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A generalized Charney–Stern theorem for semi-geostrophic dynamics

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ABSTRACT

A stability theorem for zonal flow through a channel bounded by zonally invariant topography is derived for semi-geostrophic (SG) dynamics. The stability theorem generalizes the quasi-geostrophic linear stability theorem of Charney and Stern. The SG stability theorem takes the form of a conservation law for a quadratic disturbance quantity whose sign-definiteness yields necessary and sufficient criteria for the basic flow to be stable. The disturbance quantity generalizes the linearized SG pseudomomentum wave activity for a rectangular channel derived by Kushner and Shepherd to a channel with more general topography. The conservation law is obtained by direct manipulation of the SG equations after they have been transformed to isentropic and geostrophic coordinates. The stability criteria depend on the basic state's meridional potential-vorticity gradients in the interior, and on the topographic slope and the shear of the basic-state velocity at bounding material surfaces, but do not depend on the form of the small-amplitude disturbance.

1. Introduction

The purpose of this paper is to generalize one of the results of Kushner and Shepherd (1995a, hereafter referred to as KS1; 1995b, hereafter referred to as KS2). In that study, we show how a body of quasi-geostrophic (QG) theory, describing the evolution of disturbances to basic states, can be extended to semi-geostrophic (SG) dynamics, whose three-dimensional form was first set out by Hoskins (1975). The extension is achieved by deriving the pseudomomentum and pseudoenergy wave-activity invariants for the SG equations, and then by using the invariants to find linear and nonlinear stability theorems, small-amplitude and finite-amplitude local wave-activity conservation laws, and Eliassen-Palm (E-P) flux diagnostics for SG dynamics. (A "wave-activity" invariant is defined to be a conserved quantity that is second order in the amplitude of a disturbance to a basic state with some continuous symmetry. The "pseudomomentum" is the wave-activity invariant associated with the zonal symmetry of the system, and is defined with respect to a zonal basic state. The pseudomomentum for the SG equations is derived in KS1. The "pseudoenergy" is the wave-activity invariant associated with the temporal symmetry of the system, and is defined with respect to a steady basic state. The pseudoenergy for the SG equations is derived in KS2.)

One of the principal results of KS1 is a linear Charney–Stern stability theorem for SG dynamics. The theorem gives conditions under which a zonal flow in a zonally periodic channel bounded meridionally and vertically by flat walls is necessarily stable. The theorem takes the form of a conservation law for a second-order disturbance quantity that, as a formal variational calculation reveals, is the linearized pseudomomentum for the system. Basic states for which the disturbance quantity is sign definite are necessarily stable. The stability criteria are similar to those of the classical theorem of Charney and Stern (1962) for perturbations of normal-mode form. The theorem of KS1 is expressed in terms of perturbations of arbitrary form and includes interior and boundary terms that at small Rossby number scale out of the QG theory. The theorem generalizes Magnusdottir and Schubert's (1990; hereafter referred to as MSc90) version of the SG Charney–Stern theorem, which includes boundary contributions only implicitly through a "massless layer" argument.

In this paper, the SG Charney–Stern theorem of KS1 is generalized to basic-state flow in a zonally

Tellus A/47/5:1-2

periodic channel bounded by a zonally invariant material boundary of arbitrary shape. This class of basic states includes flow along a ridge or valley. As in KS1, the relatively simple form of the SG equations under transformation to isentropic geostrophic coordinates (IGC) is exploited. The transformation to IGC reduces the interior SG dynamics to a prognostic equation describing material conservation of potential vorticity (PV) together with a diagnostic nonlinear "invertibility principle" (Hoskins et al., 1985) relating the PV to the advecting velocity field. The system in IGC is completed by a prognostic equation at the boundaries. The equations of motion linearized about a zonal basic state are directly manipulated to find a conservation law. Reference is made both to the IGC and to the physical coordinates in the derivation of the global conservation law. The f-plane Boussinesq form of SG (Hoskins, 1975) is used, although the SG Charney–Stern theorem may be extended to the β-plane compressible system of MSc90.

The paper is structured as follows. In Section 2, the f-plane Boussinesq system of Hoskins (1975) is presented and transformed to IGC. The SG Charney–Stern theorem is derived in Section 3, and the results are discussed in Section 4.

2. Semi-geostrophic dynamics

In this section, the hydrostatic, Boussinesq, inviscid, adiabatic equations of motion under the geostrophic momentum approximation (Eliassen, 1948; Hoskins, 1975) are presented. The system is discussed at greater length in Section 2 of KS1. In physical coordinates, the zonal and meridional momentum equations, the continuity equation, the thermodynamic equation, and the hydrostatic balance condition are

$$\begin{aligned}
&\frac{\mathrm{D}u_g}{\mathrm{D}t} - fv = -\phi_x,\\
&\frac{\mathrm{D}v_g}{\mathrm{D}t} + fu = -\phi_y,\\
&\nabla\cdot\boldsymbol{u} + w_z = 0,\\
&\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0,\\
&\phi_z = \frac{g\theta}{\theta_0}.
\end{aligned} \tag{2.1a–e}$$

The coordinates are (x, y, z), where the pseudoheight z is defined by

$$z = \left[1 - \left(\frac{p}{p_0}\right)^{\kappa}\right]\frac{H_s}{\kappa}, \tag{2.2}$$

with p the pressure, and the scale height $H_s = p_0/(\rho_0 g)$ is defined with respect to a reference-level pressure p_0 and density ρ_0 (Hoskins and Bretherton, 1972). The material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \boldsymbol{u}\cdot\nabla + w\,\partial_z, \tag{2.3}$$

with velocity components

$$\boldsymbol{u} = (u, v) = \left(\frac{\mathrm{D}x}{\mathrm{D}t}, \frac{\mathrm{D}y}{\mathrm{D}t}\right), \qquad w = \frac{\mathrm{D}z}{\mathrm{D}t}. \tag{2.4}$$

The geopotential is ϕ, f is a fixed reference value of the Coriolis parameter, the geostrophic velocity $\boldsymbol{u}_g$ is defined by

$$\boldsymbol{u}_g = (u_g, v_g) = \left(-\frac{\phi_y}{f}, \frac{\phi_x}{f}\right), \tag{2.5}$$

and θ is the potential temperature with reference value θ_0. Coordinate subscripts denote partial derivatives, and $\nabla = (\partial_x, \partial_y)$.

Domains bounded by material surfaces with outward normal $\hat{\boldsymbol{n}} = (n_{(x)}, n_{(y)}, n_{(z)})$ are considered. The boundary condition is that of "no normal flow",

$$(u, v, w)\cdot\hat{\boldsymbol{n}} = 0 \qquad \text{at material surfaces.} \tag{2.6}$$

Since z is a pseudoheight coordinate, the vertical contribution $wn_{(z)}$ to (2.6) is approximate and consistent with Boussinesq scaling (Hoskins and Bretherton, 1972; White, 1977). (Other boundary conditions that will be considered below take the flow to be periodic or unbounded in a particular direction.)

The system is transformed to IGC

$$(X, Y, Z, T) = \left(x + \frac{v_g}{f}, y - \frac{u_g}{f}, \frac{g\theta}{f^2\theta_0}, t\right), \tag{2.7}$$

and the Montgomery-Bernoulli potential, Ψ, is defined as

$$\Psi = \phi - f^2 Zz + \frac{u_g^2 + v_g^2}{2}. \tag{2.8}$$

It can be shown that

$$\begin{aligned}(\Psi_X, \Psi_Y, \Psi_T) &= (\phi_x, \phi_y, \phi_t)\\ &= (f^2(X-x), f^2(Y-y), \phi_t), \\ \Psi_Z = -f^2 z, &\qquad \phi_z = f^2 Z.\end{aligned} \tag{2.9}$$

Then (2.1)–(2.6) may be expressed in IGC as

$$\frac{\mathrm{D}\sigma}{\mathrm{D}t} = 0 \qquad \text{in the interior,}$$
$$\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}\cdot\hat{\boldsymbol{n}} = 0 \qquad \text{at material boundaries,} \tag{2.10a–b}$$

where the material derivative is

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t} &= \partial_T - \frac{\Psi_Y}{f}\partial_X + \frac{\Psi_X}{f}\partial_Y \\ &= \partial_T + \frac{1}{f}\partial_{XY}(\Psi, \cdot),\end{aligned} \tag{2.11}$$

and the diagnostic invertibility relations are

$$(x, y, z, \sigma) = \left(X - \frac{\Psi_X}{f^2}, Y - \frac{\Psi_Y}{f^2}, -\frac{\Psi_Z}{f^2}, \partial_{XYZ}(x, y, z)\right), \tag{2.12}$$

where the last component on the right-hand side of (2.12) is the three-dimensional Jacobian determinant of the transformation $(x, y, z) \to (X, Y, Z)$. The materially conserved dimensionless quantity σ (called the "potential pseudodensity" by MSc90) is proportional to the inverse of the SG PV. The transformation has eliminated explicit reference to ageostrophic advection terms in (2.10)–(2.12). The result is a system whose dynamics is determined by prognostic equations describing material conservation of PV and advection of boundary quantities, together with the invertibility relations (2.12). In this way, the SG and the QG systems are analogous. There are two principal differences between the two systems. First, unlike the QG invertibility relation, the SG invertibility relation between σ and Ψ is nonlinear, as can be seen from the Jacobian in (2.12). Second, (2.10b) and (2.11) imply that fixed boundaries in physical space are variable in the transformed space. By using (2.10b) in the calculations in Section 3, the so-called "massless layer" (Lorenz (1955); also used, e.g., by Andrews (1983)) approach to modelling the intersection of isentropic surfaces with material surfaces is avoided, for reasons discussed in KS1. For the linear theory discussed in this paper, technical problems brought about by boundary variability, such as those found in the nonlinear stability theory of KS1, do not arise.

Since for small Rossby number $\sigma \approx z_Z \approx (f/N)^2$, where N is the buoyancy frequency, σ is taken to be positive and finite (KS1). Since σ is the Jacobian of the transformation from (x, y, z) to (X, Y, Z), this ensures that the transformation to IGC is globally well defined.

3. A generalized Charney–Stern theorem

In this section, a linear Charney–Stern theorem is derived for small disturbances to a zonal steady basic flow in a channel bounded by a zonally invariant curved material surface. As in KS1, the theorem takes the form of a conservation law for a second-order disturbance quantity, and stable basic states are those for which this disturbance quantity is sign definite. In KS1 we show that there are two related approaches to deriving such a conservation law. The first approach is to find the conservation law by directly manipulating the equations of motion linearized about a zonal basic state (KS1, Section 3). The second approach, which is more systematic and formal, is to derive the finite-amplitude pseudomomentum invariant using a variational principle, and then to look at its small-amplitude reduction, or, equivalently, its second variation (KS1, Section 5). The first, direct approach is followed in this paper because it is a straightforward generalization of the approach taken in KS1. The conserved quantity derived below is, in fact, the linearized pseudomomentum for this geometry (Kushner, in preparation). The direct approach taken here parallels that of Eliassen (1983) and MSc90, but unlike the latter, treats boundary contributions explicitly. This result includes as a special case KS1's Charney–Stern theorem, which was derived for a rectangular and zonally peridiodic channel bounded meridionally and vertically by flat walls.

The domain considered is a zonally periodic channel bounded by a material surface whose meridional-vertical cross section is defined by the curve $c(y, z)=0$. The curve $c(y, z)=0$ is taken to be simply connected in the y-z plane, but multiply connected curves may also be considered.

For zonally invariant geometry $\hat{\boldsymbol{n}}=(n_{(y)}, n_{(z)})$. The motion is linearized about a zonal steady basic state $\tilde{u}_g(\tilde{y}, \tilde{z})$. Here and henceforth quantities with tildes (such as $\tilde{x}$) indicate the basic state and quantities with primes (such as x') indicate the disturbance quantities. In the basic state $\partial_{\tilde{x}}=0$, and since

$$\partial_{\tilde{x}}=X_{\tilde{x}}\,\partial_X+Y_{\tilde{x}}\,\partial_Y+Z_{\tilde{x}}\,\partial_Z,$$

$\partial_X=0$ using (2.7). Since from (2.5) $\tilde{v}_g=0$, from (2.7) $\tilde{x}=X$. Therefore averages in $\tilde{x}$ and X are the same at the basic state. Linearizing equations (2.10) yields

$$\begin{aligned}&\frac{\tilde{\mathrm{D}}\sigma'}{\mathrm{D}t}+\tilde{\sigma}_Y v_g'\\&\quad=0, \qquad \text{interior},\\&\left[\frac{\tilde{\mathrm{D}}}{\mathrm{D}t}(y', z')+(\tilde{y}_Y, \tilde{z}_Y)\,v_g'\right]\cdot\hat{\boldsymbol{n}}\\&\quad=0, \qquad \text{boundaries},\end{aligned} \tag{3.1a–b}$$

where the linearized advection operator is

$$\frac{\tilde{\mathrm{D}}}{\mathrm{D}t}=\frac{\partial}{\partial T}+\tilde{u}_g\frac{\partial}{\partial X}. \tag{3.2}$$

Multiplying (3.1a) by $(\sigma'/\tilde{\sigma}_Y)$ and integrating in $X=\tilde{x}$ gives

$$\frac{\partial}{\partial T}\left(\frac{\overline{(\sigma')^2}}{2\tilde{\sigma}_Y}\right)+\overline{v_g'\sigma'}=0, \tag{3.3}$$

where zonal integration is denoted by an overbar.

The term in large parentheses in (3.3) is proportional to the linearized interior pseudomomentum wave-activity density (KS1). As in MSc90, the second term, representing the flux of inverse PV by the meridional geostrophic flow, may be expressed as the divergence of a flux. To first order in disturbance amplitude,

$$\begin{aligned}\sigma'&=\partial_{XYZ}(x', \tilde{y}, \tilde{z})+\partial_{XYZ}(\tilde{x}, y', \tilde{z})\\&\quad+\partial_{XYZ}(\tilde{x}, \tilde{y}, z')\\&=-\frac{1}{f^2}\,[\Psi'_{XX}\tilde{\sigma}+\partial_{YZ}(\Psi'_Y, \tilde{z})\\&\quad+\partial_{YZ}(\tilde{y}, \Psi'_Z)],\end{aligned} \tag{3.4}$$

using $\tilde{x}=X$ and $\tilde{\sigma}=\partial_{YZ}(\tilde{y}, \tilde{z})$ at the basic state, together with (2.12). Then, multiplying (3.4) by $v_g'=\Psi'_X/f$ yields

$$\begin{aligned}v_g'\sigma'&=-\frac{1}{f^3}\Bigg[\left(\tilde{\sigma}\frac{(\Psi'_X)^2}{2}\right)_X\\&\quad+\partial_{YZ}(\Psi'_X\Psi'_Y, \tilde{z})+\partial_{YZ}(\tilde{y}, \Psi'_X\Psi'_Z)\\&\quad-\Psi'_Y\,\partial_{YZ}(\Psi'_X, \tilde{z})-\Psi'_Z\,\partial_{YZ}(\tilde{y}, \Psi'_X)\Bigg]\\&=\frac{\partial}{\partial X}\left\{\frac{1}{2f^3}\,[-\tilde{\sigma}(\Psi'_X)^2+\tilde{y}_Y(\Psi'_Z)^2\right.\\&\quad\left.-2\tilde{y}_Z\Psi'_Y\Psi'_Z+\tilde{z}_Z(\Psi'_Y)^2]\right\}\\&\quad-\partial_{YZ}\left(\frac{\Psi'_X\Psi'_Y}{f^3}, \tilde{z}\right)-\partial_{YZ}\left(\tilde{y}, \frac{\Psi'_X\Psi'_Z}{f^3}\right),\end{aligned} \tag{3.5}$$

where in the second equality the thermal wind relation $\tilde{z}_Y=\tilde{y}_Z$ has been used (see (2.12)). The final expression in (3.5) is the linearized E-P flux divergence (MSc90; KS1). Invoking periodicity in X then gives, using (2.12),

$$\overline{v_g'\sigma'}=\overline{\partial_{YZ}(v_g' y', \tilde{z})}+\overline{\partial_{YZ}(\tilde{y}, v_g' z')}. \tag{3.6}$$

Consider now the global integral of (3.6) over the domain in IGC:

$$\begin{aligned}\int_{\tilde{D}} v_g'\sigma'\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z&=\int_{\tilde{D}}[\partial_{XYZ}(\tilde{x}, v_g' y', \tilde{z})\\&\quad+\partial_{XYZ}(\tilde{x}, \tilde{y}, v_g' z')]\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\\&=\int_D[\partial_{\tilde{x}\tilde{y}\tilde{z}}(\tilde{x}, v_g' y', \tilde{z})\\&\quad+\partial_{\tilde{x}\tilde{y}\tilde{z}}(\tilde{x}, \tilde{y}, v_g' z')]\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{z}\\&=\oint_{c=0}(\overline{v_g' y'}, \overline{v_g' z'})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}\tilde{l},\end{aligned} \tag{3.7}$$

where the variables of integration have been changed in the second step. The limits of integration of volume integrals may change as fields evolve at the boundary (KS1). This yields a variation of first order in the disturbance amplitude. The contribution from this linear variation is neglected in (3.7), since $v'_g \sigma'$ is a second-order disturbance quantity. Thus, the limits of integration are fixed at their basic-state values in IGC. These basic-state limits of integration are symbolized by $\tilde{D}$, while the limits of integration in physical coordinates are constant and the region is denoted D. Like $\hat{\boldsymbol{n}}$, the arclength element $d\tilde{l}$ is defined in the physical $\tilde{y}$-$\tilde{z}$ plane. The line integral in the final step of (3.7) is taken over the bounding curve $c = 0$. Note that y' and z' are fluctuations in IGC and may be defined as functions of the physical coordinates.

Following Eliassen (1983) and MSc90 the perturbations may be expressed in terms of particle displacements η' brought about by the meridional geostrophic wind. Defining η' by

$$v'_g = \frac{\tilde{\mathrm{D}}\eta'}{\mathrm{D}t}, \tag{3.8}$$

integration of (3.1) gives

$$\begin{aligned} &\sigma' = -\tilde{\sigma}_Y \eta', \qquad \text{interior,} \\ &(y', z') \cdot \hat{\boldsymbol{n}} = -(\tilde{y}_Y, \tilde{z}_Y) \cdot \hat{\boldsymbol{n}}\eta', \\ &\quad \text{boundaries.} \end{aligned} \tag{3.9}$$

Substituting (3.8)–(3.9) into (3.3) and (3.7) yields the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\tilde{D}} \frac{\tilde{\sigma}_Y}{2} \overline{(\eta')^2}\, \mathrm{d}Y\, \mathrm{d}Z - \oint_{c=0} \frac{(\tilde{y}_Y, \tilde{z}_Y) \cdot \hat{\boldsymbol{n}}}{2} \overline{(\eta')^2}\, \mathrm{d}\tilde{l}\right) = 0. \tag{3.10}$$

Eq. (3.10) leads to a statement of the linearized Charney–Stern theorem for this system. If $\tilde{\sigma}_Y$ in the interior and $-(\tilde{y}_Y, \tilde{z}_Y) \cdot \hat{\boldsymbol{n}}$ at the boundary share the same sign, then the disturbance may not grow (in a mean-square sense). This statement of linear stability holds independently of the form of the small-amplitude disturbance. In particular, the disturbance is not restricted to be of normal-mode form.

For isovortical perturbations to a constant-PV basic state with $\tilde{\sigma}_Y = 0$, $\sigma' = 0$ and a conservation law for the surface integral alone in (3.10) can be derived.

For a rectangular channel bounded by flat walls from below and above at z_1 and z_2 and to the south and north at y_1 and y_2, (3.10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\tilde{D}} \frac{\tilde{\sigma}_Y}{2} \overline{(\eta')^2}\, \mathrm{d}Y\, \mathrm{d}Z - \left[\int_{z_1}^{z_2} \frac{\tilde{y}_Y}{2} \overline{(\eta')^2}\, \mathrm{d}\tilde{z}\right]_{y_1}^{y_2} - \left[\int_{y_1}^{y_2} \frac{\tilde{z}_Y}{2} \overline{(\eta')^2}\, \mathrm{d}\tilde{y}\right]_{z_1}^{z_2}\right) = 0. \tag{3.11}$$

KS1 derive (3.11) and go on to show that the finite-amplitude pseudomomentum for the rectangular channel, constructed from a variational principle, reduces at small amplitude to the conserved quantity in (3.11), to within a constant factor of f. It can also be shown that (3.10) is the linearized pseudomomentum for this more general geometry (Kushner, in preparation).

In KS1, we note that the boundary conditions used by MSc90 capture only the interior term in the first integral of (3.11), and that at small Rossby number $\varepsilon = U/fL$, where U and L are characteristic velocity and length scales, the theorem reduces to the classical Charney–Stern theorem. We also find that the presence of two meridional boundaries prevents the system from satisfying the SG Charney–Stern stability criteria. This can be seen from (3.11) as follows. Since

$$\tilde{y}_Y = 1 + \frac{(\tilde{u}_g)_Y}{f} = 1 + \mathcal{O}(\varepsilon), \tag{3.12}$$

then it is reasonable to expect that $\tilde{y}_Y > 0$, i.e., that $\tilde{y}$ behave as a sensible meridional coordinate for $\varepsilon < 1$. For $\varepsilon > 1$, the flow is likely to be inertially unstable and/or take on negative PV values, in which case the SG equations are not well-defined. Thus, the two y-boundary contributions in (3.11) are of opposite sign and violate the stability criteria of the Charney–Stern theorem. The criteria are violated even when the flow is at rest or has no barotropic shear, in which case $\tilde{y}_Y = 1$. This issue is discussed further in KS1 and KS2.

Considering the boundary term in (3.10),

$$-(\tilde{y}_Y, \tilde{z}_Y) \cdot \hat{\boldsymbol{n}} = \tilde{y}_Y n_{(z)} \left(-\frac{n_{(y)}}{n_{(z)}} + \left.\frac{\partial \tilde{z}}{\partial \tilde{y}}\right|_{Y,Z} \right)$$

$$= \tilde{y}_Y n_{(z)} \,(\text{topographic slope} + \text{isentropic slope}). \tag{3.13}$$

Thus, stability depends critically on the sum of the topographic and isentropic slopes, and a maximum or a minimum in topography can change the sign of the boundary contribution, violating the stability criteria even in the absence of interior PV gradients.

Effects of zonally invariant topography on the linear stability of zonally invariant flows have been studied in the QG context by, for example, Blumsack and Gierasch (1972) and Mechoso and Sinton (1981); and in the SG context by Flagg and Beardsley (1978).

4. Discussion

Using the method of Section 3, statements similar to (3.10) and (3.13) can be derived for the β-plane QG system and the β-plane compressible system of MSc90. The SG results are less restrictive than the QG results since the SG equations include terms in the interior and at the boundary that vanish under QG scaling. For the β-plane basic state with a postive interior meridional PV gradient, violation of the stability criteria occurs in the presence of a single meridional wall to the south. It should be noted that the Boussinesq and geostrophic momentum approximations place limits on the allowed size of topographic slope (Craig, 1993; Flagg and Beardsley, 1978).

The issue of whether basic states that violate the stability criteria are actually unstable is difficult to address in a general way. Additional information can be obtained from other stability results, such as the SG version of Arnol'd's (1966) first stability theorem based on pseudoenergy conservation (KS2). The SG Arnol'd stability criteria depend in a more complex way on the basic-state velocity at the boundaries than the SG Charney–Stern stability criteria. In the end, however, it is necessary to perform a linear stability analysis on a given basic state to fully determine its stability properties. Stability theorems of this sort are therefore most useful as a guide to looking for potential sources of instability.

The derivation of the boundary contributions in (3.10), using (3.7) and (3.9), demonstrates a useful approach to treating explicitly the boundary dynamics in IGC. The use of nonlinear coordinates that, like IGC, simplify the interior dynamics but complicate the boundary dynamics could well be a feature of other balanced models as they are developed (see, e.g., Schubert and Magnusdottir, 1994). Therefore, the insight gained by the approach to boundary dynamics used here and in KS1 and KS2 should apply more widely than to SG dynamics alone.

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