 1967)] might involve a separation from the wall and re-attachment (Cherniawsky and LeBlond, 1986) but whether or not such a separation exists has no bearing on our calculations.

As the light current exits the channel it must turn to the right and hug the coast because this is where Kelvin waves will propagate to and this is the only place where such a current can have a finite cross-sectional area. We shall now (temporarily) follow the traditional oceanographic view of this problem and assume that a steady state can be reached. One can then integrate the steady nonlinear x momentum equation over the (fixed) region (S) bounded by the dashed line ABCDEFA shown in Fig. 1,

$$\iint_S \left[h u \frac{\partial u}{\partial x} + h v \frac{\partial u}{\partial y} - f v h + g' h \frac{\partial h}{\partial x} \right] dx \, dy = 0, \quad (2.1)$$

where $f = f_0 + \beta y$, h is the thickness of the light water (i.e., $h \equiv 0$ outside the current), S is the integration area, and the remaining notation is conventional. (Note that, for convenience, the variables are defined both in the text and Section 9.)

Using the continuity equation and a streamfunction ψ (defined by $\partial\psi/\partial x = v h$; $\partial\psi/\partial y = -u h$), (2.1) can also be written as

$$\iint_S \left[\frac{\partial}{\partial x} (h u^2) + \frac{\partial}{\partial y} (h u v) \right] dx \, dy - \int_S \int f \frac{\partial \psi}{\partial x} dx \, dy + \frac{g'}{2} \int_S \int \frac{\partial (h^2)}{\partial x} dx \, dy = 0, \quad (2.1a)$$

which, with the aid of the Stokes' theorem, can be simplified to,

$$\oint_{\phi} h u v \, dx - \oint_{\phi} (h u^2 + g' h^2 / 2 - f \psi) \, dy = 0, \quad (2.1b)$$

where ϕ is the boundary of S .

Noting that along the zonal boundary (i.e., $dy \equiv 0$) at least one of the three variables h , u , v is always zero and defining $\psi = 0$ where $h = 0$, one finds that (2.1b) can also be written as,

$$\int_C^D [h u^2 + g' h^2 / 2 - f \psi] \, dy = 0. \quad (2.2)$$

Assuming (and later verifying with our numerical experiments) that the flow is parallel to the wall and, hence, geostrophic in CD (so that $f u = -g' \partial h / \partial y$ which, upon multiplication by h and integration in y from a point within the current to its edge, gives $f \psi = g' h^2 / 2 - \beta \int_y^L \psi \, dy$), relation (2.2) reduces to the simple relationship,

$$\int_0^L h u^2 \, dy + \beta \int_0^L \left(\int_y^L \psi \, dy \right) dy = 0, \quad (2.3)$$

where L is the width of the current downstream.

The curious result is that condition (2.3) cannot be satisfied because y and ψ are always positive along CD and the integration is done from small to large y . Note also that, for most boundary currents [i.e., $U \sim O(g'H)^{1/2}$; $L \sim O(g'H)^{1/2}/f$], the second term $[O(\beta L/f_0)]$ is much smaller than the first term $[\sim O(1)]$. The impossibility to satisfy (2.3) implies that there cannot be a steady state of the kind originally assumed because the integrated momentum (or flow-force) imparted (by the fluid exiting through CD) on the control volume (bounded by ABCDEF) cannot be balanced.

One way that the flow field can balance this momentum flux is by somehow creating a flow in the opposite sense (i.e., the negative x direction). Since a light westward current with a finite cross-section is impossible without a wall on the north side, the balance cannot be achieved with a steady westward flow. It can, however, be achieved via the generation of anticyclonic eddies which would propagate to the left due to β (Fig. 2). For our modeling purposes, it will be assumed that the generated eddies are formed offshore so that they are not in direct contact with the wall. Our numerical simulations (Section 7) will later show that the actual interaction of the eddies with the wall is indeed weak. Furthermore, we shall see

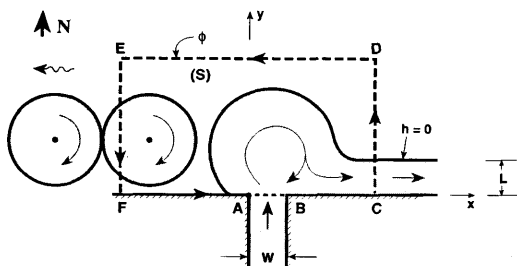


Fig. 2. Schematic diagram showing the resolution of the momentum imbalance paradox. Because of the imbalance shown in Fig. 1, anticyclonic eddies are generated on the left-hand side (looking off-shore). Through the β effect these eddies are forced to propagate to the left. Under such conditions the flow force exerted by the flow to the right (i.e., in the positive x direction) is now balanced by the force corresponding to the eddy's transport to the left. The "wiggly" arrow denotes migration and should be distinguished from solid arrows which denote flow direction.

later that for various applications, such as the Gulf of Mexico, the assumption is certainly valid because the observed rings are relatively far from the wall which is situated to the south of the inflow. Also, note that integration of the y -momentum equation over the area shown in Fig. 1 does not lead to an additional paradox nor does it give any additional information because it involves an additional unknown force acting on the wall.

Before proceeding it is appropriate to point out that, from a strict mathematical point of view, our problem is really over-constrained (due to the forced steadiness) rather than paradoxical. We chose the term "paradox" because, from a physical point of view, one would expect a steady solution of the kind shown in Fig. 1 to exist and yet our considerations show that it cannot. Note also that our paradox may not exist in the case of a curved coastline (or a topography which depends on x) because, under such conditions, the flow-force may be exerted on the wall rather than the fluid. In addition, it is pointed out that our scenario is consistent with Davey and Killworth's (1989) suggestion that a steady source can at times produce a discrete set of eddies. Note, however, that the point source considered by Davey and Killworth is very different from our inflow configuration.

2.2. The periodic integrated momentum theorem

To solve for the size, number and periodicity of the generated eddies we shall develop a new integrated momentum theorem for periodic flows. This equation, which is the heart of this article, will enable us to circumvent the time dependent generation process itself. It will give us the information that we need regarding the eddies exiting the domain and the outcome of the generation process without giving us information about the detailed formation process itself.

To derive the desired equation we begin with an integration of the time dependent x momentum equation (multiplied by h) once in time,

$$\int_0^T h \frac{\partial u}{\partial t} dt + \int_0^T hu \frac{\partial u}{\partial x} dt + \int_0^T hv \frac{\partial u}{\partial y} dt - \int_0^T hfv dt = - \int_0^T g' \frac{\partial}{\partial x} (h^2/2) dt, \quad (2.4)$$

where T is the period of eddy generation.

Leaving (2.4) aside for a moment we note that multiplication of the time-dependent continuity equation by u and integration in time over one period gives,

$$\int_0^T u \frac{\partial h}{\partial t} dt + \int_0^T u \frac{\partial}{\partial x} (hu) dt + \int_0^T u \frac{\partial}{\partial y} (hv) dt = 0.$$

Adding this equation to (2.4) gives,

$$\int_0^T \frac{\partial}{\partial x} (hu^2) dt + \int_0^T \frac{\partial}{\partial y} (huv) dt - \int_0^T hfv dt + \int_0^T \frac{g'}{2} \frac{\partial}{\partial x} (h^2) dt = 0, \quad (2.4a)$$

because

$$\int_0^T \frac{\partial}{\partial t} (hu) dt = hu|_T - hu|_0 = 0$$

(i.e., the system always returns to its original structure after one period). We now integrate (2.4a) over S (Fig. 2). Since the domain S is fixed we can exchange the order of integration with respect to time and space. One finds,

$$\int_0^T \int_S \frac{\partial}{\partial x} (hu^2) dx dy dt$$

$$\begin{aligned}
& + \int_0^T \iint_S \frac{\partial}{\partial y} (huv) \, dx \, dy \, dt \\
& - \int_0^T \iint_S hfv \, dx \, dy \, dt \\
& + \frac{g'}{2} \int_0^T \iint_S \frac{\partial}{\partial x} \left(\frac{h^2}{2} \right) \, dx \, dy \, dt = 0.
\end{aligned} \quad (2.5)$$

Next, we define a time-integrated streamfunction $\tilde{\psi}$ (to be distinguished from the steady streamfunction ψ which is defined in the usual manner),

$$\left. \begin{aligned} \partial \tilde{\psi} / \partial y &= - \int_0^T hu \, dt \\ \partial \tilde{\psi} / \partial x &= \int_0^T hv \, dt \end{aligned} \right\}. \quad (2.6)$$

This definition stems from an integration of the continuity equation in time (from zero to T), noting again that $h(T) = h(0)$ and changing the order of integration so that,

$$\frac{\partial}{\partial x} \left(\int_0^T hu \, dt \right) + \frac{\partial}{\partial y} \left(\int_0^T hv \, dt \right) = 0.$$

Using the streamfunction and Stokes' theorem (and noting again that along the zonal boundary at least one of the three variables h , u and v is always zero), enables us to express (2.5) as,

$$\begin{aligned}
& \int_0^T \oint_{\phi} hu^2 \, dy \, dt - \oint_{\phi} f \tilde{\psi} \, dy \\
& + \frac{g'}{2} \oint_{\phi} \int_0^T h^2 \, dy \, dt = 0,
\end{aligned} \quad (2.7)$$

where the arrowed circle denotes counterclockwise integration.

It is now assumed (and later verified by our numerical experiments) that, even though eddies are generated near the mouth, the downstream flow across CD is approximately steady and geostrophic. This implies that (2.7) can also be written as,

$$\begin{aligned}
& \int_0^T \int_F^E (hu^2 + g'h^2/2) \, dy \, dt - \int_F^E f \tilde{\psi} \, dy \\
& = T \int_C^D (hu^2 - f\psi + g'h^2/2) \, dy,
\end{aligned} \quad (2.8)$$

where, as before, ψ is the steady streamfunction defined in the usual manner.

Equation (2.8) is our desired flow-force balance and is the essence of this article. In an analogy to a rocket and a spinning sprinkler, the right-hand side corresponds to the flow-force exerted on the domain by the water exiting on the right. It is the flow-force (or integrated momentum) that we argued is not balanced unless eddies are shed on the left. The term on the left-hand side of (2.8) corresponds to the flow-force exerted by the eddies moving to the left out of the domain. It is a simple matter to compute the contribution of the steady current on the right to (2.8) but computation of the eddies' contribution (which is time-dependent) is not so trivial. It is, therefore, discussed in detail in the next section.

Before proceeding, it is appropriate to point out that not all situations corresponding to the momentum imbalance paradox obey (2.8). This results from the fact that, due to the lack of an eddy removal mechanism, a flow may not be periodic but rather may evolve constantly in time. Examples are, a channel emptying into an otherwise stagnant ocean on an f -plane or a channel flow which is directed southward rather than northward. Our numerical simulations will later show that, in the case of a channel on an f -plane, a forever growing eddy is established near the mouth. The eddy is constantly growing because the curving flow bifurcates as it impinges on the wall (between points B and C , Fig. 2). As a result of the bifurcation, part of the flow turns backwards causing the volume of the eddy to constantly grow. On a β plane, this growth causes a steady increase in the tendency of the eddy to migrate westward; ultimately, this tendency overcomes the growth and the eddy detaches. This means that the role of β in our problem is merely to arrest the growth of the eddies and remove them from the generation area.

2.3. Other constraints

In addition to the above momentum constraint, the field must, of course, satisfy the continuity equation,

$$Q = \int_C^D hu \, dy + \frac{1}{T} \int_{\text{eddy}} h \, dx \, dy, \quad (2.9)$$

where Q is the mass flux of the channel. Also, the potential vorticity equation,

$$\frac{D}{Dt} \left(\frac{\partial v / \partial x - \partial u / \partial y + f}{h} \right) \equiv 0, \quad (2.10)$$

must be satisfied. In addition, we shall assume (and later verify with our numerical simulation) that along the right bank, away from the eddy generation region (see Fig. 2), the flow is approximately steady so that the Bernoulli integral is satisfied and the upstream and downstream fields can be connected, i.e.,

$$\left[\frac{V^2}{2} + g'H \right] = \left[\frac{U_l^2}{2} + g'H_l \right], \quad (2.11)$$

where the subscript "l" indicates that the variable in question is associated with the steady long-shore current on the right-hand side, and H is the upstream current depth next to the right wall (looking downstream).

It will become clear shortly that the above system of equations is not sufficient to close the problem with our integrated balances approach. A reasonable closure condition is that the eddies are approximately circular and are touching each other as they leave the domain. Although there are known solutions for noncircular elliptical eddies, anticyclones that are not strongly interacting with each other (as is the case here due to the vanishing depth along the rim) tend to be approximately circular on a β -plane (Killworth 1983). Also, although there is no physical reason for the eddies to touch each other as they leave the domain, our closure condition gives us a reasonable upper bound on the eddy mass flux, the generation frequency, and the eddies' size*. Our numerical simulations (Section 6) will later show that, although there is some spacing between the exiting eddies, the above bounds are useful and the analytically predicted mass fluxes are not very far from the actual numerical values.

Finally, note that the "touching assumption"

does not mean that, during the formation process, the current is always in contact with the separated ring. All it says is that the eddies touch each other when they leave the generation area. Since the upstream conditions in the channel are known and the β -induced eddy drift is also known (Nof, 1981a), the use of (2.10) and (2.11) will ultimately lead to two unknowns (the long-shore current width and the eddies' size). The solution to these two unknowns will be provided by (2.8) and (2.9).

3. The integrated momentum balance

As mentioned, this calculation is not straightforward because all the terms associated with the left-hand side of equation (2.8) are time-dependent. To simplify the procedure, we shall transform those terms to a new coordinate system $(\hat{x}, \hat{y}, \hat{t})$ moving steadily westward at the speed of the exiting eddies.

3.1. Transformation to a moving coordinate system

The relationship between the old fixed system (x, y, t) and the new moving system $(\hat{x}, \hat{y}, \hat{t})$ is:

$$t = \hat{t}, \quad x = \hat{x} + C\hat{t}, \quad y = \hat{y}, \quad (3.1)$$

where C is the migration rate (which is taken to be positive when it is directed in the positive x direction). According to (3.1), the transformed integrations are associated with,

$$dt = d\hat{t} \quad (3.2a)$$

$$dy = d\hat{y} \quad (3.2b)$$

$$dx = d\hat{x} + C d\hat{t}. \quad (3.2c)$$

Since the integration is done across a *fixed* meridional boundary, $dx = 0$ and (3.2c) gives $d\hat{x} = -C d\hat{t}$ which implies,

$$d\hat{t} = -\frac{1}{C} d\hat{x}. \quad (3.3)$$

In view of this, the left-hand side of (2.8) which describes the exit of the eddy takes the form,

$$\begin{aligned} & -\frac{1}{C} \int_0^a \int_{b_1}^{b_2} h(\hat{u} + C)^2 d\hat{y} d\hat{x} - \int_{b_1}^{b_2} f\tilde{\psi} d\hat{y} \\ & - \frac{g'}{2C} \int_0^a \int_{b_1}^{b_2} h^2 d\hat{y} d\hat{x}. \end{aligned} \quad (3.4)$$

* The fact that the "touching assumption" gives an upper bound on the eddy mass flux, frequency and size can be easily demonstrated using the well-known formulas for zero potential vorticity lenses. For such lenses, the westward eddy mass flux is given by $\beta\pi f^2 R^6/128g'D$ (where R is the eddy radius and D is the distance between the center of two consecutive eddies) so that, for a given D , it is maximum when $R = D/2$.

To get an expression for $\tilde{\psi}$ we note that,

$$\begin{aligned}\frac{\partial \tilde{\psi}}{\partial \hat{y}} &= \frac{1}{C} \int_0^a h(\hat{u} + C) d\hat{x} \\ &= \frac{1}{C} \int_0^a -\frac{\partial \tilde{\psi}}{\partial \hat{y}} d\hat{x} + \int_0^a h d\hat{x},\end{aligned}\quad (3.5)$$

which can be integrated in $\hat{y}$ to give,

$$\tilde{\psi}(b_2) - \tilde{\psi}(\hat{y}) = \frac{1}{C} \int_0^a \tilde{\psi} d\hat{x} + \int_{\hat{y}}^{b_2} \int_0^a h d\hat{x} d\hat{y}, \quad (3.5a)$$

where we defined $\tilde{\psi}$ to be zero outside the eddy. Also, without loss of generality, we can choose $\tilde{\psi}(b_2) = 0$; if this choice is not made the terms involving $\tilde{\psi}(b_2)$ will ultimately drop out anyway. Note that the integration described by (3.4) is done over a bounding square that touches the eddy (Fig. 3). Also, note that, here, $\tilde{\psi}$ is the *time integrated streamfunction* which is different from both the streamfunction in the moving system $\hat{\psi}$ (originating from the continuity equation $\partial(h\hat{u})/\partial\hat{x} + \partial(h\hat{v})/\partial\hat{y} = 0$) and the usual streamfunction ψ . In addition, note that b_2 is taken to be constant because, in a layer-and-a-half "reduced-gravity" model, eddies drift directly toward the west.

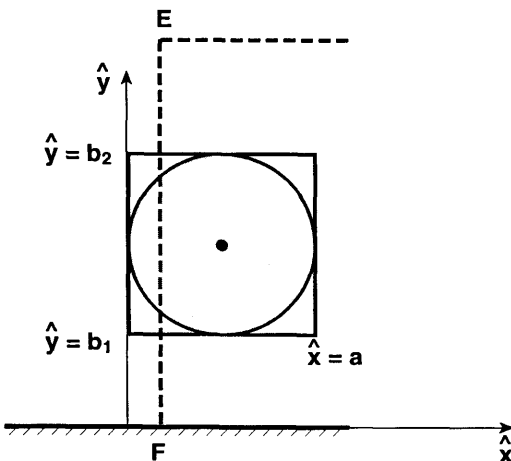


Fig. 3. Schematic diagram of the eddy's integration area in the moving coordinate system $(\hat{x}, \hat{y}, t)$. The integration is done over the square bounded by $\hat{x} = 0$; $\hat{x} = a$ and $\hat{y} = b_1$; $\hat{y} = b_2$, i.e., a bounding square that touches the eddy. The dashed line denotes the fixed integration boundary through which the eddy is exiting (see Fig. 2).

3.2. Simplifications

Although the computation of (3.4) is easier than that of the original time-dependent left-hand side of (2.8) because it is no longer time-dependent, it is still not so simple. We shall now show that (3.4) can be simplified to,

$$\begin{aligned}- \int_0^a \int_{b_1}^{b_2} hC d\hat{y} d\hat{x} \\ + \int_{b_1}^{b_2} f \left[\int_{\hat{y}}^{b_2} \left(\int_0^a h d\hat{x} \right) d\hat{y} \right] d\hat{y}.\end{aligned}\quad (3.6)$$

To show this we note that integration of the transformed $\hat{x}$ momentum equation over part of the eddy bounded by $h = 0$ on the west and a meridional line on the east gives,

$$\begin{aligned}\int \int \left\{ \frac{\partial}{\partial \hat{x}} (h\hat{u}^2) + \frac{\partial}{\partial \hat{y}} (h\hat{u}\hat{v}) - f\hat{v}h \right. \\ \left. + \frac{g'}{2} \frac{\partial}{\partial \hat{x}} (h^2) \right\} d\hat{x} d\hat{y} = 0.\end{aligned}\quad (3.7)$$

Using Stokes' theorem and the fact that $h = 0$ and $\hat{\psi} = 0$ outside the eddy shows that for any $\hat{x}$,

$$\int_{b_1}^{b_2} (h\hat{u}^2 + g'h^2/2 - f\hat{\psi}) d\hat{y} = 0.$$

This, together with the condition

$$\int \int h\hat{u} d\hat{x} d\hat{y} = 0 \quad \left(\text{resulting from } \oint \hat{\psi} d\hat{x} \equiv 0 \right),$$

completes our demonstration that (3.4) can be replaced by (3.6).

3.3. The resulting equation

To compute the right-hand side of (2.8) we need to compute the term $(g'h^2/2 - f\psi)$. To do so, we note that $\psi = \tilde{\psi}/T = 0$ along the steady current edge ($y = L$) because there is no northward mass flux in between the eddy and the current edge where $h \equiv 0$ (i.e., $\tilde{\psi} = 0$ everywhere outside the eddy and the current). Since we take the flow to be geostrophic across CD it follows that $g'h^2/2 = f\psi - \beta \int_L^y \psi dy$. Using this condition and substituting (3.6) into the left-hand side of (2.8) gives the desired relationship:

$$\begin{aligned}
& -C \int_{b_1}^{b_2} \int_0^a h \, d\hat{x} \, d\hat{y} \\
& + \underbrace{\int_{b_1}^{b_2} \left[f \int_{\hat{y}}^{b_2} \left(\int_0^a h \, d\hat{x} \right) d\hat{y} \right] d\hat{y}}_{\text{I}} \\
& = \underbrace{T \int_0^L hu^2 \, dy}_{\text{II}} + \beta T \int_0^L \int_y^L \psi \, dy \, dy. \quad (3.8)
\end{aligned}$$

As mentioned, the left-hand side of (3.8) corresponds to the flow-force exerted on the region by the exiting eddy whereas the right-hand side is associated with the flow-force of the steady long-shore jet. The first term on the left-hand side represents the flow-force analogous to that of a moving solid body whereas the second term (I) is the force exerted by the fluid. The first term on the right-hand side (II) is the dynamical flow-force analogous to that of a moving rocket whereas the second is its modification due to β . By noting that $\beta R_d/f_0 \ll 1$ and that the eddies' migration rate is $O(\beta R_{de}^2)$ so that $C/f_0 R_{de} \sim O(\beta R_{de}/f_0) \ll 1$ (where R_{de} and R_d are the eddy and approaching current Rossby radii), it is a simple matter to show that the dominant terms in (3.8) are I and II.

Two comments should be made with regard to (3.8). First, it is important to realize that, even though (3.8) represents a balance of momentum flux integrated over a period T , the time-dependent momentum flux varies as each eddy passes (through the western boundary) leaving an instantaneous and periodic imbalance. This means that the curving part of the eastern flow also cannot be steady. It must advance and retreat into the control volume in accordance with the eddy formation process. This does not imply, however, that the one-dimensional flow far downstream is strongly time dependent. We shall see later (with the aid of our numerical simulation) that the downstream flow is approximately steady as originally assumed.

Second, as already stated, (3.8) does not necessarily imply that the eddies are in contact with each other as they leave the control volume. Spacing between the eddies is clearly possible but one has to make some sort of an assumption about the exiting eddies in order to close the problem. It is difficult to say where the edge of actual eddies

in the ocean is and the Gulf of Mexico is too small to allow the presence of more than two rings at a time. Both of these aspects make it impossible to say whether the rings would have touched each other had they been in a much larger basin. Our numerical experiments (Section 6) suggest that, in the open ocean case, there is, in fact, quite a bit of spacing between the eddies. Nevertheless, the assumption that the eddies are in contact with each other is in reasonable agreement because it gives us an upper bound on the desired unknowns. It appears to be in agreement with the observations of Kassler (1981, see (his) Fig. 7) probably because the relatively small Gulf forces the rings to be close to each other.

4. Scaling and expansion

4.1. Scales

Since the eddies are slowly removed from the generation area by β (at a speed of the order of βR_{de}^2 , where R_{de} is the eddy's Rossby radius) one expects their length scale to be much greater than the current's length scale. Indeed, a comparison of the second term on the left and the first term on the right of (3.8) (i.e., the largest terms on each side) gives,

$$f H_e R_{de}^3 \sim O \left[\frac{R_{de}}{\beta R_{de}^2} H(g'H) R_d \right],$$

which yields

$$R_{de} \sim O(R_d/\varepsilon^{1/6}), \quad (4.1)$$

where $\varepsilon \equiv \beta R_d/f_0 \ll 1$, R_d is the current Rossby radius, and it has been taken into account that $T \sim O(R_{de}/C)$. (As mentioned, we shall later assume that the generated eddies are in contact with each other as they leave the control box but it is not necessary to make this assumption at this stage.) Note that in obtaining (4.1) it has also been taken into account that the ratio of the eddy depth scale, H_e , and the approaching current depth scale, H , is given by R_{de}^2/R_d^2 . In view of (4.1), the periodicity at which the eddies are generated (T) is,

$$T \sim O(\varepsilon^{1/6}/\beta R_d). \quad (4.2)$$

These scales are sensible for the Gulf of Mexico as the parameter ε is (0.01) implying that the eddies are somewhat larger than most rings; the

eddies' depth scale h_e is of the order of the current depth divided by $\varepsilon^{1/3}$.

Two additional comments should be made with regard to (4.1) and (4.2). First, as is frequently the case, the scaling may conceal potentially large numbers such as powers of the known $2\sqrt{2}$ ratio between the eddy radius and the Rossby radius (see e.g., Nof 1981b, Killworth 1983). Second, the $1/6$ power of ε implies that, for most cases, R_{de} will not be greatly different from R_d .

Given the above scales we now introduce the following nondimensional parameters. For the long-shore current leaving the control volume through CD (i.e., region 2) the scaled variables are,

$$\begin{aligned} x_l^* &= x/R_d, & y_l^* &= y/R_d, \\ v_l^* &= v/(g'H)^{1/2}, & h_l^* &= h/H, \\ R_d &\equiv (g'H)^{1/2}/f_0, & \psi^* &= \psi/g'H^2/f_0. \end{aligned} \quad (4.3)$$

Similarly, the scaled variables for the eddies are,

$$\begin{aligned} x_e^* &= x/R_{de}, & y_e^* &= y/R_{de}, & h_e^* &= \frac{h}{H(R_{de}/R_d)^2}, \\ C^* &= C/\beta R_{de}^2, & R_{de} &= \alpha R_d/\varepsilon^{1/6}, & \alpha &\sim O(1), \\ \varepsilon &\equiv \beta R_d/f_0 \ll 1, & T^* &= \frac{T}{\varepsilon^{1/6}/\beta R_d}, \end{aligned} \quad (4.4)$$

where α is an unknown number of order unity that measures the size of the resulting eddies. Note that the introduction of α in (4.4) is necessary because the Rossby radius of the eddies determines their size and speed.

The above scaling implies that the ratio of the eddy depth to the parent current depth is $O(\varepsilon^{1/3})$ which means that the eddy is deeper than the current. Since the eddy must have an anticyclonic vorticity, this means that the parent current must also have anticyclonic vorticity and that it must be large. Under such conditions, the large initial anticyclonic vorticity can compensate for the stretching of the water columns which occurs during the eddy formation and introduces cyclonic vorticity. We shall see that, for the Gulf of Mexico, the anticyclonic vorticity of the parent current is indeed large so that our scaling is appropriate. Of course, one can think of parent currents with cyclonic rather than anticyclonic vorticity but, evidently, such currents must include some processes which are not considered here. We shall return to this point briefly in Section 6 where,

with the aid of our numerical experiments, it will be shown that through the action of small friction the potential vorticity of a (parent) cyclonic current is gradually altered so that anticyclones are again produced. A thorough investigation of this situation is beyond the scope of this study.

4.2. The nondimensional equations

Substitution of (4.3) and (4.4) into the integrated momentum (3.8) gives,

$$\begin{aligned} C^* \alpha^6 \varepsilon^{5/6} \int_{b_1^*}^{b_2^*} \int_0^{a^*} h_e^* dx_e^* dy_e^* \\ + \alpha^5 \int_{b_1^*}^{b_2^*} (1 + \alpha \varepsilon^{5/6} y^*) \\ \times \int_{y^*}^{b_2^*} \left(\int_0^{a^*} h_e^* dx_e^* \right) dy_e^* dy_e^* \\ = T^* \int_0^{L^*} h_l^* (u_l^*)^2 dy_l^* \\ + \varepsilon T^* \int_0^{L^*} \int_{y^*}^{L^*} \psi_l^* dy_l^* dy_l^*. \end{aligned} \quad (4.5)$$

Similarly, the continuity equation can be written as,

$$Q^* = \int_0^{L^*} h_l^* u_l^* dy_l^* + \frac{\alpha^4 \varepsilon^{1/6}}{T^*} \int_0^{a^*} \int_{b_1^*}^{b_2^*} h_e^* dy_e^* dx_e^*, \quad (4.6)$$

where Q^* is the known nondimensional mass flux out of the channel defined by $Q/[g'H^2/f_0]$.

Recall that, (i) in addition to the above two equations, the field is governed by the potential vorticity eq. (2.10), and (ii) we have assumed that the steady Bernoulli principle is satisfied along the right bank (2.11). We shall shortly see that these, together with our earlier closure condition stating that the eddies are touching each other as they leave (and the fact that we know how fast eddies migrate on a β plane), will enable us to close the problem. Note that the problem is not subject to any additional constraints and that the Bernoulli integral cannot be applied to the outward edge of the current ($h=0$) because the flow there is clearly *unsteady*.

4.3. Expansions

To obtain the solution, all the dependent variables are now expanded in a power series in $\varepsilon^{1/6}$, e.g.,

$$\left. \begin{aligned} h_e^* &= h_e^{(0)} + \varepsilon^{1/6} h_e^{(1)} + \dots \\ h_l^* &= h_l^{(0)} + \varepsilon^{1/6} h_l^{(1)} + \dots \\ u_l^* &= u_l^{(0)} + \varepsilon^{1/6} u_l^{(1)} + \dots \end{aligned} \right\}. \quad (4.7)$$

Substitution of (4.7) into (4.5) and (4.6) gives the following leading-order balances,

$$\begin{aligned} \int_{b_l^{(0)}}^{b_e^{(0)}} \int_y^{b_e^{(0)}} \left(\int_0^{a^{(0)}} h_e^{(0)} dx_e^* \right) dy_e^* dy_e^* \\ = \frac{T^{(0)}}{(\alpha^{(0)})^5} \int_0^{L^{(0)}} h_l^{(0)} (u_l^{(0)})^2 dy_l^*, \end{aligned} \quad (4.8)$$

$$Q^* = \int_0^{L^{(0)}} h_l^{(0)} u_l^{(0)} dy_l^*. \quad (4.9)$$

These balances imply that the mass removed by the eddies is small and negligible (to leading order) but the momentum exerted by the translating eddies is not small and cannot be neglected. Note also that up to $O(\varepsilon)$ the edge of the eddy is a circle (Killworth, 1983). Both of these aspects will be later supported with our numerical simulations. Finally, note that the periodicity $T^{(0)}$ is related to the eddies' β -induced drift through the diameter of the eddies.

5. Solution

As mentioned, since our scaling implies that the eddies must be much deeper than the parent current [$H_e \sim O(H/\varepsilon^{1/3})$] it follows that the parent current must have strong anticyclonic vorticity. This results from conservation of potential vorticity. It implies that the eddies' vorticity will be anticyclonic (despite the severe stretching of the water column which occurs during the formation) only if the parent current vorticity is strongly anticyclonic. Both the Loop Current and the subsequent Florida Current satisfy this condition of strong anticyclonic vorticity (see e.g., the surface velocity structure discussed by Brooks and Niiler, 1977; Molinari and Morrison, 1988; and Richardson et al., 1969). As already mentioned, what happens when the parent current vorticity is not strongly anticyclonic is not entirely clear. We shall see later, however, that, according to our

numerical simulations, friction plays a very important role and changes the potential vorticity so that anticyclones are ultimately produced.

5.1. Zero potential vorticity eddies

We shall first look at the case where both the current and the eddies have zero potential vorticity because it is the simplest possible case. Under such conditions, the solutions for both the long-shore current and the eddy are straightforward despite the nonlinearity. Specifically, the leading-order solution for the long-shore current is,

$$\left. \begin{aligned} u &= f_0 y + U_l \\ h &= \frac{f_0^2}{2g'} (L^2 - y^2) + U_l (L - y) \frac{f_0}{g'} \end{aligned} \right\}, \quad (5.1)$$

where U_l is the near-wall speed, L is the (unknown) long-shore current width and, for simplicity, we have returned to dimensional variables. Hence, the long-shore current near-wall depth H_l is

$$H_l = \frac{f_0^2 L^2}{2g'} + f_0 L \frac{U_l}{g'}. \quad (5.2)$$

Note that the variables H_l and U_l are also connected to each other via the application of the Bernoulli to the wall on the right-hand side,

$$\left[\frac{1}{2} (V^2) + g'H \right] = \left[\frac{1}{2} (U_l)^2 + g'H_l \right], \quad (5.3)$$

because both V and H are known upstream parameters near the right bank. This implies that both U_l and H_l can now be easily expressed as a function of the unknown L .

Similarly, the leading-order solution for the eddy is,

$$\left. \begin{aligned} v_\theta &= -fr/2 \\ h &= \frac{f_0^2}{8g'} (8R_{de}^2 - r^2) \end{aligned} \right\}, \quad (5.4)$$

where, for simplicity, we use here a polar coordinates system (r, θ) whose origin coincides with the center of the eddy; R_{de} is the (unknown) Rossby radius of the eddy. The above general solution has *two* unknowns, the width of the long-shore current, L , and the eddy size which is related to its Rossby radius R_{de} .

Insertion of the nondimensional form of (5.1–5.4) into (4.8) and (4.9) gives the following

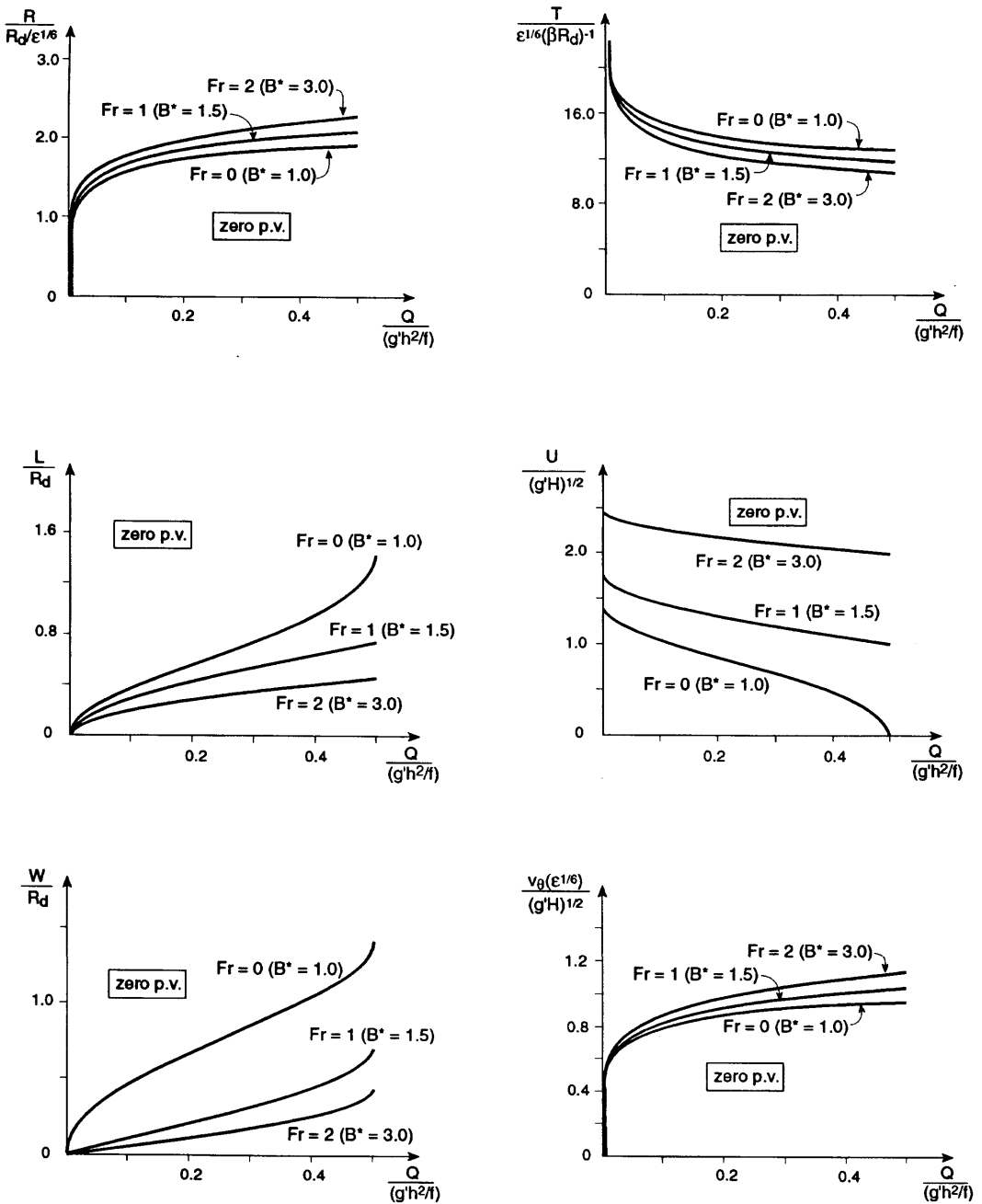


Fig. 4. Upper panel. The eddy diameter (R) and generation period (T) as a function of the channel discharge (Q) for zero potential vorticity flow. Middle panel. The corresponding long-shore current width (L) and near-wall speed (U) as a function of the channel discharge. Lower panel. The channel width (W) and the eddy speed along the edge (v_θ) as a function of the channel discharge. Note that Fr is the channel Froude number defined by $V/(g'H)^{1/2}$ where V and H are the speed and depth next to the right wall ($x = W/2$; $y \rightarrow -\infty$). It is related to the non-dimensional

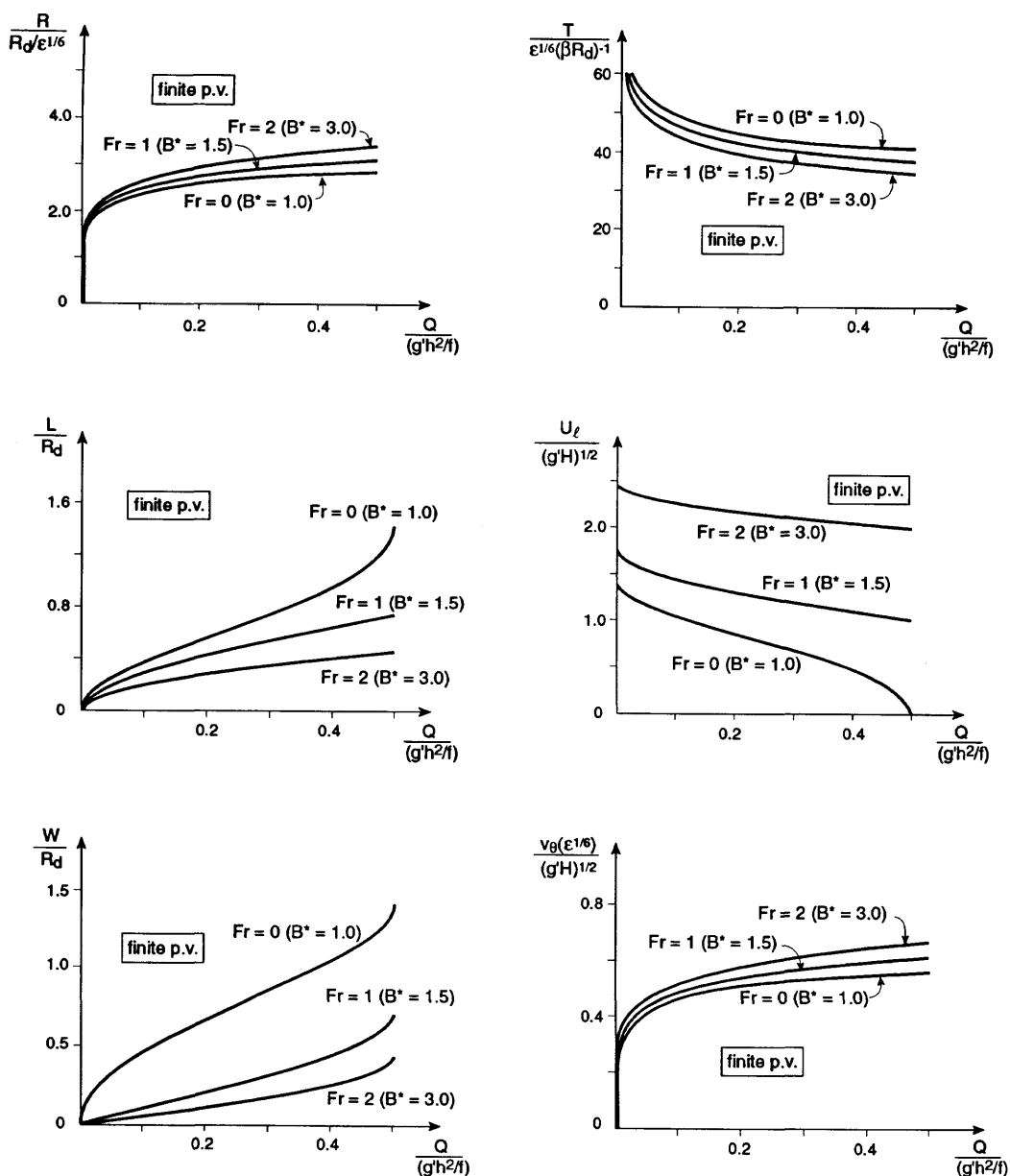


Fig. 5. The same as Fig. 4 except that the resulting eddies now have a finite potential vorticity corresponding to a vorticity $[1/r(d/dr)(rv_\theta)]$ of $-0.05f$. Such a vorticity is typical for many oceanic rings. Note that the results are very similar to the zero potential vorticity case (Fig. 4) except that T^* and R^* are greater.

Bernoulli B^* through $B^* = B/g'H = 1 + Fr/2$. Note that $Q^* \leq 1/2$ because along the left bank $h^* \geq 0$. For $Q^* = 1/2$, the depth near the left wall vanishes and the structure of the flow in the channel is identical to that of the long-shore current. Note that T increases as Q^* decreases, whereas R decreases as Q^* decreases.

leading-order algebraic equations,

$$\frac{4\pi}{3}(\alpha^{(0)})^6 = \frac{(L^{(0)})^5}{15} + \frac{U_l^{(0)}(L^{(0)})^4}{3} + \frac{2}{3}(U_l^{(0)})^2(L^{(0)})^3 + \frac{1}{2}(U_l^{(0)})^3(L^{(0)})^2, \quad (5.5)$$

$$Q^* = \frac{(L^{(0)})^4}{8} + \frac{(L^{(0)})^3}{2} U_l^{(0)} + (L^{(0)})^2 \frac{(U_l^{(0)})^2}{2}, \quad (5.6)$$

where U_l , the long-shore current near-wall speed, is given by,

$$U_l^{(0)} = -L^{(0)} + (2B^*)^{1/2}. \quad (5.7)$$

Here, B^* is the *known* upstream Bernoulli ($V^2/2 + g'H$) along the right wall nondimensionalized by $g'H$ (which is also known). Eqs. (5.6) and (5.7) can be combined to give

$$L^{(0)} = (2B^*)^{1/2} - [2B^* - 2(2Q^*)^{1/2}]^{1/2}, \quad (5.8)$$

$$U_l^{(0)} = [2B^* - 2(2Q^*)^{1/2}]^{1/2}, \quad (5.9)$$

so that $Q^* \leq (B^*)^2/2$ which means that a minimum amount of energy is required in order to force a given amount of fluid out of the channel. Note that in deriving (5.5–5.7) it has been taken into account that the westward migration of a zero potential vorticity eddy due to β is $-\frac{2}{3}\beta R_d^2$ (Nof, 1981a). The set (5.5–5.7) gives the desired solution for both $\alpha^{(0)}$ and $L^{(0)}$ in terms of the known upstream flux Q and the known Bernoulli energy along the upstream wall B . The appropriate root of $L^{(0)}$ was chosen such that $U_l^{(0)}$ is always positive.

For the special case of $U_l^{(0)} = 0$ (i.e., no long-shore current speed along the wall) the dimensional solution for the eddy radius (R), the current width (L), the periodicity (T) and the eddy mass flux (Q_e) is,

$$\left. \begin{aligned} R &= \frac{2^{43/24}}{(5\pi)^{1/6}} \left(\frac{Qg'}{f_0^3 R_d^4} \right)^{5/24} \frac{R_d}{\varepsilon^{1/6}} \\ L &= \left(\frac{8g'Q}{f_0^3} \right)^{1/4} \\ T &= 3(5\pi)^{1/6} 2^{29/24} / f_0 \varepsilon^{5/6} \left(\frac{Qg'}{f_0^3 R_d^4} \right)^{5/24} \\ Q_e &= \frac{\pi^{1/6} 2^{47/24} \varepsilon^{1/6}}{3 \times 5^{5/6}} \left(\frac{g'}{f_0^3 R_d^4} \right)^{1/24} Q^{25/24} \end{aligned} \right\}, \quad (5.10)$$

where, as before, R_d is the Rossby radius based on the depth along the channel right bank and Q

is the channel mass flux. For the more general case of $U_l^{(0)} \neq 0$, the results can also be obtained analytically but they are somewhat more complicated; they are given in Fig. 4. This completes our discussion of zero potential vorticity flows.

5.2. Finite potential vorticity eddies

For finite potential flow, the final calculations must be done numerically. Such calculations were performed for the case where the potential vorticity depth is 5% greater than the eddies' maximum depth (Fig. 5). This corresponds to a relative vorticity of $-0.05f$ which is typical for many eddies in the ocean. Because of the stretching that occurs during the formation process, the corresponding parent current potential vorticity is close to zero. Note that the eddy β -induced westward speed was calculated from the formula given by Nof (1981a) and the structure of the eddies from the relationships given originally by Flierl (1979). Note that the main difference between the zero and finite potential vorticity eddies is that, in the finite potential vorticity case, both the periodicity and the eddy size are larger than those in the zero potential vorticity case.

6. Numerical simulations

To further analyze the validity of our (many) assumptions (e.g., that the eddies' interaction with the wall is unimportant and that the flow is parallel to the wall downstream), *quantitative* numerical simulations were performed.

6.1. Model description

We used the Bleck and Boudra (1986) reduced gravity isopycnic model with a passive lower layer. For clarity, we began with a numerical simulation of a channel on an f -plane (Fig. 6) which shows that a forever growing eddy is established near the channel mouth. As mentioned, the eddy is established because of the backward force of the downstream current. We then proceeded to examine the role of β (Figs. 7–14). We used the Orlanski (1976) second-order radiation conditions for the open boundary on the right. One can see from Figs. 6–14 that this condition is satisfactory because the downstream streamlines are not disturbed when they cross the boundary.

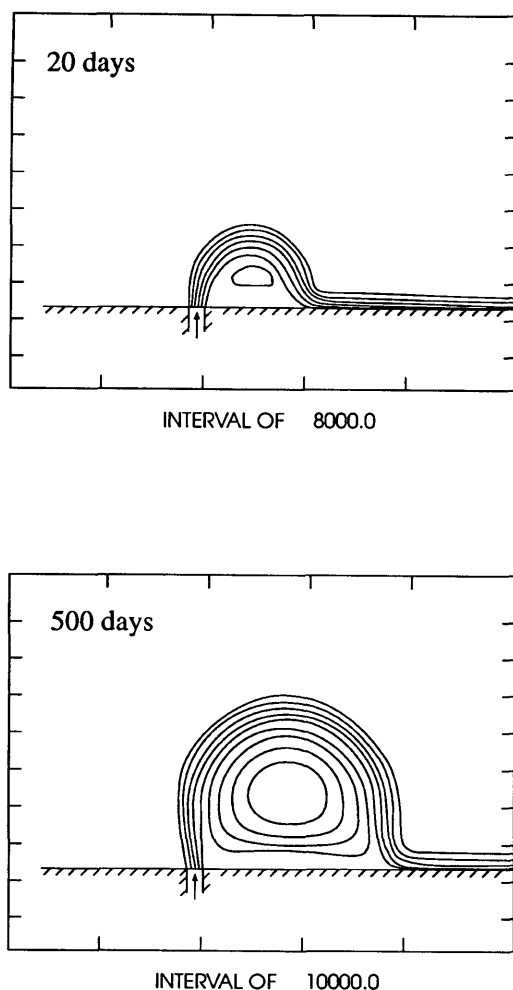


Fig. 6. Depth contours (in centimeters) of the f -plane flow ($\beta \equiv 0$). We see that the eddy is continuously growing near the mouth; this continuous growth absorbs the backward force exerted by the downstream current. Physical constants: $f_0 = 5 \times 10^{-5} \text{ s}^{-1}$; $g' = 10^{-2} \text{ m s}^{-2}$; $H = 450 \text{ m}$; $Q = 20 \text{ Sv}$. The basin is $750 \times 1125 \text{ km}$.

Furthermore, placing the boundary in different locations did not alter our results.

Our grid size was 15 km which is adequate for processes such as ours where the Rossby radius is roughly 40 km . It would, of course, have been better to use a higher resolution (because during the pinching process the scale of the flow is very small) but our computing time was limited and, consequently, we could not perform such experi-

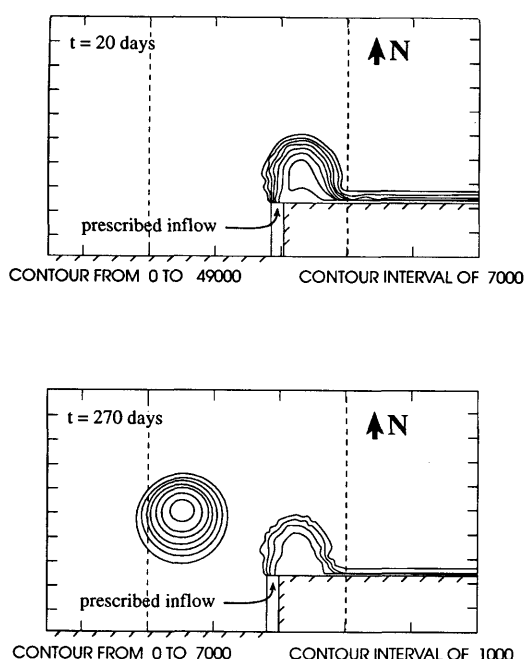


Fig. 7. Depth contours (in centimeters) of our "reference simulation" with zero potential vorticity. To avoid any interaction between the eddies and the boundary, the zonal western wall to the left of the channel (Section AF, Fig. 2) has been displaced 170 km to the south. We shall see later that this interaction is negligible, i.e., runs with an undisplaced zonal boundary show similar results (Fig. 14). The physical constants are the same as those discussed earlier except that β is not zero. In fact, to speed up the experiments we chose an artificially large value for β ($8 \times 10^{13} \text{ cm}^{-1} \text{ s}^{-1}$). The shown basin size is $750 \times 2250 \text{ km}$; Q is 20 Sv ; $g' = 1 \text{ cm s}^{-2}$; $H = 450 \text{ m}$; and $f_0 = 5 \times 10^{-5} \text{ s}^{-1}$. The meridional dashed lines on the right and left represent the sections across which the mass and momentum budgets were calculated (Figs. 8, 9).

ments. The few higher resolution experiments that we did perform showed that our error is of the order of (10–20%). Our time step was 1500 s and, for numerical stability, we introduced a small (horizontal) Laplacian friction of $\nu = 3 \times 10^6 \text{ cm}^2 \text{ s}^{-1}$. Our walls were slippery and, as is usually the case, we took the vorticity to be zero next to the walls. As is typical for the Gulf of Mexico, we took $f_0 = 5 \times 10^{-5} \text{ s}^{-1}$ and $g' = 1 \text{ cm s}^{-2}$.

Our initial numerical experiment (Figs. 7–12) corresponds to large negative (relative) vorticity

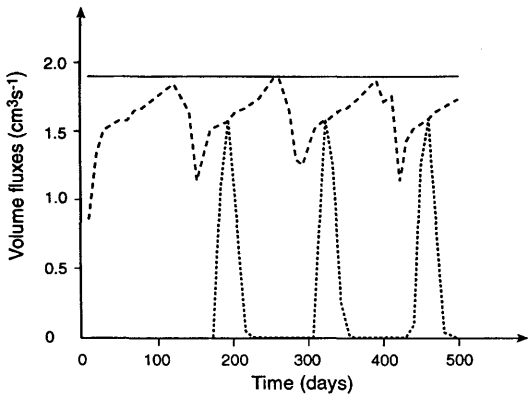


Fig. 8. Evolution of the mass fluxes for the reference simulation. The solid line represents the inflow; the dashed line is the mass flux of the downstream current (through the meridional line shown in Fig. 7) and the dotted line is the mass flux of the eddies. Note that, although some time dependency is present in the downstream current, the average flux there (1.5 ± 0.4) is approximately constant. Note that an error of 20–30% is acceptable with our kind of modeling.

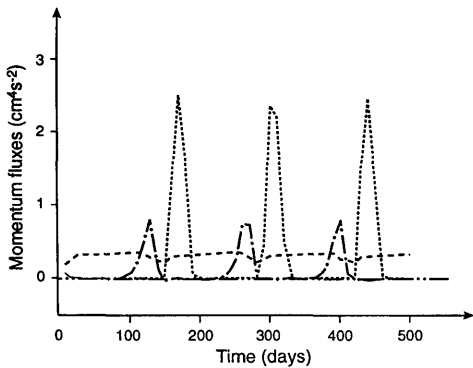


Fig. 9. Evolution of the momentum fluxes for the reference simulation. As in Fig. 8, the dashed line represents the downstream current and the dotted line represents the eddies. The dashed-dotted line represents the flow force on the short meridional wall immediately to the left of the strait (neglected in the computation). The dashed-double-dotted line (corresponding to an almost zero flux) represents the viscous effects.

($-f$) and is referred to as the “reference simulation”. We also performed some preliminary experiments with smaller relative vorticity ($-0.2f$, Figs. 13–14) and experiments with cyclonic, rather than anticyclonic, relative vorticity (Fig. 15). To examine the importance of the eddies’ interaction

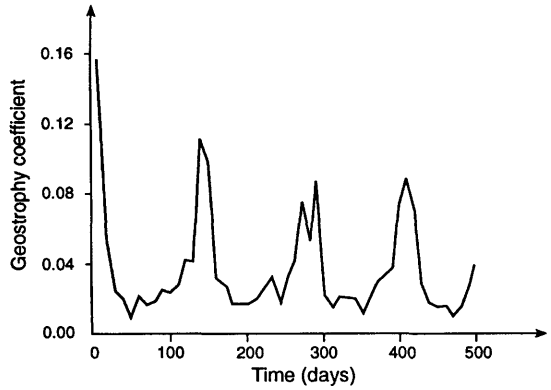


Fig. 10. The geostrophy coefficient of the downstream current [defined by $\Sigma u^2|(u - u_g)/u|/\Sigma u^2$ where u is the instantaneous speed and u_g is the geostrophic speed computed from the depth profile. (Note that the inclusion of u^2 in the coefficient emphasizes the high speeds rather than the weak flow.)] as a function of time. The figure shows that the downstream deviations from geostrophy and, hence, the deviations from one-dimensionality, are usually less than 10%.

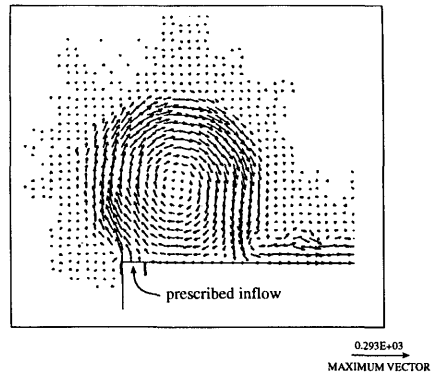


Fig. 11. A close-up showing the velocity field of the growing eddy shown in Fig. 7. Note the bifurcation of the jet impinging on the wall.

with the zonal western wall to the left of the mouth (neglected in our calculations), we performed some of our experiments with the western wall displaced 170 km to the south and some of our experiments with a nondisplaced wall (Fig. 14).

6.2. General results

The general features described by our analytical theory are clearly present in our numerical simula-

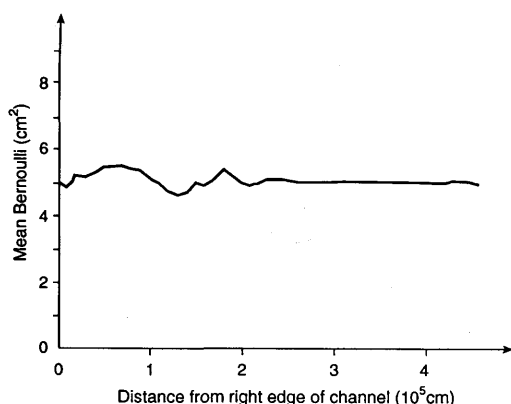


Fig. 12. The mean Bernoulli ($[u^2 + v^2]/2 + g'h$) along the right bank during the reference simulation. As assumed in the theoretical calculations, its value is approximately constant.

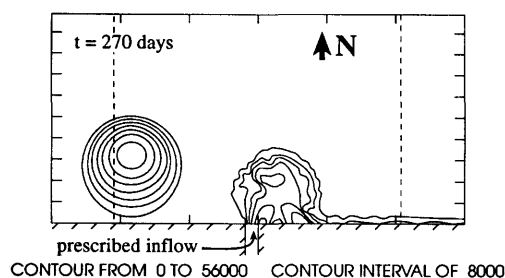
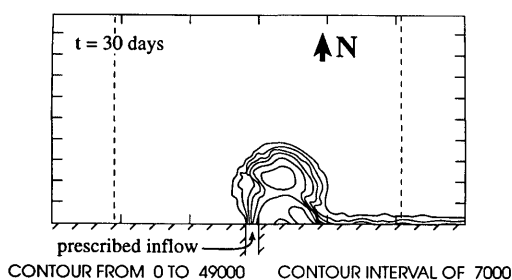


Fig. 14. The same as Fig. 7 except that the wall to the left of the inflow is not displaced to the south so that there is an interaction between the eddy and the wall. This interaction is minimal as can be seen by comparing the two figures. Note that the first detached eddy shown in the upper panel has left the displayed area (which is somewhat smaller than that shown in Fig. 7) by day 270. The shown basin size is 625×1800 km.

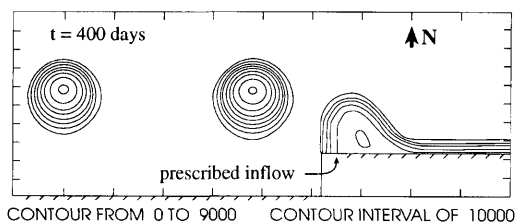
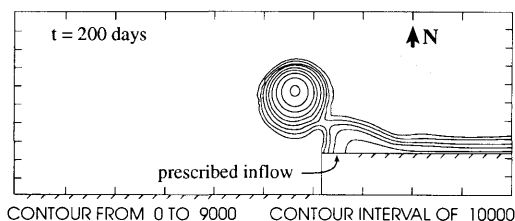
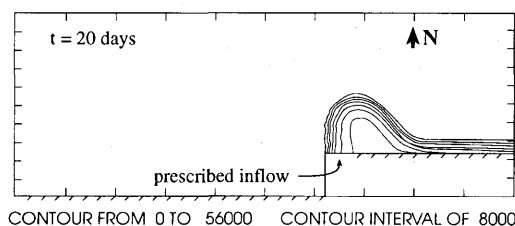


Fig. 13. The same as Fig. 7 except that $H = 620$ m and the initial negative relative vorticity is $-0.2f$ instead of $-f$. Note that the spacing between the eddies is much smaller than that displayed in Fig. 7.

tions. In all of our experiments, the flow turns to the right (eastward) after leaving the channel and, as expected, sheds eddies on the left. The reference simulation shows that, as assumed in the theoretical calculations, the flow on the right is approximately steady and geostrophic (Figs. 8–10). Our averaged observed eddy drift speed is approximately 12 cm s^{-1} which compares well with the value obtained from Nof's (1981b) analytical formula (approximately 10 cm s^{-1}).

The (averaged) reference simulation transports (with a displaced western wall) are 19 Sv for the inflow, 16 Sv for the downstream current and 3 Sv for the eddies. With an aligned wall that is not displaced to the south (not shown) the corresponding values are quite similar: 19 Sv, 15 Sv and 4 Sv. The increased eddy mass transport probably results from the image effect which increases the westward propagation speed. The analytically predicted upper bound for the eddies transport is 5 Sv which is quite close to the observed values (3 and 4 Sv) even though the eddies are definitely

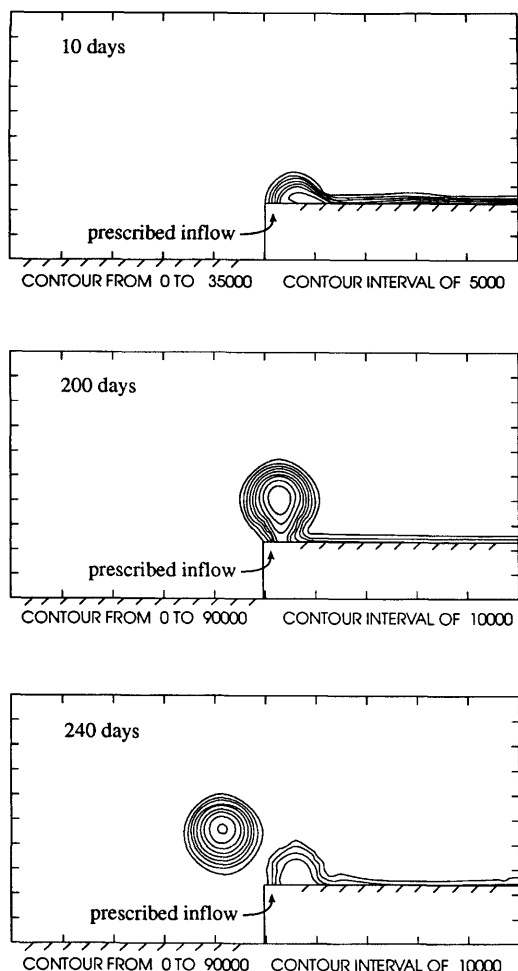


Fig. 15. Depth contours for the cyclonic outflow experiment. Note that, despite the fact that the original vorticity is cyclonic and despite the stretching of the water column during the eddy formation, anticyclonic eddies are formed and shed. This is due to friction which, even though it is small, can change the vorticity of the flow because of the extremely long formation process. The periodicity is 210 days and the radius of the eddies is 220 km. The shown basin size is 1000×2250 km. The numerical values of the various parameters were: $f_0 = 5 \times 10^{-5} \text{ s}^{-1}$; $\beta = 2 \times 10^{-1} \text{ m}^{-1} \text{ s}^{-1}$; $g' = 10^{-2} \text{ m s}^{-2}$; $2w = 45 \text{ km}$; $H = 300 \text{ m}$. The upstream velocity profile is given by $V = 2e^{(x-w)/30,000}$, where v is in m/s and x in m. The resulting transport is 14 Sv.

not touching each other as they leave the domain. It is of interest to point out here that the spacing between the eddies decreases when the inflow's relative vorticity is decreased (Fig. 13).

In contrast to the eddies' westward mass flux which is small compared to the inflow's mass flux, the eddies' momentum flux is not small (Fig. 9); this is also in agreement with the theory. It is appropriate to comment on the periodicity and size of the (numerically) observed eddies. In our reference simulation (Figs. 7–12), the periodicity was 130 days and the average eddy radius was 175 km. With a nondisplaced western wall (i.e., with a weak eddy-wall interaction) the periodicity decreased to 85 days and the eddy size increased to 210 km. All of these values agree with the lower analytical bound of the periodicity (30 days) and the upper bound of the eddy size (1200 km). Note that the periodicity and eddy size of the displaced and nondisplaced wall appear to be quite different but, as pointed out earlier, the eddy-induced transports associated with these cases are very similar. This is due to the fact that the depth along the eddies' rim is small so that it does not contribute much to the transport.

We also checked the validity of our parallel-streamlines-at-the-mouth assumption. To do so, we compared the momentum flux across the mouth ($\int hu v dx$) of an inflow where the streamlines were not parallel to the channel walls (i.e., the inflow was specified several channel widths upstream rather than at the mouth itself) to the downstream momentum flux ($\int hu^2 dy$) and the eddy momentum flux. We chose a case where $f_0 = 5 \times 10^{-5} \text{ s}^{-1}$; $\beta = 8 \times 10^{-11} \text{ m s}^{-1}$; $g' = 10^{-2} \text{ m s}^{-2}$; $W = 45 \text{ km}$; $H = 350 \text{ m}$ and $V = 0.2 \text{ m s}^{-1}$ so that the Rossby radius is 37 km implying that the channel is neither very broad nor is it very narrow (compared to the Rossby radius). We found that the contribution of the non-parallel streamlines ($\int hu v dx$) was less than 19% of either the eddies' momentum flux or the current momentum flux. The relatively small error is partly due to the large scale ($R_d/\epsilon^{1/6}$) immediately to the north of the mouth. Still better agreement is expected in the cases where the channel is narrow compared to the Rossby radius and it is concluded that the assumption is certainly adequate.

6.3. Changing the vorticity of the inflow

As pointed out earlier, it is not at all obvious what happens when the inflow's vorticity is cyclonic rather than anticyclonic. The difficulty with

this situation is that our lens-like eddies can only *exist* when their vorticity is anticyclonic. To examine this peculiar aspect we have made a preliminary numerical simulation with an inflow whose vorticity is cyclonic rather than anticyclonic (Fig. 15). Surprisingly, we see that anticyclonic eddies are still being formed even though the original vorticity is cyclonic and even though the fluid stretches during the eddy formation process. This is due to friction which, though small, can act for a long time and alters the initial vorticity. This is supported by the (numerical) observation that the formation process in this cyclonic case is considerably longer than that in the anticyclonic case previously discussed and by a detailed examination of the change in potential vorticity that the particles experience (Fig. 16).

6.4. Sensitivity to variation of ν and boundary conditions

To examine the possible influence of the frictional parameter, we increased the eddy viscosity coefficient ν step by step from the original value of $3 \times 10^6 \text{ cm}^2/\text{s}$. Initially, the physical processes remained almost unaltered. When we reached the value of $3 \times 10^7 \text{ cm}^2/\text{s}$, the eddies were destroyed by viscosity before separating from the current. Finally, for a viscosity of $3 \times 10^8 \text{ cm}^2/\text{s}$, the code exploded because it violated the stability criterion.

In the interior of the real ocean, the viscosity is very small and the ocean can be considered inviscid. Consequently, the main effect of viscosity is near the boundaries, and a good way to examine its role is to check how the potential vorticity is altered. Potential vorticity maps (not shown) illustrate that the potential vorticity remains at its zero-upstream value everywhere except along the edges of the flow. Along the edges there is creation of (positive) potential vorticity. Specifically, the viscosity smoothes the velocity fields and generates a positive relative vorticity all along the edge of the growing eddy and the downstream current.

It is not a priori obvious whether a free-slip or a non-slip boundary condition is the most appropriate to use. The no-slip condition reproduces the fact that the actual speed is zero at the boundary but cannot reproduce the particular dynamics of the frictional boundary layers (because this would require a much smaller length scale). On the other hand, the free-slip condition

does not reproduce the zero speed at the wall but can be taken as the appropriate condition *at the outer edge* of the frictional boundary layer. Since the only coastal process involved in our work is the inviscid detachment of the current near the mouth of the channel, frictional boundary layers should play a minor role. To verify that this is indeed the case, we also performed a numerical experiment with a no-slip boundary condition. We found that, as expected, the results were within 10% of the slippery boundary case.

7. Discussion and summary

The foregoing theory is applicable to numerous situations because almost all oceans and marginal seas are connected to each other by channels. For example, the Indian and Pacific Oceans are connected via a series of channels through which between 5 and 20 Sv are regularly exchanged (Field and Gordon, 1992) and the Sea of Japan is connected to the Pacific through a number of passages (Conlon, 1982). While these and other exchange processes clearly occur via channels, the case for which there are the most data available on both the origin of the flow and its final fate is probably the Gulf of Mexico. For this reason, we shall attempt to compare our results to the way that loop-current eddies are formed.

The flow through the Yucatan Channel carries about 30 Sv of warm water into the Gulf of Mexico (Fig. 17). On average, a ring is separated from the current once every 8–13 months. The reader is referred to Sturges (1994) and Maul and Vukovich (1993) and the references mentioned therein for information regarding the observational aspects of the Gulf and its rings. Despite extensive numerical studies of the region (Sturges, 1992; Sturges et al., 1993; Hurlburt and Thompson, 1980) and an examination of the influence of β on the maximum northward protrusion of the loop (Reid, 1972), it is not at all clear *why* rings are generated in the Gulf of Mexico in the first place. The idea that the zonal current is unstable does not provide an adequate explanation for the formation process as the rings are always generated in the same general area suggesting some sort of a geographical control.

Our present theory suggests that, without the generation of loop-current rings, the momentum

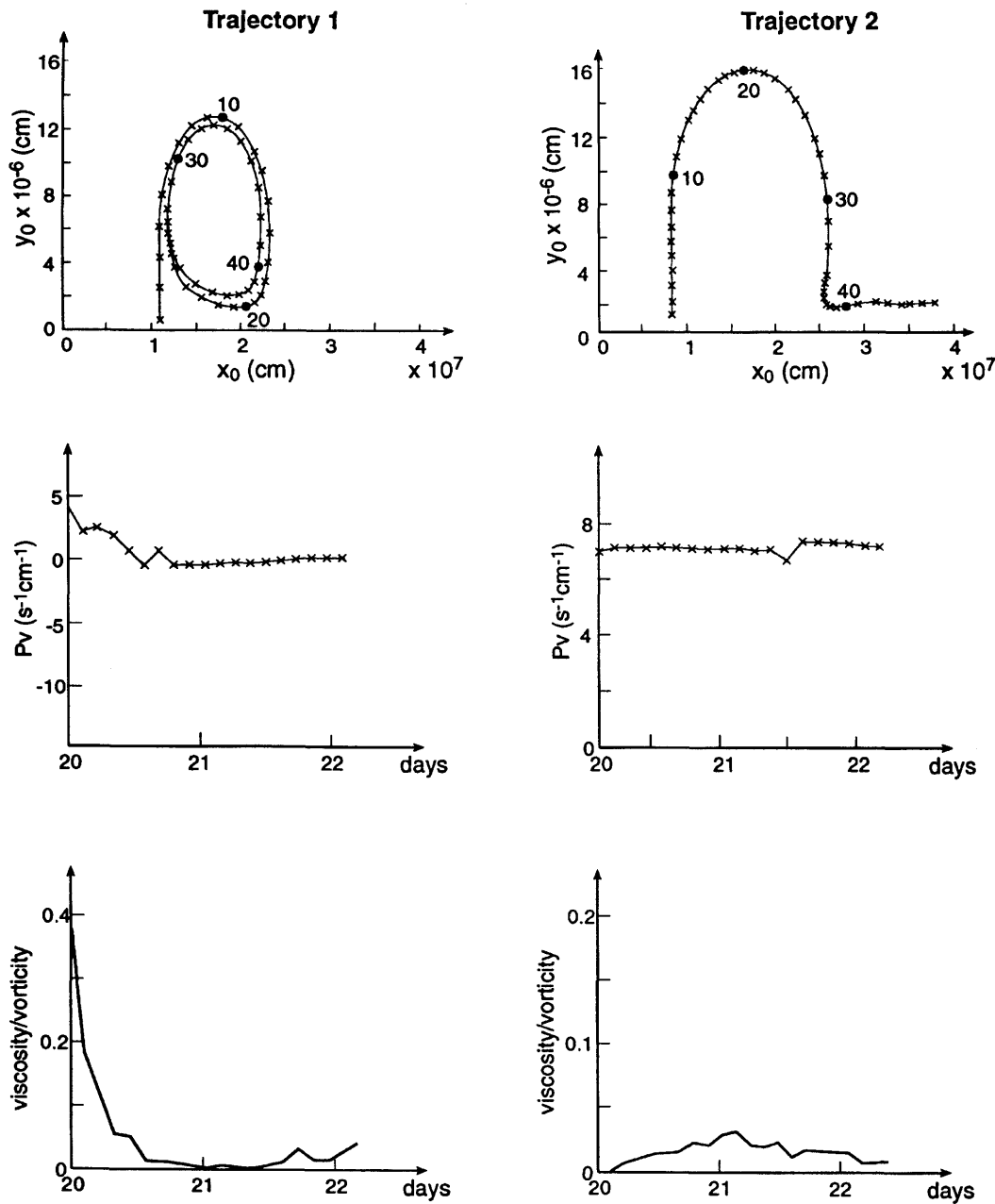


Fig. 16. Trajectories of two particles and the associated changes in potential vorticity. Particle 1 (left panels) is trapped in the eddy, whereas particle 2 (right panels) is situated outside the eddy and ends up in the downstream current. The spacing between two consecutive crosses is one day; times corresponding to 10, 20, 30 and 40 days are indicated with solid dots. The lower panel represents the importance of friction as a function of time. The parameter viscosity/vorticity (shown on the vertical axis of the lower panel) is defined by $[(F_x^2 + F_y^2)/(f_x^2 + f_y^2)]^{1/2}$, where F_x and F_y are the frictional terms in the x and y directions, and f_x and f_y are the Coriolis terms in the x and y directions. It is a global measure of the importance of viscosity. Note that particle 1 experiences a dramatic reduction in its potential vorticity (due to friction), whereas particle 2 experiences no such reduction.

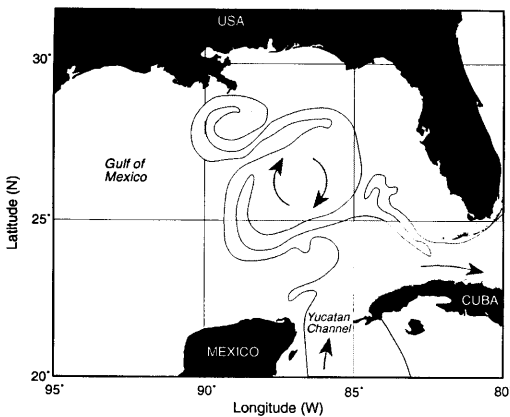


Fig. 17. A diagram showing the typical position of the loop current and its associated ring (shaded regions). Adapted from NOAA/National Ocean Service plot for May 10, 1988 (analyzed by Jenifer Clarke). Such a ring is formed every 8–13 months.

associated with the eastward flow north of Cuba could not have been balanced. This is an outcome of the momentum imbalance paradox discussed earlier in Section 2. It implies that rings are generated in order to off-set the flow-force associated with the eastward turning jet. As mentioned, we chose the following typical values for the Gulf of Mexico: $g' \approx 1 \times 10^{-2} \text{ m s}^{-2}$; $f \approx 5 \times 10^{-5} \text{ s}^{-1}$, and $\beta \approx 2 \times 10^{-11} \text{ s}^{-1} \text{ m}^{-1}$. Because of the tendency of nonlinear frontal models such as ours, to produce large speeds along the front, we have chosen here a relatively low value for g' . For the same reason, we shall also use a relatively low value for the discharge Q and take it to be 20 Sv. For simplicity we shall take the channel width W to be identical to the long-shore current width (roughly $1.4R_d$) and the upstream near-wall speed (at $x = W/2$) to be zero ($Fr = 0$).

For such values, the Rossby radius is 42 km so that both the channel and current widths are 59 km. For a zero potential vorticity inflow, the corresponding speed along the left edge is 2.95 m s^{-1} . Such high speed is typical for the Florida Current where values of 4–5 knots are regularly reported. Our predicted analytical lower bound for the radius of the resulting eddy is 240 km. The corresponding central depth is 640 m, the orbital speed near the edge is 2 m s^{-1} , and the westward drift is 2.1 cm s^{-1} . Most importantly, according to our model and calculations, the

upper bound for the eddies' generation period is once every 297 days! All of the above values [save perhaps the eddies' orbital speed near the edge which, as is usually the case for such models, is relatively high due to the presence of the front (Flierl, 1979)] are in very good agreement with the observations. Also, the observations of Maul and Vukovich (1993) suggest that there is no correlation between the flow in the Florida Current and the generation of rings in the Gulf of Mexico supporting our choice of a steady long-current on the right-hand side. In addition, the mass transport carried westward by drifting eddies (a few Sverdrups) is much smaller than the mass flux out of the channel, indicating that our perturbation expansion is valid. Finally, our calculations show that the generation period decreases with increasing mass flux (Fig. 5) suggesting that the two observed peaks (once every 8 months and once every 13 months) may be associated with varying transports. The above agreements are, at least partly, fortuitous because of the relative simplicity of the model. In particular, the agreement is better than it would have been had the Gulf of Mexico been a completely open basin. Under such conditions, the rings would have probably not touched each other (see Figs. 7–12) and, consequently, their generation period would have been longer.

In summary, it can be said that the primary aim of our theory and numerical experiments was to examine a new eddy generation mechanism which is related to the flow-force of a current exiting from a northward channel. The new inviscid generation process results from what we have termed “the momentum imbalance paradox.” It implies that a light current exiting from a channel perpendicular to the coastline (Fig. 1) exerts a flow-force (parallel to the wall) which cannot be balanced without the generation of eddies on the opposing side (Fig. 2). For the eddies to be generated periodically it is necessary that there be an agent responsible for their removal from the generation area. We have focused on the case where β is the removing agent but one can imagine that other processes such as an interaction with the wall can also be present.

Using a new integrated momentum technique that involves nonlinear time-dependent processes, we have computed the structure of the resulting eddies and their generation frequency (Figs. 4, 5).

A new length scale for eddies emerges from our calculations. In contrast to most eddies whose length scale is equal to the Rossby radius of the parent current (R_d), our eddies have a length scale of $R_d/\varepsilon^{1/6}$ (where $\varepsilon = \beta R_d/f_0$), and hence, are larger than most eddies. Quantitative numerical simulations (Figs. 6–16) that show the shedding process in detail were also performed. Finally, we applied our model to the Gulf of Mexico and found that Loop Current eddies (Fig. 17) must be generated every 10 months or so. This, as well as other predictions regarding the eddy size and speed, are in very good agreement with observations. Similar techniques have recently been applied to the generation of Meddies and Brazil Current eddies (Nof and Pichevin, 1996; Pichevin and Nof, 1996).

8. Acknowledgments

This study was supported by the National Science Foundation under contract OCE 9503816, National Aeronautics and Space Administration grant NAGW-4883 and Office of Naval Research grant N00014-89-J-1606. T. Pichevin was funded by SHOM-DRET. We had useful discussions with G. Weatherly and W. Sturges. Computations were carefully checked by S. Van Gorder. Drawings and typing were done by Beth Raynor. Two anonymous reviewers provided very useful comments on an earlier version of this paper.

9. Appendix A. List of symbols

a eddy width along the $\hat{x}$ axis (see Fig. 3)
 a^* nondimensional eddy width along the $\hat{x}$ axis
 B^* known nondimensional upstream Bernoulli along the right channel wall
 b_1, b_2 distances between the eddy edges and the coast (Fig. 3)
 b_1^*, b_2^* nondimensional distances between the eddy edges and the coast
 C eddy migration rate

f Coriolis parameter ($f_0 + \beta y$)
 Fr channel Froude number $V(g'H)^{1/2}$, where V and H are the upstream speed and depth next to the right bank
 g' “reduced gravity,” $(\Delta\rho/\rho)g$
 h thickness of light water
 H upstream current depth next to the right wall (looking downstream)
 H_l near-wall depth of long-shore current
 L long-shore current width
 Q mass flux out of the channel
 Q^* nondimensional mass flux out of the channel
 r, θ polar coordinates
 R eddy diameter
 R_d Rossby radius of the channel flow, $(g'H)^{1/2}/f$
 R_{de} Rossby deformation radius of the eddy
 T period of eddy generation
 T^* nondimensional periodicity
 u, v velocities in Cartesian coordinates
 v_θ orbital velocity in the eddy
 U_l near-wall speed of the long-shore current
 V upstream velocity along the right bank
 W width of channel
 $\hat{x}, \hat{y}, \hat{t}$ moving coordinates system
 $x_l^*, y_l^*, v_l^*, h_l^*$ scaled variables associated with the steady long-shore current on the right-hand side
 $x_e^*, y_e^*, v_e^*, h_e^*$ scaled variables associated with the eddies on the left
 α unknown of order unity that measures the size of the resulting eddies
 β variation of the Coriolis parameter with latitude
 ε small parameter equal to $\beta R_d/f_0$
 ϕ boundary of integration area S
 $\rho, \Delta\rho$ density and density difference between the layers
 ν eddy viscosity coefficient
 $\hat{\psi}$ streamfunction in the moving coordinate system
 $\tilde{\psi}$ time-integrated streamfunction
 ψ usual streamfunction (defined by $\partial\psi/\partial y = -uh; \partial\psi/\partial x = vh$)

REFERENCES

- Batchelor, G. K. 1967. *An introduction to fluid dynamics*. Cambridge University Press, 615 pp.
- Bleck, R. and D. Boudra, 1986. Wind-drive spin-up in eddy-resolving ocean models formulated in isopycnic and isobaric coordinates. *J. Geophys. Res.* **91**, 7611–7621.
- Brooks, I. H. and P. P. Niiler, 1977. Energetics of the Florida Current. *J. Mar. Res.* **35**, 163–191.
- Cherniawsky, J. and P. H. LeBlond, 1986. Rotating flows along indented coastlines. *J. Fluid Mech.* **169**, 379–407.
- Conlon, D. M. 1982. On the outflow modes of the Tsugaru warm current. *La Mer*, **20**, 60–64.
- Davey, M. K. and P. D. Killworth, 1989. Flows produced by discrete sources of buoyancy. *J. Phys. Oceanogr.* **19**, 1279–1290.
- Ffield, A. and A. L. Gordon, 1992. Vertical mixing in the Indonesian thermocline. *J. Phys. Oceanogr.* **22**, 184–195.
- Flierl, G. 1979. A simple model of the structure of warm and cold-core rings. *J. Geophys. Res.* **84**, 78–85.
- Hurlburt, H. E. and J. D. Thompson, 1980. A numerical study of Loop Current intrusions and eddy shedding. *J. Phys. Oceanogr.* **10**, 1611–1651.
- Kassler, R. D. 1981. *Variability of the anticyclonic circulation in the western Gulf of Mexico*. M.S. Thesis, Florida State University, 74 pp.
- Killworth, P. D., 1983. On the motion of isolated lenses on a beta-plane. *J. Phys. Oceanogr.* **13**, 368–376.
- Maul, G. A. and F. M. Vukovich, 1993. The relationship between variations in the Gulf of Mexico Loop Current and Straits of Florida volume transport. *J. Phys. Oceanogr.* **23**, 785–796.
- Molinari, R. L. and J. Morrison, 1988. The separation of the Yucatan Current from the Campeche Bank and the intrusion of the Loop Current into the Gulf of Mexico. *J. Geophys. Res.* **93**, 10,645–10,654.
- Nof, D. 1978a. On geostrophic adjustments in sea straits and wide estuaries: Theory and laboratory experiments. Part I: One-layer system. *J. Phys. Oceanogr.* **8**, 690–702.
- Nof, D. 1978b. On geostrophic adjustment in sea straits and wide estuaries. Theory and laboratory experiments. Part II: Two-layer system. *J. Phys. Oceanogr.* **8**, 861–872.
- Nof, D. 1981a. On the β -induced movement of isolated baroclinic eddies. *J. Phys. Oceanogr.* **11**, 1662–1672.
- Nof, D. 1981b. On the dynamics of equatorial outflows with application to the Amazon's basin. *J. Mar. Res.* **39**, 1–29.
- Nof, D. 1988. Outflows dynamics. *Geophys. and Astrophys. Fluid Dynamics*, **40**, 165–193.
- Nof, D. 1996. What controls the origin of the Indonesian throughflow? *J. Geophys. Res.*, **101**, 12,301–12,314.
- Nof, D. and Pichevin, T., 1996. The retroreflection paradox. *J. Phys. Oceanogr.*, in press.
- Orlanski, I. 1976. A simple boundary condition for unbounded hyperbolic flows. *J. Comp. Phys.* **21**, 251–269.
- Pichevin, T. and D. Nof, 1996. The eddy cannon. *Deep-Sea Res.*, in press.
- Reid, R. O., 1972. A simple dynamic model of the Loop Current. *Contributions on the physical oceanography of the Gulf of Mexico*, (2). L. R. A. Capurro and Joseph L. Reid (eds.). Texas A & M Oceanographic Studies, Gulf Publishing, 157–159.
- Richardson, W. S., W. J. Schmitz, Jr., and P. P. Niiler, 1969. The velocity structure of the Florida Current from the Straits of Florida to Cape Fear. *Deep-Sea Res.* **16**, 225–231.
- Sturges, W. 1992. The spectrum of Loop Current variability from gappy data. *J. Phys. Oceanogr.* **22**, 1245–1256.
- Sturges, W. 1994. The frequency of ring separations from the Loop Current. *J. Phys. Oceanogr.* **24**, 1647–1651.
- Sturges, W., J. C. Evans, S. Welsh and W. Holland, 1993. On the separation of a warm core ring in the Gulf of Mexico. *J. Phys. Oceanogr.* **23**, 250–268.