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Finite difference approximations to the advection-diffusion equation

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ABSTRACT

Fiadeiro and Veronis (1977) use a weighted mean scheme to develop a finite difference approximation to advection-diffusion type partial differential equations. The resulting numerical approximation tends to a centered difference formulation for strongly diffusive cases and to an upstream formulation for strongly advective cases. A simpler alternative derivation of their scheme is presented here which clarifies its basis. Using the new approach, the extension to a spatially variable grid and a similar scheme based on average cell properties are obtained. Errors associated with temporal variations are discussed and a stability criterion appropriate to forward differencing in time is presented.

1. Introduction

An interesting finite difference approximation to advection-diffusion type partial differential equations was developed by Fiadeiro and Veronis (1977). The method has now been used in several oceanographic studies (Musgrave, 1985; Lin, 1990; Wright and Stocker, 1991), and it has proven to be a very useful tool. A practical advantage of the scheme is that it yields solutions for relatively coarse grids allowing efficient use of computer resources. A particular appeal of the method is that it automatically adjusts from a centered-difference scheme to an upstream scheme as the influence of advection increases.

A disadvantage of the scheme is that it is restricted to a spatially uniform grid, and the somewhat intricate derivation makes the extension to a non-uniform grid appear to be a very tedious task. This restriction is somewhat limiting where boundary layers are present. Below, we will present a straightforward derivation of the Fiadeiro and Veronis (1977) scheme from which its simple basis will be more evident. The approach is similar to that used by Young and Gregory (1973) to derive the exact equivalent solution to the one dimensional advection-diffusion equation referred to by Fiadeiro and Veronis (1977). The extension to a spatially variable grid will also be presented, and an analogous scheme based on properties averaged over a grid cell will be derived. In addition, a stability criterion appropriate to a time-dependent model using forward differencing in time will be determined.

2. Generalization of the Fiadeiro and Veronis scheme to a spatially varying grid

Fiadeiro and Veronis (1977) begin their analysis with the three-dimensional advection-diffusion equation with non-constant coefficients. They make considerable progress within this general framework, but their weighting factors are ultimately determined through consideration of the one-dimensional, constant coefficient equation, and subsequently assuming that this analysis applies locally in each spatial dimension. Although we are also interested in time-dependent, multi-dimensional, advection-diffusion problems (see section 4), guided by the work of Fiadeiro and Veronis (1977), we begin with the steady, one-dimensional, constant coefficient advection-diffusion equation:

$$uC_x = KC_{xx}. \tag{1}$$

Integrating (1) from some arbitrary level x_0 (x_0 will be identified with the boundary of a grid cell) to any other position x yields

$$C_x(x) = C_x(x_0) \exp\left[\frac{u}{K}(x - x_0)\right], \tag{2}$$

and integrating again over the same interval yields

$$C(x) = C(x_0) + \frac{K}{u} C_x(x_0) \times \left\{\exp\left[\frac{u}{K}(x - x_0)\right] - 1\right\}. \tag{3}$$

Now, substituting $x = x_+ = x_0 + \Delta x_+/2$, then $x = x_- = x_0 - \Delta x_-/2$ (x_+ and x_- are the centers of the grid cells immediately above and below x_0) into (3), gives

$$[uC_0 - KC_x|_0] + KC_x|_0 \exp\left[\frac{u\,\Delta x_+}{2K}\right] = uC_+, \tag{4}$$

$$[uC_0 - KC_x|_0] + KC_x|_0 \exp\left[-\frac{u\,\Delta x_-}{2K}\right] = uC_-, \tag{5}$$

where C_+ (C_-) is the value of C at $x = x_+$ (x_-). The local flux per unit area across $x = x_0$ is thus

$$F_0 = uC_0 - KC_x|_0 = \alpha_0 C_- - \beta_0 C_+, \tag{6}$$

where

$$\alpha_0 = \frac{u \exp\left[\frac{u\,\Delta x_+}{2K}\right]}{\exp\left[\frac{u\,\Delta x_+}{2K}\right] - \exp\left[-\frac{u\,\Delta x_-}{2K}\right]} = \frac{u}{2}\left(1 + \coth\left[\frac{u}{4K}(\Delta x_- + \Delta x_+)\right]\right), \tag{7}$$

$$\beta_0 = \frac{u \exp\left[-\frac{u\,\Delta x_-}{2K}\right]}{\exp\left[\frac{u\,\Delta x_+}{2K}\right] - \exp\left[-\frac{u\,\Delta x_-}{2K}\right]} = -\frac{u}{2}\left(1 - \coth\left[\frac{u}{4K}(\Delta x_- + \Delta x_+)\right]\right). \tag{8}$$

The corresponding finite difference approximation to (1) for the cell centered on $x = x_i$ is simply $(F_{i+1/2} - F_{i-1/2})/\Delta x_i = 0$, where the subscripts $i \pm 1/2$ indicate evaluation at $x_i \pm \Delta x_i/2$ (i.e., at the upper and lower boundaries of cell i). This yields

$$\Gamma_{i-1} C_{i-1} + \Gamma_i C_i + \Gamma_{i+1} C_{i+1} = 0, \tag{9}$$

where

$$\Gamma_{i-1} = -\frac{u}{2\,\Delta x_i}\left\{1 + \coth\left[\frac{u(\Delta x_{i-1} + \Delta x_i)}{4K}\right]\right\}, \tag{10a}$$

$$\Gamma_i = \frac{u}{2\,\Delta x_i}\left\{\coth\left[\frac{u(\Delta x_i + \Delta x_{i+1})}{4K}\right] + \coth\left[\frac{u(\Delta x_i + \Delta x_{i-1})}{4K}\right]\right\}, \tag{10b}$$

$$\Gamma_{i+1} = \frac{u}{2\,\Delta x_i}\left\{1 - \coth\left[\frac{u(\Delta x_i + \Delta x_{i+1})}{4K}\right]\right\}. \tag{10c}$$

Taking $\Delta x_{i-1} = \Delta x_i = \Delta x_{i+1} = h$ in (9) and (10) gives results precisely equivalent to (17) and (22) in Fiadeiro and Veronis (1977). However, (9) and (10) hold for arbitrary values of the Δx's and hence represent the appropriate generalization to a spatially varying grid. Fiadeiro and Veronis (1977) suggest that to allow for non-constant coefficients, one may simply replace u and K by $u_{i\pm1/2}$ and $K_{i\pm1/2}$ in $F_{i\pm1/2}$, and to consider more than one dimension, the above formula for the flux across a cell boundary may be simply applied independently in each direction. Errors resulting from this approach are discussed in Sections 4 and 5.

3. An analogous scheme based on average cell concentrations

The scheme derived in Section 2 gives the evolution of a concentration field in terms of the values at the centers of specified cells. A disadvantage of it is that the total amount of C within a cell is not obtained simply by multiply C by the cell size: it must be obtained from (3) applied over the upper and lower halves of each cell. While this is clearly possible, it is cumbersome and somewhat tedious. The situation is further complicated when additional dimensions are introduced. Fortunately, the approach used above is easily modified to give

an analogous scheme in terms of cell-averaged properties. The basic analysis is not significantly different from that outlined above, and with only minor modifications the method may be used to determine analogous numerical schemes for more general constant coefficient differential operators. To demonstrate this, consider the more general constant coefficient differential equation

$$uC_x = KC_{xx} - \lambda C. \tag{11}$$

The solution may be written as

$$C = A\exp[\gamma^+(x-x_0)] + B\exp[\gamma^-(x-x_0)], \tag{12}$$

where

$$\gamma^\pm = \frac{u}{2K} \pm \left[\left(\frac{u}{2K}\right)^2 + \frac{\lambda}{K}\right]^{1/2}. \tag{13}$$

From (12) we have the relations

$$C_0 = A + B, \tag{14}$$

$$C_x|_0 = \gamma^+ A + \gamma^- B, \tag{15}$$

so, solving for A and B and substituting in (12) we have

$$C = (\gamma^- - \gamma^+)^{-1}\{C_0[\gamma^- E^+ - \gamma^+ E^-] - C_x|_0\,[E^+ - E^-]\}, \tag{16}$$

where

$$E^\pm = \exp[\gamma^\pm(x-x_0)]. \tag{17}$$

Eq. (16) reduces to (3) in the limit $\lambda \to 0$. To obtain equations in terms of cell-averaged quantities, we average (16) over the intervals $[x_0 - \Delta x_-, x_0]$ and $[x_0, x_0 + \Delta x_+]$ to obtain

$$C_\pm = (\gamma^- - \gamma^+)^{-1}\{C_0[\gamma^- E^\pm_\pm - \gamma^+ E^\mp_\pm] - C_x|_0\,[E^\pm_\pm - E^\mp_\pm]\}, \tag{18}$$

where a subscript $+$ $(-)$ indicates an average over the upper (lower) interval. Note that superscripts refer to the spatial structure functions, while subscripts refer to position. Thus,

$$E_\pm = (\pm\gamma\,\Delta x_\pm)^{-1}\,[\exp(\pm\gamma\,\Delta x_\pm) - 1], \tag{19}$$

with $\gamma = \gamma^\pm$ for $E = E^\pm$.

Finally, straightforward algebraic manipulation of (18) gives

$$F_0 = uC_0 - KC_x|_0 = \alpha_0 C_- - \beta_0 C_+, \tag{20}$$

where

$$\alpha_0 = \frac{(u-\gamma^- K)\,E^+_+ - (u-\gamma^+ K)\,E^-_+}{E^+_+ E^-_- - E^+_- E^-_+}, \tag{21}$$

$$\beta_0 = \frac{(u-\gamma^- K)\,E^+_- - (u-\gamma^+ K)\,E^-_-}{E^+_+ E^-_- - E^+_- E^-_+}. \tag{22}$$

If $\lambda = 0$, then $\gamma^- = 0$, $E^-_\pm = 1$, and $\gamma^+ = u/K$, so that (21) and (22) reduce to

$$\alpha_0 = \frac{uE^+_+}{E^+_+ - E^+_-}, \tag{23}$$

$$\beta_0 = \frac{uE^+_-}{E^+_+ - E^+_-}, \tag{24}$$

which are the analogs of (7) and (8) for a numerical scheme based on cell-averaged properties. The corresponding finite difference approximation to (11) is now easily obtained.

4. Errors and corrections for non-constant K and u

The numerical schemes discussed in Sections 2 and 3 give exact results for the solutions to (1) and (11) no matter how coarse the resolution is chosen to be. However, they are clearly not exact if u and K are spatially varying. We now consider the errors associated with such variations. We begin with the appropriate generalization of (1):

$$(uC - KC_x)_x = 0. \tag{25}$$

Temporarily treating C_0 and F_0 $[\equiv (uC - KC_x)|_0]$ as known quantities, (25) may be integrated twice to obtain

$$C = \exp\left[\frac{u}{K}(x-x_0)\right] \times \left\{C_0 - F_0\int_{x_0}^{x} K^{-1}\exp\left[-\frac{u}{K}(x'-x_0)\right]dx'\right\}. \tag{26}$$

Proceeding exactly as in Section 2, but replacing (3) by (26), one obtains the (exact) relation

$$F_0 = u_0 C_0 - K_0 C_x|_0 = \alpha_0 C_- - \beta_0 C_+, \tag{27}$$

where

$$\alpha_0 = \frac{\exp\left[\left(\frac{u}{2K}\right)_- \Delta x_-\right]}{\int_{x_0-\Delta x_-/2}^{x_0+\Delta x_+/2} \left\{K^{-1}\exp\left[-\frac{u}{K}(x'-x_0)\right]\right\} dx'}, \tag{28}$$

$$\beta_0 = \frac{\exp\left[-\left(\frac{u}{2K}\right)_+ \Delta x_+\right]}{\int_{x_0-\Delta x_-/2}^{x_0+\Delta x_+/2} \left\{K^{-1}\exp\left[-\frac{u}{K}(x'-x_0)\right]\right\} dx'}. \tag{29}$$

The integrals in these expressions are undesirable in a finite difference scheme, but (28) and (29) are useful for the estimation of errors associated with the scheme obtained by simply replacing u and K by their values at x_0. Expanding u/K in a Taylor series about $x = x_0$, substituting the result in the arguments of each exponential, and expanding the resulting expressions, one easily verifies that the relative errors due to variations in u/K are of order $(u/2K)_x \Delta x^2$. Similarly, variations in K (with u/K fixed) give rise to relative errors of order

$$2(\Delta x_+ + \Delta x_-)^{-1} \int_{x_0-\Delta x_-/2}^{x_0+\Delta x_+/2} \left\{\left.\frac{K_x}{K}\right|_0 (x'-x_0) + \left.\frac{K_{xx}}{2K}\right|_0 (x'-x_0)^2\right\} dx' + \text{smaller terms.} \tag{30}$$

If $\Delta x_+ = \Delta x_-$, then the first term in the integrand vanishes identically, leaving a relative error of order $(K_{xx}/K)\,\Delta x^2/12$, but for a non-uniform grid the relative error associated with spatial variations in K is of order $(K_x/K)(\Delta x_+ - \Delta x_-)/4$. Clearly, on a spatially varying grid, spatial variations in K can give rise to larger relative errors than similar variations in u. Simply using the trapezoid rule to estimate the denominator in (28) and (29) would yield a second order scheme for general K and u on a spatially varying grid.

Analysis of the scheme based on cell-averaged properties shows that spatial variations in u/K and K give rise to errors of the same order as those discussed above. In this case, the scheme may be upgraded to second order accuracy when K varies spatially by replacing the denominator in (23) and (24) by

$$E_+^+\left[1 - \frac{K_x}{K}\left(\frac{K}{u} - \frac{\Delta x_-}{2}\right)\right] - E_-^+\left[1 - \frac{K_x}{K}\left(\frac{K}{u} + \frac{\Delta x_+}{2}\right)\right]. \tag{31}$$

The corresponding upgrade for (7) and (8) is obtained by replacing the denominator in these equations by

$$\exp\left[\frac{u\,\Delta x_+}{2K}\right]\left[1 - \frac{K_x}{K}\left(\frac{K}{u} - \frac{\Delta x_-}{2}\right)\right] - \exp\left[-\frac{u\,\Delta x_-}{2K}\right]\left[1 - \frac{K_x}{K}\left(\frac{K}{u} + \frac{\Delta x_+}{2}\right)\right]. \tag{32}$$

The latter modification may be determined from (28) and (29) by replacing u/K by $u/K|_0$ in each exponential (relative error of order $(u/K)_x \Delta x^2$), expanding K^{-1} as $K_0^{-1}[1 - (K_x/K)|_0 (x - x_0)]$ (relative error of order $(K_{xx}/K)\,\Delta x^2/2$), and evaluating the integral analytically.

5. Time-dependence and a stability criterion for forward differencing

The above finite difference schemes may be immediately applied to steady state advection-diffusion type problems. The scheme may, of course, also be used to estimate fluxes in time-dependent problems, but results during any transient phase will only be accurate if the spatial structure remains well approximated by the steady state solution. To quantify this statement, consider the time-dependent version of (11):

$$C_t + uC_x = KC_{xx} - \lambda C. \tag{33}$$

Assuming harmonic time-dependence ($C = \hat{C}(x)\,e^{i\omega t}$), the general solution of (33) is of the form given by (12) and (13), but with λ replaced by $\lambda + i\omega$. Temporal variations have little influence on

the spatial structure (and hence on the fluxes across cell boundaries) if

$$\omega \ll \lambda + \frac{1}{4}\frac{u^2}{K}. \tag{34}$$

However, for higher frequencies, the estimation of fluxes using a scheme which simply neglects temporal variations may result in significant errors, particularly in the phase. It is thus important to realize that the Fiadeiro and Veronis (1977) scheme and the modifications discussed above are most appropriate to steady-state (or nearly steady-state in the sense defined above) problems. In particular, no claim is made that these schemes will have small implicit diffusion or dispersion if they are used to study transient problems. This point is emphasized by the fact that each of these schemes reduces to upwind differencing for $K/u \ll 1$. This is appropriate for steady-state solutions, but for transient problems the resulting errors can be substantial. In cases where the transient behavior is of interest and low implicit diffusion is required, numerical schemes such as those discussed by Smolarkiewicz (1983) are recommended.

For situations where (34) is valid for all frequencies of interest, or when one simply wishes to integrate forward to a final steady state, one must determine an appropriate stability criterion for the time-stepping procedure. Here we will derive criteria appropriate to schemes based on either (7) and (8) or on (23) and (24), for a spatially variable grid and forward differencing in time. We will consider the two-dimensional problem (the extension to three dimensions is trivial), and allow the velocity (u, v) and diffusion coefficients $(K^x, K^y$; not necessarily equal) to vary in both space and time. We emphasize, however, that any temporal variations are assumed to be slow compared to either the explicit decay time-scale or the advection time-scale.

Note that we have chosen not to use the forms (21) and (22) which include the influence of λ in the determination of the difference operator. This is partly for simplicity, but even when $\lambda \neq 0$, the scheme based on (23) and (24) will sometimes be more appropriate for time dependent problems. To understand why, consider (33), and define $\mathscr{C}$ by $C = \mathscr{C}\exp(-\lambda t)$. Then $\mathscr{C}$ satisfies (33) without the radioactive decay term. It thus appears that (23) and (24) would be more appropriate than (21) and (22) to represent the spatial structure of $\mathscr{C}$. But C has exactly the same spatial structure as $\mathscr{C}$, so (23) and (24) must also be better representations for C. How can this be?

The answer is simply that we are assuming that the function we are attempting to approxinate is near steady state so that a steady state approximation can be used to obtain a reasonable representation of its spatial structure. For an example such as radioactive waste dumped into the ocean, $\mathscr{C}$ might well be very near steady state, while C could be far from it: the spatial part of the difference scheme wold then be more appropriately based on the equation for $\mathscr{C}$ (for this case, the term $-\lambda C$ is nearly balanced by C_t, explaining why this scheme is also more appropriate for C). It should be noted, however, that if there was a steady source, C could well be near steady state, and $\mathscr{C}$ far from it. Then it would be more appropriate to use (22) and (23). The most appropriate choice may not always be obvious. However, if decay is rapid we would generally expect $\mathscr{C}$ to be closer to steady state than C, and if decay is weak, the choice will not be critical.

Consider then, the time-dependent, two-dimensional, advection-diffusion equation:

$$C_t + (uC)_x + (vC)_y = (K^x C_x)_x + (K^y C_y)_y - \lambda C. \tag{35}$$

The corresponding finite difference scheme based on cell-averaged properties, with forward differencing in time, is

$$\begin{aligned}\frac{C^{n+1}_{i,j} - C^n_{i,j}}{\Delta t^n} &= \frac{\left\{\begin{array}{l}[\alpha^n_{i-1/2} C^n_{i-1,j} - \beta^n_{i-1/2} C^n_{i,j}] \\ \quad -[\alpha^n_{i+1/2} C^n_{i,j} - \beta^n_{i+1/2} C^n_{i+1,j}]\end{array}\right\}}{\Delta x_i} \\ &+ \frac{\left\{\begin{array}{l}[\alpha^n_{j-1/2} C^n_{i,j-1} - \beta^n_{j-1/2} C^n_{i,j}] \\ \quad -[\alpha^n_{j+1/2} C^n_{i,j} - \beta^n_{j+1/2} C^n_{i,j+1}]\end{array}\right\}}{\Delta y_j} \\ &- \lambda C^n_{i,j},\end{aligned} \tag{36}$$

where the α's and β's are defined in accordance with (23) and (24). A subscript i (j) indicates association with the x (y) direction, and a superscript n refers to the nth time step. The α's, β's, and

the time step Δt may all vary with time, but the time index on these quantities will henceforth be omitted for notational convenience. Note that (36) is also appropriate to the scheme based on point values (the α's and β's are then obtained from (7) and (8)), but this scheme should not be used for $\lambda \neq 0$ since the last term in (36) will not be an accurate representation for this scheme.

To examine the stability of this scheme, we use the local von Neumann stability analysis, essentially as outlined by Press, et al. (1986). Noting that, since (36) is linear, any errors in the approximate solution will also satisfy (36), we consider an error which is locally of the form

$$\xi^n_{i,j} = \exp[i(lx_i + my_j - \omega n\,\Delta t)], \tag{37}$$

where l, m are the local wavenumbers in the x-, y-directions, and ω is the corresponding frequency. Substituting this form into (36) and dividing by $\xi^n_{i,j}$ gives

$$\begin{aligned} E \equiv e^{i\omega\,\Delta t} = 1 - \lambda\,\Delta t - \frac{\Delta t}{\Delta x_i}\,[&\beta_{i-1/2} - \alpha_{i-1/2} \\ &\times e^{-il\,\Delta X_-} - \beta_{i+1/2} e^{il\,\Delta X_+} + \alpha_{i+1/2}] \\ -\frac{\Delta t}{\Delta y_j}\,[&\beta_{j-1/2} - \alpha_{j-1/2} e^{-im\,\Delta Y_-} \\ &- \beta_{j+1/2} e^{im\,\Delta Y_+} + \alpha_{j+1/2}], \end{aligned} \tag{38}$$

where $\Delta X_- = (\Delta x_i + \Delta x_{i-1})/2$, $\Delta X_+ = (\Delta x_i + \Delta x_{i+1})/2$, etc.

For stability we need $|E| < 1$, which is assured if both real and imaginary parts are less than one for all ω, l, and m (this follows from the special exponential form of E). With a little algebra, these can be rewritten as:

$$\begin{aligned} E_r = 1 - \Delta t[&(\beta_{j-1/2} - \alpha_{j-1/2} - \beta_{j+1/2} \\ &+ \alpha_{j+1/2})/\Delta y_j + (\beta_{i-1/2} - \alpha_{i-1/2} \\ &- \beta_{i+1/2} + \alpha_{i+1/2})/\Delta x_i] \\ -2\,\Delta t[&\lambda/2 + [\alpha_{j-1/2} \sin^2(m\,\Delta Y_-/2) \\ &+ \beta_{j+1/2} \sin^2(m\,\Delta Y_+/2)]/\Delta y_j \\ &+ [\alpha_{i-1/2} \sin^2(l\,\Delta X_-/2) \\ &+ \beta_{i+1/2} \sin^2(l\,\Delta X_+/2)]/\Delta x_i], \end{aligned} \tag{39}$$

$$\begin{aligned} E_i = -\Delta t[&[\alpha_{j-1/2} \sin(m\,\Delta Y_-) \\ &+ \beta_{j+1/2} \sin(m\,\Delta Y_+)]/\Delta y_j \\ &+ [\alpha_{i-1/2} \sin(l\,\Delta X_-) \\ &+ \beta_{i+1/2} \sin(l\,\Delta X_+)]/\Delta x_i]. \end{aligned} \tag{40}$$

Noting that $\alpha_{i\pm 1/2} - \beta_{i\pm 1/2} = u_{i\pm 1/2}$, and $\alpha_{j\pm 1/2} - \beta_{j\pm 1/2} = v_{j\pm 1/2}$, we see that the first square bracketed term on the right side of (39) vanishes identically by continuity. Thus, representing the second bracketed term in (39) by ϑ, the condition $E_r^2 \leqslant 1$ may be written as:

$$E_r^2 = 1 - 4\,\Delta t\vartheta[1 - \Delta t\vartheta] \leqslant 1. \tag{41}$$

Finally, it is easily shown that the α's and β's are positive quantities, so ϑ is also positive, and the condition $E_r^2 < 1$ is guaranteed if the square bracketed term in (41) is greater than zero for all l and m. This gives the condition

$$\Delta t < \left[\frac{\lambda}{2} + \frac{(\alpha_{i-1/2} + \beta_{i+1/2})}{\Delta x_i} + \frac{(\alpha_{j-1/2} + \beta_{j+1/2})}{\Delta y_j}\right]^{-1}. \tag{42}$$

Requiring that $E_i^2 < 1$ for all l and m gives the same condition but without the $\lambda/2$ term. Thus (42) is the required stability criterion, which must be satisfied for each grid cell and at each time step.

6. Examples with variable spatial resolution

In the derivations presented above, there was very little difficulty allowing for a spatially varying grid in the analysis, and there are many examples where such a grid is of use: essentially any case where a boundary layer is present. Here we will present two simple examples motivated by studies of the ocean's thermohaline circulation. Note that these results are not intended as a study of the thermocline problem. Any reasonable study of the latter must, among other things, determine the velocity field as part of the analysis: here it will simply be prescribed. The intention is simply to illustrate that the scheme corresponding to a spatially varying grid does work.

Consider an idealized basin with a two-dimensional velocity field given by $u = -\psi_z$, $w = \psi_x$, where x, z are the horizontal and vertical coordinates, u, w are the corresponding velocity

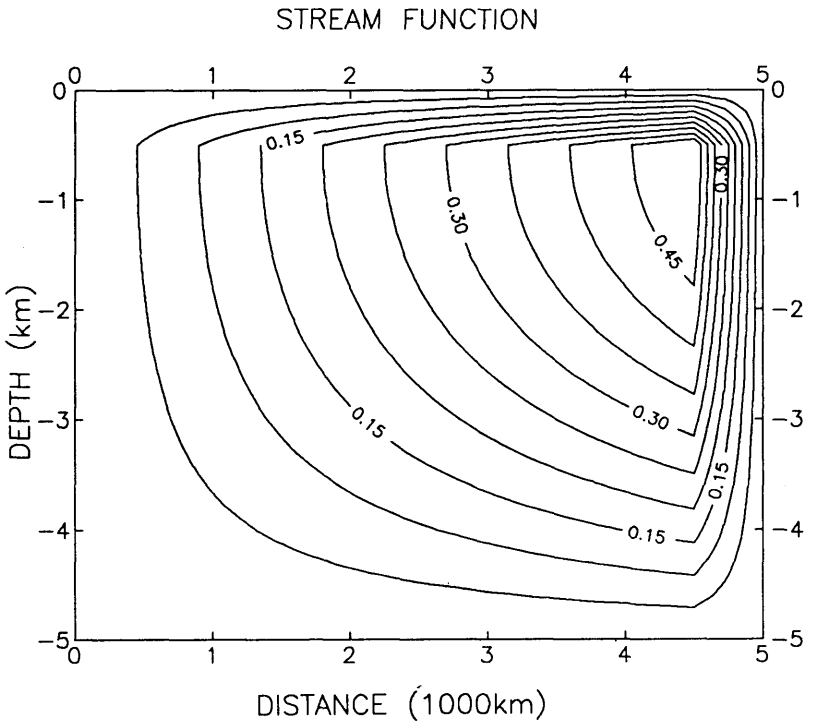

Fig. 1. The specified velocity stream function in units of $m^2\,s^{-1}$.

components, and ψ is as illustrated in Fig. 1. The maximum vertical velocities occur 500 m below the surface, with upwelling at a rate of $10^{-7}\,ms^{-1}$ for $x < 4.5 \cdot 10^6$ m and downwelling at a rate of $9 \cdot 10^{-7}\,ms^{-1}$ for larger values of x. The horizontal diffusivity is taken to be $10^3\,m^2\,s^{-1}$ and two choices for the vertical diffusivity are considered:

$$K^z = \begin{cases} 10^{-4}\,m^2\,s^{-1}, & \text{(43a)} \\ 10^{-4}\left(1 - \dfrac{1.25}{\pi}\tan^{-1}[4.5 \cdot 10^{-3} \times (z + 2.5 \cdot 10^3)]\right) m^2\,s^{-1}, & \text{(43b)} \end{cases}$$

where $-z$ is the distance below the surface of the basin. The latter form has K^z reduced to $3.75 \cdot 10^{-5}\,m^2\,s^{-1}$ at $z = 0$. Finally, the surface temperature is specified by

$$T_s = 25 \cdot [1 + \cos(\pi x / 5 \cdot 10^6\,m)], \quad (44)$$

and the system is spun up to steady state using (36) with $\lambda = 0$, and α and β given by (23) and (24).

Figs. 2a, b show the solutions in the upper 3000 m of the basin, based on a uniform spatial grid with 50 grid cells in each of the x and z directions. Decreasing the resolution further does not influence these solutions. Figs. 2c, d show the corresponding solutions based on a 10×10 grid. (Results were interpolated to a 50×50 grid using the kriging technique available as part of the SURFER software package marketed by Golden Software, Inc. Kriging is an interpolation method based on the autocorrelation between data points which produces an unbiased, minimum variance estimate.) Significant errors are evident in the results based on the coarser grid, particularly in Fig. 2d. In both cases, the thermocline tends to be too deep in the upwelling region and too shallow where the water is downwelling. Encorporating the change corresponding to (31) did not improve the results. Further, in each case the largest errors

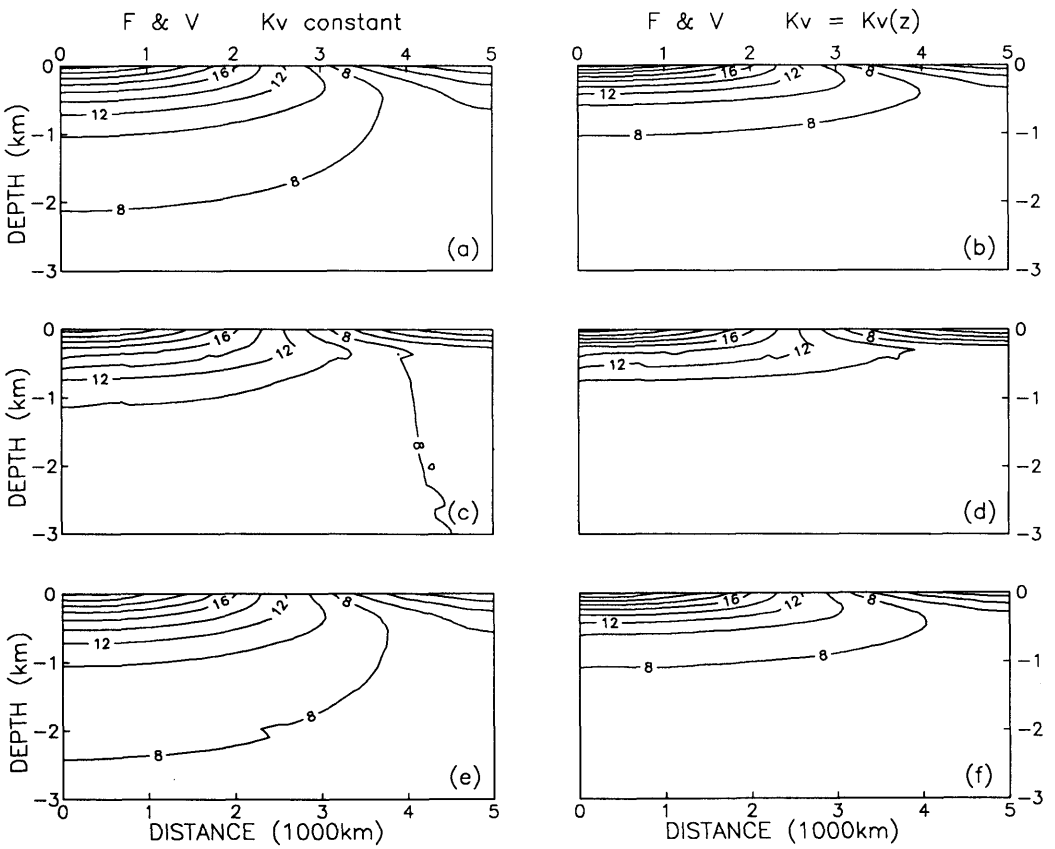

Fig. 2. The "temperature" field determined by solving the advection diffusion equation with the velocity field determined by Fig. 1 and the surface temperature given by (44). Panels (a), (b) correspond to a uniform 50×50 grid over the domain and give accurate approximations to the solutions. Panels (c), (d) are the corresponding results based on a uniform 10×10 grid, and panels (e), (f) correspond to a spatially varying grid chosen to have improved resolution in the upper and right hand portions of the domain.

occur in the upper right corner, where the velocity changes direction by 90° in one grid cell for the coarse resolution runs, suggesting that this is the primary source of error.

To test the finite difference scheme with a spatially varying grid, the solutions were recalculated based on a grid with cell boundaries at $x = 0$, 1000, 2000, 3000, 3800, 4400, 4500, 4600, 4700, 4800, and $5000 \cdot 10^3$ m, and at $z = 0, -100, -200, -400, -600, -900, -1200, -1500, -2000, -3500$, and -5000 m. The intention was to have good resolution in the upper couple of km and around the downwelling region, without going to more than a 10×10 grid. The results based on this grid are shown in Figs. 2e, f. The improvement over the results shown in Figs. 2c, d is obvious.

7. Summary

Four basic objectives have been achieved:

(1) the basis of the Fiadeiro and Veronis (1977) finite difference scheme has been clarified;

(2) the generalization to a spatially variable grid has been developed;

(3) an analogous formulation based on cell-averaged properties has been presented;

(4) a stability criterion has been obtained for the finite difference schemes based on the new

spatial formulations together with forward differencing in time.

In addition, errors due to spatial variations in u and K have been considered and a modification has been derived to give a second order scheme when a variable grid is used through a region where K is spatially varying. Examples are presented to illustrate the utility of the spatially varying grid.

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