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The location of thermal shelf fronts and the variability of the heights of tidal benthic boundary layers

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ABSTRACT

It is often hypothesized that the locus of a thermal shelf fronts is where the water depth (D) is equal to the thickness of the tidal frictional bottom boundary layer h. To determine the proper scales for tidal benthic boundary layers, we present simple, but rather general arguments which demonstrate that benthic boundary layers (BBL) in neutrally stratified environment must be defined by Ekman scale $L_e = u_*/\Omega$, where u_* is friction velocity, based on the bottom stress $\tau_b = \rho u_*^2$, ρ-water density, and Ω-Coriolis parameter. This result differs from those suggested by the numerical simulation of the formation of BBL in initially continuously stratified fluid, subject to a suddenly imposed barotropic pressure gradient as well as by direct observations of the intensity of turbulence close to the sea bottom, which indicated that the thickness of the well-mixed turbulent region close to the bottom of the sea is very often significantly less than L_e. Recently, Stigebrandt has argued that it can be explained by introducing the numerical constant λ in the expression $h = \lambda L_e$ ($\lambda \simeq 0.2$). We are trying here, as before to explain this by the influence of the imposed background stable stratification (with buoyancy frequency N), which limits the effective growth of BBL according to classical Ekman dynamics and creates a quasi-equilibrium thickness of BBL better characterized as $h = bu_*/N$, where b is a numerical constant ($b \approx 4-9$ in agreement with Kitaigorodskii and Joffre). We also demonstrate that the latter result is not in disagreement with observations of the locus of shelf thermal fronts in the Irish Sea and the Gulf of Maine.

1. Introduction

In a recent note Stigebrandt (1988), it was suggested that the locus of the thermal shelf front is where the water depth D is equal to the thickness of the tidal frictional bottom boundary layer. The basic idea behind this suggestion is natural; the front will be found where the turbulent BBL becomes thinner than the water column and "dives" below the sea surface. Under such circumstances, the role of surface-heating process (positive surface buoyancy flux Q_s) will be limited mainly by creating higher temperatures on the seaward side of the front, which make the front visible in, e.g., satellite infrared imagery.

The idea that the locus of a shelf front is where the water depth is equal to the thickness of the tidal BBL was first briefly discussed by Garret et al. (1978). These authors considered the thickness of the tidal BBL to be determined by the Ekman scale $L_e (h \sim L_e)$ with an empirical constant λ in the expression

$$h = \lambda L_e = \lambda \frac{u_*}{\Omega} = \lambda \frac{(\tau_b/\rho)^{1/2}}{\Omega}, \qquad (1)$$

equal to 0.4.

However, it is known since the work of Charney (1969), (see Kitaigorodskii and Joffre, 1976), that the correct value of the height of the Ekman boundary layer in neutrally stratified conditions (in homogeneous fluid) is about half this. Charney's prediction ($\lambda = 0.2$), based on the results of determination of stability criteria Re_{cr} for boundary layers in a rotating fluid, was confirmed recently by many authors, in particular when applied for both *tidal* currents (Mofjeld and Lavelle, 1984), and wind-driven surface currents (Stigebrandt, 1985). Further with this value of λ in (1), Stigebrandt (1988) was able to demonstrate that the water depth $D = 0.2L_e$ locates the shelf fronts in the Irish Sea and the Gulf of Maine region at about the correct place (cf. Simpson and Hunter, 1974, and Loder and Greenberg, 1986, respectively).

In contrast with such a viewpoint, Weatherly and Martin (1978), as early as in 1978, discussed and modelled the thickness of BBL in presence of stratification, and proposed the BBL thickness h to be dependent on the buoyancy frequency N. It was suggested that stable stratification can sufficiently reduce the thickness of tidal-generating bottom boundary layer.

The most convincing evidence of the latter fact can be found in the results of the recent numerical simulation of the formation of BBL in initially stably stratified fluid (Rahm and Svensson, 1989). They are in agreement with Kitaigorodskii's (1988) prediction that the rotation is probably not important in the formation of a well-mixed turbulent region in the presence of stable stratification; and the thickness is better characterized by the scale $L_N = (\tau/\rho)^{1/2}/N \ll L_e$. The direct experimental evidence of this, based on the measurements of the intensity of bottom turbulence, was presented by Nabatov and Ozmidov (1987), who as well as Rahm and Svensson (1989), demonstrate that in the presence of even weak stable stratification, the bottom boundary layers are usually thinner than 10 m. Thus, it is apparent that the turbulent bottom boundary layer caused by a *fluctuating tidal* current may in sufficiently deep water not reach the sea surface, while in shallow water the turbulent layer will fill whole water column. Therefore, the predictions of the locus of a *shelf fronts* strongly depends on the validity and accuracy of our estimates of the thickness of the tidal BBL. In Section 2, I present the general theoretical background for such estimates.

2. The thickness of the tidally generated BBL in the presence of imposed stable stratification

The purpose of this section is to show that the benthic boundary layer, generated by tidal waves can be strongly influenced by the presence of imposed background stable stratification. Since the thickness of BBL is much less than the wavelength, the Prandl model for laminar BL can be described by the equation

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial z^2} = \frac{\partial U_\infty}{\partial t}, \tag{2}$$

where U_∞ is the free stream velocity, which in the case of tidal wave for finite depth can be represented as:

$$U_\infty = V_0 \cos(kx - \sigma t), \tag{3}$$

$$V_0 = \frac{a\sigma}{\mathrm{sh}kD}; \tag{4}$$

here a is wave amplitude, k and σ are the wave number and frequency, respectively, and ν is kinematic viscosity.

The thickness of the *laminar* benthic boundary layer in such cases is given by the usual expression

$$\delta = \left(\frac{2\nu}{\sigma}\right)^{1/2} = \left(\frac{\nu T}{\pi}\right)^{1/2}, \tag{5a, b}$$

where T is the wave period.

Let us underline here the fact that this definition of boundary layer is based on the attenuation of the *rotational* part of the velocity field (in $e^{-1.0} \approx 0.36$) from *its value at the bottom* (Phillips, 1977). Such definition of the boundary-layer thickness is based on the assumption that the vortical boundary layer flow can, to first order, simply be superimposed on the irrotational motion.

It has been well-known since the work of Collins (1963) that in natural conditions, BBL is always turbulent (even with relatively small tidal amplitudes). Since the period of the tidal wave T and the characteristic length $\lambda = 2\pi/k$ exceeds the typical time and length scales of turbulence in the BBL, the tidal motion can be considered as a continuous sequence of stationary realizations of the BBL, or, at least the parameters of BBL can be treated as slowly-varying functions of time and position (compare with T and $\lambda = 2\pi/k$). This justifies the approach of when to estimate the height of a turbulent benthic boundary layer effective value, as the eddy viscosity K_e can be introduced in such a way that, instead of (3), we can write

$$h = \left(\frac{2K_e}{\sigma}\right)^{1/2} = \left(\frac{K_e T}{\pi}\right)^{1/2}. \tag{5b}$$

For *effective turbulent viscosity* K_e in shear driven tidal BBL, we can use the simplest expression (based on dimensional considerations)

$$K_e = K_e(u_*, h) = \text{constant } u_* h = a_1 u_* h \tag{6}$$

where a_1 is a numerical constant, supposedly *of order one*. With (6), the expressions (5a, b) can be rewritten as

$$h = 2a_1 L_\sigma, \tag{7}$$

where the scale L_σ is defined as

$$L_\sigma = u_*/\sigma. \tag{8}$$

The scaling length L_σ is a *fundamental* estimate for the thickness of turbulent BL generated by periodic tidal motion. The application of the expression (6), and therefore the scale L_σ to the estimates of the thicknesses of benthic boundary layers needs first of all comparison with the internal Ekman scale $L_e = U_*/\Omega$. Since as a rule, for semidiurnal and diurnal tides

$$\frac{L_e}{L_\sigma} = \frac{\sigma}{\Omega} \simeq 1, \tag{9}$$

we can arrive at the very important conclusion that BBL generated by tidal motion in neutrally stratified environment can in principle reach the *Ekman height*. This means that rotation can prevent further growth of shear generated tidal BBL (but possibly rotation is not important for small-scale three-dimensional turbulence, which determines the value of K_e).

For semidiurnal tides, $\sigma = 0.00014\ \mathrm{s}^{-1}$ and $\sigma/\Omega = 1.4$, and according to (7)

$$h = 2a_1 L_\sigma = 2a_1 \frac{u_*}{\sigma} = 1.42 a_1 L_e. \tag{10}$$

Comparing this expression with Charney's (1969) prediction of the well-founded value of $\lambda = 0.2$ in (1), we find that

$$a_1 \approx 0.14. \tag{11}$$

This value is very different from the value of the universal von Karman constant $\kappa \cong 0.4$ in the expression

$$K = \kappa u_* z, \tag{12}$$

which can be applied successfully only for the constant flux region of the Ekman boundary layer. However, it is evident that the *average* value of K according to (12) in the boundary layer h ($h \gg z_0$, z_0 the roughness parameter) is very close to its effective value K_e, since $\frac{1}{2}\kappa$ is close to a_1. Thus, in neutral conditions, it is quite appropriate to neglect the effect of rotation on the value of effective vertical turbulent viscosity K_e for BBL.

A typical range of values of u_* in tidal flows with different bottom roughness conditions is 0.7–4 $\mathrm{cm\ s^{-1}}$ (Kagan, 1968), which gives the heights h of the tidal BBL according to (10, 11) in the range 14–80 m. The value of $u_* = 2.3\ \mathrm{cm\ s^{-1}}$ (the average between 0.7 and 4 $\mathrm{cm\ s^{-1}}$) gives the height of BBL as $h \approx 46$ m.

Such a value of h locates the shelf fronts in the Irish Sea and the Gulf of Maine region at *about* the correct water depth. Indeed, if we use the value of the drag coefficient C_d for tidal flow according to Stigebrandt (1988), i.e., C_d in the expression

$$\tau_b = \rho u_*^2 = \rho C_d U_t^2, \tag{13}$$

where U_t is the *depth* mean tidal speed and the value $h/U_t \approx 80$ s, obtained by Stigebrandt (1988), we will get the value of $u_* = 3.1\ \mathrm{cm\ s^{-1}}$, which probably better represents the roughness conditions in Gulf of Maine and the Irish Sea; the value of u_* is then 2.3 $\mathrm{cm\ s^{-1}}$. Hence, we can conclude that for normal (average) bottom roughness conditions in shelf regions, the height of the tidally-generated boundary layers in a *neutrally* stratified environment must be *at least few tens of meters*, and thus indeed must be limited by effect of the Earth's *rotation*, or in other words, the appropriate scale for h must be L_e.

Application of the scale L_e to the estimates of the thicknesses of BBL also needs its comparison with the Monin-Obuchov Length L_*, based on the *surface* buoyancy flux Q_s and the tidal friction velocity $u_* = (\tau_b/\rho)^{1/2}$,

$$L_* = -\frac{(\tau_b/\rho)^{3/2}}{\kappa Q_s}. \tag{14}$$

This is needed because Q_s in the period of heating in shallow waters can also prevent growth of well-mixed tidally generated BBL.

For friction velocities of a few cm/s (tidal velocities $U_t \approx 1\ \mathrm{m\ s^{-1}}$), and Q_s for the heating season of $10^{-8}\ \mathrm{m^2\ s^{-1}}$ ($Q_s = 4 \cdot 10^{-8}\ \mathrm{m^2\ s^{-1}}$

corresponds to the heat flux $H = 100$ W m^{-2} typical for the Irish Sea (Stigebrandt, 1988)), the ratio $L_e/L_* = \mu_0$ (first introduced into the description of the atmospheric boundary layer by Kazanskii and Monin, (1960), and in the description of oceanic mixed layers by Kitaigorodskii (1960)); for tidally generated BBL, this is sufficiently different from atmospheric and oceanic boundary layers. Indeed, L_e is one or more orders of magnitudes smaller than L_*, which gives $|\mu_0| = L_e/L_* \sim 10^{-1}$(!) for Q_s *positive* (for atmospheric boundary layers for positive Q_s, typical values of μ_0 are about 100). That means that the penetration depth of the turbulent boundary layer generated by bottom friction in the presence of surface heat flux (positive buoyancy flux) is primarily determined by the scale L_e according to (1). The locus of the shelf front will then be found where the turbulent bottom boundary layer "dives" below the sea surface. This result is in conflict with the very recent observations of BBL in the Norwegian Sea (Nabatov and Ozmidov, 1987). Furthermore, the numerical simulation of the formation of BBL in conditions characteristic for estuaries, including the Baltic Sea (initially stable stratification close to two layer structure) (Rahm and Svensson, 1989), demonstrates quite convincingly that the thickness of the well-mixed turbulent region close to the bottom of the sea is sufficiently less than that given by the scales L_* or L_e (according to references, cited above the height of BBL in the presence of stable stratification, rarely exceeds 10 m). Following recent work by the author (Kitaigorodskii, 1988) this can be explained in general terms by taking into account the effect of *initial* stable stratification, which can prevent the effective growth of BBL even before rotation can become as important as another concurrent stabilizing factor. The simplest way to estimate the effect of initial stable stratification on BBL height is to use the natural condition on the existence of the limiting value of the Richardson flux number $\mathrm{Rf_h}$ in the entrainment zone. The basis for the estimation of the Richardson flux number $\mathrm{Rf_h}$ is the following simple formula for the entrainment flux (Kitaigorodskii, 1988; Kitaigorodskii and Joffre, 1988)

$$Q_h = -K_h N^2, \tag{15}$$

where K_h is the effective turbulent diffusivity responsible for the overall entrainment process (Kitaigorodskii, 1988; Kitaigorodskii and Joffre, 1988). The above-mentioned condition for $\mathrm{Rf_h}$ can be written as

$$\mathrm{Rf_h} = -\frac{Q_h}{u_*^2(\mathrm{d}u/\mathrm{d}z)} \leqslant \mathrm{Rf_{lim}}. \tag{16}$$

To proceed further, we need more detailed characteristics of the turbulence involved in the growth of BBL. The simplest assumption is still that rotation is not significant in determining the shear $\mathrm{d}u/\mathrm{d}z$ in (16), so that we can write

$$\frac{\mathrm{d}u}{\mathrm{d}z} \sim \frac{u_*^2}{K_h}, \tag{17}$$

and use for K_h the most reliable value-effective turbulent viscosity K_e in (6)

$$K_h \approx K_e = a_1 u_* h. \tag{18}$$

However, the formula (17) is written with the accuracy of the unknown factor of proportionality. This does not permit us to find the exact expression for h, even when $\mathrm{Rf_{lim}}$ is known. Nevertheless, formulae (15–18) lead to the following results

$$h = bL_n, \qquad L_N = \frac{u_*}{N}, \tag{19}$$

where the scaling length L_N corresponds to the scale of the well-mixed region developed by upward diffusion of turbulent energy in the presence of background stable stratification, and b is a numerical coefficient. Its value can be estimated from the expression

$$b \approx \sqrt{\frac{\mathrm{Rf_{lim}}}{a_1}}. \tag{20}$$

For the range of $\mathrm{Rf_{lim}} \approx 0.1$–$0.15$, and value of $a_1 \simeq 0.14$, we get $b \simeq 0.85$–1.0. From the other hand, the empirical findings, based on the data from stratified atmospheric boundary layers (Kitaigorodskii, 1988) give about four times larger values of b (see Fig. 2 in Kitaigorodskii, 1988 and Fig. 4 in Kitaigorodskii and Joffre, 1988), where the values of the function $Y_+(\mu_N)$ introduced in Kitaigorodskii (1988):

$$h = L_N Y_+(\mu_N), \qquad \mu_N = \frac{L_N}{L_*}, \tag{21}$$

for $\mu_N \leqslant 5.0$ are about $Y_+(\mu_N) \approx \text{const.} = 5 - 10$. Thus, this implies that assumptions (17, 18) can give the right values of the BBL height (comparable with its atmospheric analog-shear driven ABL in the presence of initial stratification), only if in the expression (17) we include a constant of proportionately close to 4–5! This may be implies that (17, 18) are oversimplifications, compared with more detailed calculations performed in Kitaigorodskii and Joffre (1988). We return to this question later, but let us now mention the fact that the application of the scale L_N to the estimates of BBL heights generated by tidal waves need first of all its comparison with the scale $L_\sigma = \bar{v}_*/\sigma$. The expression (19) can be rewritten as:

$$h_{\max} = bL_\sigma \left(\frac{\sigma}{N}\right), \tag{21}$$

and since, as a rule for semidiurnal and diurnal tides, $\sigma/N \ll 1$, we can conclude that in the presence of imposed stable stratification, the BBL generated by tidal motion never reaches equilibrium for the typical Ekman scale $L_e \approx L_\sigma$, but rather achieves a quasi-equilibrium thickness characterized as $h_N = bu_*/N$(!). This is a very *important* conclusion showing that the BBL depth can be much smaller than a few tens of meters. Fig. 7b in Rahm and Svensson (1989) demonstrates that for values of $N/\Omega = 28 - 88$, the quasi-equilibrium BBL height approaches of 4–10 m in about 20 h. It is interesting that for the same values of N/Ω, the atmospheric boundary layer, determined by the scale L_N (the value of b in (10) is only twice as small as for atmospheric data), has height in the range 50 to 200 m (see Fig. 4 in Kitaigorodskii and Joffre, (1988)). This is of course natural, because, with not *a big differences in values of N*, the typical ratio of friction velocities in atmospheric and benthic boundary layers are equal, at a value of at least 20(!). Thus, the results, presented in Kitaigorodskii (1988), and Kitaigorodskii and Joffre (1988), seem to be in agreement with Rahm and Svensson (1989), as well as with direct observation of h in the Norwegian Sea (Nabatov and Ozmidov, 1987). Therefore, the parameterization of the buoyancy flux Q_h according to Kitaigorodskii (1988) and Kitaigorodskii and Joffre (1988) is relevant and together with conditions for flux Richardson number Rf_h, it gives a rather good description of experimental data but only for $\mu_N < 5$(!). For larger values of μ_N (smaller values of N and thus higher entrainment rates), the picture described by (18, 19) is probably too simplistic and we will turn attention to this question in Section 3.

The other way of using Richardson number criteria was also suggested in Kitaigorodskii (1988). It is simply to take in (16),

$$\frac{d\bar{u}}{d\bar{z}} \sim \frac{U_\infty}{h}. \tag{22}$$

The latter expression *also* contains an unknown numerical factor of proportionality and thus cannot be quite *representative* for the process of growth of BBL due to entrainment.

Nevertheless, using (22) instead of (17), we can write (16) by using (3, 4) and (15, 18) as

$$h = bL_N \left(\frac{a\sigma}{u_* \mathrm{sh}kD}\right)^{1/2}$$

$$= bL_\sigma \left(\frac{\sigma}{N}\right)\left(\frac{a\sigma}{u_* \mathrm{sh}kD}\right)^{1/2}, \tag{23}$$

or

$$h = bL_N \left(\frac{a}{L_\sigma}\right)^{1/2} (\mathrm{sh}kD)^{-1/2}. \tag{24}$$

The latter expression demonstrates that the ratio of tidal-wave amplitudes to the scale of the benthic boundary layer (L_σ or L_e) is important in determination of its thickness, as are the type of tidal waves propagating in shelf areas, which determines the ratio kD. Nevertheless, using by analogy with (13) the classical quadratic law with effective drag coefficient C_f, defined as $C_f^{-1/2} = U_\infty/u_*$, we can rewrite (16) as

$$h = bL_N (C_f)^{-1/4}. \tag{25}$$

For values of $C_f \approx 0.003$, this leads to

$$h \cong 4.2bL_N. \tag{26}$$

This indicates that to have the boundary-layer thickness in agreement with the observations of Kitaigorodskii and Joffre (1988), $h \approx (4 - 10)\, L_N$, we must use the value of b according to (20). However, since both expressions (17, 22) include an unknown proportionality factor, this demonstrates that the weakest point in the arguments with overall Richardson number criteria in the entrainment zone (15, 16) is the determination of K_h and dU/dZ. If we follow the same line of analysis as in Kitaigorodskii (1988), we can simply write that the thickness of an equilibrium BBL in the presence of imposed stable stratification is governed only by two scales L_N and L_σ, so that

$$h = L_N Y(\mu_\sigma), \tag{27}$$

where

$$\mu_\sigma = \frac{L_N}{L_\sigma} = \frac{\sigma}{N}. \tag{28}$$

The two asymptotic expressions for $Y(\mu_\sigma)$ will correspond to $\mu_\sigma \to \infty$ (neutral BBL) when $h \sim L_\sigma$ and $\mu_\sigma \to 0$ when $h \sim L_N$.

The results presented in this section are based on the ideas of Kitaigorodskii (1988) and Kitaigorodskii and Joffre (1988) on overall scaling of the entrainment process which leads to the growth of BBL. However it is worthwhile to attempt a more accurate description of the turbulence generated by frictional BBL (and responsible for entrainment), aiming at least to find some detailization or corrections to the conclusions drawn in this section.

3. The influence of the entrainment zone on the thicknesses of the tidally generated BBL in the presence of imposed stable stratification

In our note (Kitaigorodskii, 1988), we considered the limits for growth of the turbulent boundary layer in the presence of imposed stable stratification characterized by the Brunt-Väsala frequency N. The assumption on the existence of the quasi-equilibrium height of the BBL in such cases simply means that the rate of growth of the BL up to its quasi-equilibrium value h in such cases is *much larger* than the evolution of both the well-mixed region and the entrainment zone after the quasi-equilibrium height is reached.

Recent numerical simulations of the growth of BBL by Rahm and Svensson (1989) proved this fact rather convincingly. For example, from their Fig. 7b, it follows that for the range of values of $N/\Omega = 28 - 88$ (typical for the Baltic), the BBL thickness approaches its quasi-equilibrium value $h = bL_N \approx 5 - 10$ m in about 12 h, whereas during next 24 h, there is on average only a very slight increase in h (not more than 2 m)*. Thus the idea that the quasi-equilibrium height can exist is correct, as well as the method of its determination, on the basis of consideration of the overall energetics by using external and internal parameters (Kitaigorodskii, 1988 and Kitaigorodskii, Joffre, 1988). However, just because of the fact that the BBL thicknesses can be really small (Nabotov and Ozmidov, 1987), it is worthwhile to take into consideration some of the specifics of entrainment processes in stratified fluids, as well as the contribution of entrainment zone finite thickness to the total BBL height in the presence of initially imposed stratification.

The most interesting facts about the vertical mixing processes in the entrainment zone reported by Turner (1976) are: *first*, that the effective value of K_h sufficiently exceeds the molecular diffusivity, and *second*, and most importantly, that instead of its decrease with stability typical for classical boundary layer three-dimensional turbulence (below critical Richardson's number $\mathrm{Rf} < \mathrm{Rf}_{CR}$), what is observed in the entrainment zone (as well as in oceanic thermoclines) is the opposite: K_h is larger *when* N is *larger*! The most natural explanation for this, and maybe the only one possible, is related to the effect of breaking of small-scale internal waves in the entrainment zone, created by the turbulent motions at the boundary of the well-mixed region. It is rather well-known that the larger N is, the higher is the probability of breaking. One of the ways to incorporate this fact into the description of intermittent turbulence in the thermoclines and entrainment zones is to assume (Zilitinhevich et al., 1988):

$$Q_h = Q_h(N, l) = C_Q l^2 N^3, \tag{29}$$

* This point of view is not widely accepted, and I am grateful to one of the Reviewers, who considers the results of Rahm and Svensson as supporting the view that Ekman dynamics determines the quasy-equilibrium thickness after about 1 day. Of course, in the general case of the growth of BBL, time enters the problem, but for most geophysical situations, the existence of quasi-equilibrium height is a good and accurate enough concept (see Fig. 7b in Rahm and Svensson (1989)).

where the length scale l must be determined by the fluctuation amplitude at the turbulent front, as the latter will also define the amplitudes typical of internal waves in the entrainment zone. In (28), C_{Q} is a numerical constant. In the framework of our formulae for Q_{h} (15), this leads to

$$K_{\mathrm{h}} = C_{\mathrm{Q}} l^2 N. \tag{30}$$

The latter result can also be considered as a consequence of dimensional analysis only. Both (29) and (30) are consistent with the picture of how internal waves dissipate their energy, through the creation of turbulent patches due to their breaking. The most reliable estimate for l will be to take it equal to the half of the thickness of the entrainment zone Δh

$$l = \tfrac{1}{2}\Delta h. \tag{31}$$

It was shown in Zilitinkevich et al. (1988), that assuming the self-similar form of the Kitaigorodskii–Miropolskii type (1970) in the temperature distribution in the entrainment zone during the growth of well-mixed BL till its equilibrium value in the two-layer fluid, the Δh (and correspondingly l) can be expressed as

$$\begin{aligned}\Delta h &= \frac{16}{27} C_{\mathrm{Q}}^{-2} \left(\frac{g\,\Delta\rho}{\rho}\right)^{-1} u_{\mathrm{e}}^2 \\ &= \frac{16}{27} C_{\mathrm{Q}}^{-2} h \mathrm{Ri}_{u_{\mathrm{e}}}^{-1},\end{aligned} \tag{32}$$

where $C_{\mathrm{Q}} \approx 0.028$, $\Delta\rho$ is the density difference across the entrainment zone, and $\mathrm{Ri}_{u_{\mathrm{e}}}$ is a Richardson number, based on the entrainment rate $u_{\mathrm{e}} = \dot{h} = \partial h/\partial t$, i.e., $\mathrm{Ri}_{u_{\mathrm{e}}} = (g/\rho\,\Delta\rho h)/u_{\mathrm{e}}^2$.

When u_{e} is relatively large (first stage of growth of BL), $\mathrm{Ri}_{u_{\mathrm{e}}}$ is small, and the expression (32) *agrees with the experimental* (*laboratory*) *results* on the thickness of the entrainment zone both for shear driven and convectively driven boundary layers (Zilitinkevich and Kreiman, 1988; Deardorff et al., 1980). When $\dot{h} = u_{\mathrm{e}}$ is small and $\mathrm{Ri}_{u_{\mathrm{e}}}$ is large, which corresponds to the *second* stage of growth, the more appropriate expression for Δh, according to the same authors is

$$\frac{\Delta h}{h} \approx 0.33. \tag{33}$$

When $\mathrm{Ri}_{u_{\mathrm{e}}}$ is as small as $4 \cdot 10^{-4}$ (which corresponds to the first stage of growth of BL):

$$\frac{\Delta h}{h} \approx 0.6. \tag{34}$$

The results of Zilitinkevich et al. (1988) (32, 33, 34) if used together with (29–31) and the criteria (16) for the existence of quasi-equilibrium BBL height leads to

$$\frac{C_{\mathrm{Q}} \frac{1}{4} \Delta h^2 N^3}{u_*^2 (\mathrm{d}\bar{u}/\mathrm{d}\bar{z})} \leqslant \mathrm{Rf}_{\mathrm{lim}}. \tag{35}$$

Again for $\mathrm{d}U/\mathrm{d}Z$ given by (17), with a factor of 4 (see p. 12), i.e.,

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{z}} \approx 4 \frac{u_*^2}{a_1 u_* h} \approx 28.5 \frac{u_*}{h}, \tag{36}$$

the formulae (34, 36) and $C_{\mathrm{Q}} = 0.028$, again leads to $h = bL_N$, where b is *now given* by the expression

$$b = \sqrt[3]{\frac{10^2 \mathrm{Rf}_{\mathrm{lim}}}{\Delta h/h}}. \tag{37}$$

For $\Delta h/h$ given by (33, 34) and the range of $\mathrm{Rf}_{\mathrm{lim}} = 0.10 - 0.15$, we get in (37) $b \approx 3 - 5$. This is consistent with our Section 2 estimates of the shear driven BBL height as well as with the data of Kitaigorodskii (1988) and Kitaigorodskii and Joffre (1988) *only* for very small value of $\mu_N \leqslant 1.0$. By using the formulae (29–31) for a description of entrainment flux Q_{h}, according to Kitaigorodskii and Joffre (1988), it is more appropriate to take (instead of the limiting value of the flux Richardson number of $\mathrm{Rf}_{\mathrm{lim}} \approx 0.10 - 0.15$) its critical value in the range $0.25 - 1$. In such a case, (recall that in (37), $\mathrm{Rf}_{\mathrm{CR}} \sim b^3$), we get $b \approx 4.1 - 9.7$, which is a *perfect fit to the Kitaigorodskii and Joffre* (*1988*) data on atmospheric BL in the range $\mu_N \leqslant 2$(!). This is quite understandable because for small μ_N (larger N), the results of the two-layer *model* are more appropriate than for large μ_N ($\mu_N > 2$). Fig. 2 in Kitaigorodskii (1988) indicates a value of b around 30 in the region $\mu_N \approx 5 \sim 50$(!). Actually, in *this region* the scale L_N *is no longer a good measure* of the thickness of boundary layer (small N) since in the relationship, $h = L_N Y_+(\mu_N)$, the function $Y_+(\mu_N)$ can be described as

$$Y_+(\mu_N) = C\mu_N, \tag{38}$$

so that

$$h = C\frac{Q_s}{u_* N^2}, \tag{39}$$

with $C \approx 1.0-2.0$. The latter result, if applied to BBL, implies that its height in the presence of surface heat flux (positive buoyancy flux Q_s) diminishes with increasing of bottom stress, which seems to be an unrealistic conclusion. Therefore, we come to the very important and natural conclusion that since L_N corresponds to the scale of a well-mixed region developed by shear-driven turbulence and upward diffusion of turbulent energy in the presence of a background stratification, its application to the description of the variability of the thicknesses of BBL must be limited to the value $\mu_N = Q_{sb}/Nu_*^2 \leqslant 1$, where Q_{sb} is now the value of the vertical buoyancy flux in the *vicinity of the bottom of the sea* (this quantity is very difficult to measure).

However, the typical values of μ_N for shelf regions with a seasonal cycle for the thermocline for the heating season, can be estimated as $Q_{sb} \approx Q_s$, and since as a rule $N/\Omega \gg 1$, taking typical values of $Q_s \approx 10^{-8}\ \mathrm{m^2\ s^{-3}}$ (for summer period in Gulf of Maine or Irish Sea) we will find that for N typical for thermoclines:

$$|\mu_N| \ll 1; \tag{40}$$

thus, the results with $h = bL_N$ with $b \approx 4.1-9.7$ are also perfectly applicable for tidally generated BBL.

4. Conclusions

It has been demonstrated that the theoretical background for the determination of the thicknesses of a tidally-generated frictional bottom boundary layer is formed by Ekman dynamics corrected by the influence of imposed stable stratification according to Kitaigorodskii's (1988) arguments on the existence of the quasi-equilibrium heights of turbulent boundary layers in startified rotating fluids. It was shown that these results can also be interpreted in the framework of penetration of a turbulent front in a stably stratified fluid, due to the generation of internal waves and their subsequent breaking in the entrainment zone. Thus, it was established that there is no contradiction between an approach based on the overall energetics of stratified shear driven BBL and the flux Richardson number criteria ($\mathrm{Rf_h} \leqslant \mathrm{Rf_{lim}}$) in the vicinity of the entrainment zone, and an approach based on local dynamics of turbulence in the entrainment zone related to the dissipation of internal wave energy due to formation of turbulent patches through their breaking. It was found also that there is no contradiction between experimental data on the variability of h presented in Kitaigorodskii (1988) and Kitaigorodskii and Joffre (1988) with regard to the variability of atmospheric boundary layers and results of the estimates of the BBL thicknesses, using the data on the locus of thermal shelf fronts in the Gulf of Maine and the Irish Sea (Stigebrandt, 1988). They are also consistent with the results of BBL numerical simulation in the Baltic Sea in the presence of imposed stable stratification (Rahm and Svensson, 1988) which demonstrates that whereas in situations close to neutrality (in respect to stratification), the tidal BBL easily reaches the Ekman height $h = \lambda L_e$ where λ is the Charney (1969) constant, $\lambda = 0.2$, $L_e = u_*/\Omega$, (Stigebrandt, 1988), but for $N/\Omega \sim 100$, h can be sufficiently smaller. In the latter case, the BBL thicknesses h are determined by the scale $L_N = u_*/N$, such that $h/L_N \approx 4-9$ for $|\mu_N| = L_N/L_* \leqslant 2$ (L_* is Monin-Obuchov scale based on *bottom buoyancy flux* and *bottom friction velocities*). This conclusion is also in agreement with recent direct measurements of turbulence intensity presented in Nabatov and Ozmidov (1987).

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