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# The potential vorticity equation: from planetary to small scale

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ABSTRACT

A systematic approach for specializing equations to any specific scale from planetary to fairly small scale, is refined and applied to the potential vorticity equation. By the omission of small terms from the exact equation, an applicable conservation equation is presented for each of six identified scale regimes.

## 1. Introduction

The basic dynamical equations of meteorology require specialization for the particular scale or scales of motion to which they are being applied, since this aids studies aimed at increased insight, and also since full generality of the equations allows irrelevant solutions that can cause large errors in computations when using observed data.

A process has been described (Burger, 1988), which gives a unified and systematic approach to the problem to specializing the equations to any scale from the largest (or planetary) scale to fairly small scales of only several kilometres. The basis of the approach is the known method of comparing the orders of magnitude of dimensionless combinations of the quantities that characterize the scales of motion, and omitting small terms. There is always the possibility of obtaining large deviations by the omission of small terms, but Segel (1972) has shown that the use of dimensionless combinations of scale quantities reduces the chances of that happening.

A resumé, and in fact a refinement, of the abovementioned process for the simplest physics of a dry, frictionless, adiabatic atmosphere, is presented in Section 2. To start with, some very unrestrictive order of magnitude inequalities between such combinations or ratios are assumed to hold. Then from these, consequential order equalities are deduced with the aid of the basic equations themselves. As a final preparation, a few general inequalities are stated, and then each of six relevant scale regimes is characterized by the orders of magnitude of six of the dimensionless quantities as compared to the number 1. These order of magnitude statements can be used to obtain equations adapted to each scale regime.

Note incidentally that a certain elegance is given to the treatment by the fact that the physical magnitudes of scale quantities do not feature explicitly. Instead, only dimensionless combinations of such quantities are compared. However, in any specific application, the actual magnitudes of the scale quantities do become relevant when testing for the applicability of the stated relationships.

Section 3 contains the application of the above process to the potential vorticity equation. For this purpose, not only the straightforward comparison of orders of magnitude of terms is used, but also the consideration that, for the simple physics considered, the equation is a conservation equation and has to remain so. The results provide a basis of comparison with conservation equations obtained in modelling.

## 2. Summary of basic order equalities and inequalities

As in Burger (1988), we use, as scale quantities, typical values for external quantities like the earth's radius $a$, the gravity acceleration $g$ and the Coriolis parameter $f$ (namely $2\Omega \sin \varphi$), as well as quantities referring to the state and motions of the atmosphere, like the "standard" hydrostatic parts $p_0$ and $\rho_0$ of the pressure and density depending on height only, and in addition their deviations $p_1$ and $\rho_1$; the "standard" part $\sigma_0$ of the static stability (namely $\partial \ln \theta_0/\partial z$), and its deviation $\sigma_1$; the horizontal and vertical length scales $L$ and $h$ and the "scale height" or "pressure height" $H$ (namely, $-p_0/(\partial p_0/\partial z)$); the horizontal and vertical speeds $V$ and $W$ and the phase speed $C$ of waves; and the horizontal divergence $D$. Gradients or derivatives are represented by dividing by the scale quantity in the direction concerned, for instance, $W/h$ corresponds to the vertical derivative of the vertical speed.

Note that the definitions of $p_0$, $p_1$, $\rho_0$ and $\rho_1$ imply

$$\frac{\partial p}{\partial z} + \rho g = \frac{\partial p_1}{\partial z} + \rho_1 g.$$

From the above choices, the order equality

$$p_0/gH\rho_0 \sim 1 \qquad (1)$$

follows immediately. Next, we arbitrarily choose some order statements which are easily shown to be very unrestrictive, namely the strong inequality

$$H/a \ll 1,$$

and the weak inequalities

$$fV/g \lesssim h/a$$

and

$$\rho_1/\rho_0, \quad h/H, \quad H/L, \quad L/a,$$
$$C/V, \quad W/V, \quad DL/V \quad \text{and} \quad V^2/gh \lesssim 1.$$

These relations, with the continuity equation, yield the further weak inequality

$$WL/Vh \lesssim 1.$$

In the above, the relation $C/V \lesssim 1$ implies that fast-moving waves, where the time scale in derivatives is small compared to that of the advective space derivative, are to be excluded.

The basic dynamical equations now yield useful order equalities. The horizontal equations of motion give

$$p_1/fVL\rho_0 \sim V/fL + 1,$$

the vertical equation of motion, with (1), gives

$$p_1/p_0 \sim p_1/gH\rho_0 \sim h\rho_1/H\rho_0,$$

the thermodynamic equation,

$$WL/Vh \sim \rho_1/\rho_0\sigma_0 h,$$

the continuity equation,

$$D \sim W/h + W\sigma_1,$$

and the vorticity equation,

$$(WL/Vh)(V/fL + 1) \sim V/fL + L/a + \rho_1 p_1/\rho_0^2 fVL.$$

For combining these relations to more convenient forms, we use the "convection-advection ratio"

$$A = WL/Vh, \qquad (2)$$

the Burger number

$$B = g\sigma_0 h^2/f^2L^2, \qquad (3)$$

and the Rossby number

$$\mathrm{Ro} = V/fL. \qquad (4)$$

Then the following order equalities are found:

$$p_0/gH\rho_0 \sim 1, \qquad (5)$$

$$p_1/fVL\rho_0 \sim \mathrm{Ro} + 1, \qquad (6)$$

$$p_1/p_0 \sim h\rho_1/H\rho_0 \qquad (7)$$

$$\sim (fVL/gH)(\mathrm{Ro} + 1), \qquad (8)$$

$$AB/\mathrm{Ro} \sim \mathrm{Ro} + 1, \qquad (9)$$

$$D \sim VA/L, \qquad (10)$$

$$A(\mathrm{Ro} + 1) \sim \mathrm{Ro} + L/a, \qquad (11)$$

where, in the last two, further use has been made of the assumed inequalities. In addition, by elimination of $A$ from (9) and (11), we get

$$B(1 + L/a\,\mathrm{Ro}) \sim \mathrm{Ro}^2 + 1. \qquad (12)$$

These relations generalize previous work. Relations (7) and (8) correspond, with some re-interpretation, to relations obtained earlier by Dutton (1986, eqs. (15) and (18)), while (11) generalizes his eq. (34). Further, (12) generalizes the known relations $B = 1$ or $g\sigma_0/f^2 \sim L^2/a^2$ for the synoptic scale (Prandtl, 1936), and $B(1 + L/a\,\mathrm{Ro}) \sim 1$ for the synoptic and planetary scales (Burger, 1958).

The above relations are a useful tool for specializing the meteorological equations to different scales. Note that, for all scales from planetary to "small" scale:

$$h/a, \quad H/a, \quad fVL/gh, \quad fVL\,\mathrm{Ro}/gh \ll 1, \tag{13}$$

and for scales smaller than the planetary scale, also the further strong inequality

$$fVL/gh\,\mathrm{Ro} \ll 1 \tag{14}$$

holds.

Note that the initial assumption of the simplest physics can be weakened and still yield the above relationships, provided only that the horizontal and vertical (turbulent) force per unit mass, $F_\mathrm{h}$ and $F_\mathrm{v}$, and the rate of non-adiabatic addition of heat per unit mass, $\dot{q}$, satisfy the order inequalities

$$F_\mathrm{h}/fV, \quad F_\mathrm{v}h/fVL, \quad \dot{q}\kappa h/fV^2H \ll 1 + \mathrm{Ro},$$

as is easily seen by comparison with the dominant terms in the equations of motion and the thermodynamic equation. When these conditions are violated, the terms concerned have to be added on, but the details of this lie outside our present scope. Although the relevance of this for smaller scales is known, it is interesting to note that the above factors $h/L$ with $F_\mathrm{v}$ and $h/H$ with $\dot{q}$ tend to weaken the restriction on these quantities for shallow flows, while the right-hand side does the same for the (deep or shallow) "small" scale, for which $\mathrm{Ro} \gg 1$.

To characterize every specific scale, we find that, with the above, we only need to determine the order of the six quantities $A$, $B$, Ro, $L/a$, $h/H$ and $h/L$ for every scale.

We distinguish six scale regimes, namely the planetary, synoptic, deep and shallow mesoscale, and deep and shallow small scale. The indications "deep" and "shallow" are used here to distinguish between the cases where the vertical scale $h$ is of the same or of lower order than the scale height $H$.

The following characterizations turn out to be reasonable for the different scales, although other combinations may also be useful, for instance where horizontal length scales of different orders apply in different directions, or where low latitudes cause larger values of Ro to influence the situation for the larger scales.

Planetary scale:

$$A \sim L/a \sim h/H \sim 1, \qquad B \sim \mathrm{Ro} \ll 1, \qquad h/L \ll 1. \tag{15}$$

Synoptic scale:

$$B \sim h/H \sim 1, \qquad A \sim \mathrm{Ro} \sim L/a \ll 1, \qquad h/L \ll 1. \tag{16}$$

Deep mesoscale:

$$A \sim B \sim \mathrm{Ro} \sim h/H \sim 1, \qquad L/a, \quad h/L \ll 1. \tag{17}$$

Shallow mesoscale:

$$A \sim B \sim \mathrm{Ro} \sim 1, \qquad L/a, \quad h/L, \quad h/H \ll 1. \tag{18}$$

Deep small scale:

$$\sqrt{B} \sim \mathrm{Ro} \gg 1, \qquad A \sim h/L \sim h/H \sim 1, \qquad L/a \ll 1. \tag{19}$$

Shallow small scale:

$$\sqrt{B} \sim \mathrm{Ro} \gg 1, \qquad A \sim 1, \qquad L/a, \quad h/L, \quad h/H \ll 1. \tag{20}$$

It turns out that these relations are now sufficient to determine, for each scale regime, which terms are negligibly small, and so to obtain the specialized sets of equations. Note, though, that very small scales, for instance with a horizontal extent of 1 km, are not treated here.

## 3. Application to the potential vorticity equation

The above will now be applied to find specialized forms of the potential vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{1}{\rho}\left\{\boldsymbol{k} \times \left(\frac{1}{r}\frac{\partial rV}{\partial z} - \nabla w\right) + 2\boldsymbol{\Omega}_\mathrm{h}\right\} \cdot \nabla \ln \theta + \frac{\sigma}{\rho}(\zeta + f)\right] = 0, \tag{21}$$

with fairly standard notation: $V$ and $w$ denote horizontal and vertical wind velocities; $\boldsymbol{k}$ is the vertical unit vector with $r$ the radial distance from the earth's centre and $z$ height above sea level; $\nabla$ is the horizontal gradient operator and $\zeta$ the vertical component of the vorticity vector; $f$ and $2\boldsymbol{\Omega}_{\mathrm{h}}$ are the vertical and horizontal components of the earth's rotation $\boldsymbol{\Omega}$; $\rho$ and $\theta$ density and potential temperature; $\sigma = \partial \ln \theta / \partial z$ the static stability; and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla + w \frac{\partial}{\partial z}.$$

In order to consider the magnitudes of individual terms, (21) is expanded to give:

$$\begin{aligned} 0 = {} & \frac{\nabla \ln \theta}{\rho} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \left\{ \boldsymbol{k} \times \left( \frac{1}{r} \frac{\partial r \boldsymbol{V}}{\partial z} - \nabla w \right) + 2\boldsymbol{\Omega}_{\mathrm{h}} \right\} \\ & + \frac{1}{\rho} \left\{ \boldsymbol{k} \times \left( \frac{1}{r} \frac{\partial r \boldsymbol{V}}{\partial z} - \nabla w \right) + 2\boldsymbol{\Omega}_{\mathrm{h}} \right\} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \nabla \ln \theta \\ & - \left\{ \boldsymbol{k} \times \left( \frac{1}{r} \frac{\partial r \boldsymbol{V}}{\partial z} - \nabla w \right) + 2\boldsymbol{\Omega}_{\mathrm{h}} \right\} \cdot (\nabla \ln \theta) \frac{1}{\rho^2} \frac{\mathrm{D}\rho}{\mathrm{D}t} \\ & + \frac{\sigma}{\rho} \frac{\mathrm{D}}{\mathrm{D}t} (\zeta + f) + \frac{1}{\rho} (\zeta + f) \frac{\mathrm{D}\sigma}{\mathrm{D}t} \\ & - \sigma (\zeta + f) \frac{1}{\rho^2} \frac{\mathrm{D}\rho}{\mathrm{D}t}. \end{aligned} \tag{22}$$

For convenience in characterizing the order of magnitude of each part in this equation, we will label the above terms (a) to (f), as follows:

$$(\mathrm{a}) = \frac{\nabla \ln \theta}{\rho} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \left\{ \boldsymbol{k} \times \left( \frac{1}{r} \frac{\partial r \boldsymbol{V}}{\partial z} - \nabla w \right) + 2\boldsymbol{\Omega}_{\mathrm{h}} \right\},$$

$$(\mathrm{b}) = \frac{1}{\rho} \left\{ \boldsymbol{k} \times \left( \frac{1}{r} \frac{\partial r \boldsymbol{V}}{\partial z} - \nabla w \right) + 2\boldsymbol{\Omega}_{\mathrm{h}} \right\} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \nabla \ln \theta,$$

$$(\mathrm{c}) = - \left\{ \boldsymbol{k} \times \left( \frac{1}{r} \frac{\partial r \boldsymbol{V}}{\partial z} - \nabla w \right) + 2\boldsymbol{\Omega}_{\mathrm{h}} \right\} \cdot (\nabla \ln \theta) \frac{1}{\rho^2} \frac{\mathrm{D}\rho}{\mathrm{D}t},$$

$$(\mathrm{d}) = \frac{\sigma}{\rho} \frac{\mathrm{D}}{\mathrm{D}t} (\zeta + f),$$

$$(\mathrm{e}) = \frac{1}{\rho} (\zeta + f) \frac{\mathrm{D}\sigma}{\mathrm{D}t},$$

$$(\mathrm{f}) = - (\zeta + f) \frac{\sigma}{\rho^2} \frac{\mathrm{D}\rho}{\mathrm{D}t}.$$

Some care is needed in determining the order of magnitude of specific terms. Note that horizontal and time derivatives of state variables involve only the "deviation" parts with subscript 1, while their vertical derivatives involve not only these parts, divided by the height scale $h$, but also the "standard" parts with subscript 0, divided by the scale height $H$, for instance,

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} \sim \frac{C}{L} \frac{\rho_1}{\rho_0},$$

$$\frac{1}{\rho} \frac{\partial \rho}{\partial z} \sim \frac{\rho_1}{h \rho_0} + \frac{1}{H}.$$

Thus, whereas the individual derivative operator is generally characterized by

$$\frac{\mathrm{D}}{\mathrm{D}t} \sim \frac{C}{L} + \frac{V}{L} + \frac{W}{h},$$

we have

$$\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} \sim \left( \frac{C}{L} + \frac{V}{L} + \frac{W}{h} \right) \frac{\rho_1}{\rho_0} + \frac{W}{H},$$

Note also that the deviation part of the static stability is characterized by

$$\sigma_1 \sim -\frac{1}{\rho_0} \frac{\partial \rho_1}{\partial z} \sim \frac{\rho_1}{h \rho_0}.$$

Similar care is needed with the individual derivative of the Coriolis parameter $f$, for which we get

$$\frac{\mathrm{D}f}{\mathrm{D}t} \sim \frac{Vf}{a},$$

at least for middle latitudes, and similarly for the horizontal component of the earth's rotation,

$$\frac{\mathrm{D}}{\mathrm{D}t} 2\boldsymbol{\Omega}_{\mathrm{h}} \sim \frac{Vf}{a}.$$

The order of the different parts of (22) is now given by

$$(\mathrm{a}) \sim \frac{1}{\rho_0} \frac{\rho_1}{L \rho_0} \left\{ \left( \frac{C}{L} \quad \frac{V}{L} \quad \frac{W}{h} \right) \left( \frac{V}{h} \quad \frac{V}{a} \quad \frac{W}{L} \right) \quad \frac{Vf}{a} \right\},$$

$$(\mathrm{b}) \sim \frac{1}{\rho_0} \left( \frac{V}{h} \quad \frac{V}{a} \quad \frac{W}{L} \quad f \right) \left( \frac{C}{L} \quad \frac{V}{L} \quad \frac{W}{h} \right) \frac{\rho_1}{L \rho_0},$$

$$\begin{aligned} (\mathrm{c}) \sim {} & \left( \frac{V}{h} \quad \frac{V}{a} \quad \frac{W}{L} \quad f \right) \frac{\rho_1}{L \rho_0} \frac{1}{\rho_0} \\ & \times \left\{ \left( \frac{C}{L} \quad \frac{V}{L} \quad \frac{W}{h} \right) \frac{\rho_1}{\rho_0} \quad \frac{W}{H} \right\}, \end{aligned}$$

$$(\mathrm{d}) \sim \frac{\sigma_0}{\rho_0}\left\{\left(\frac{C}{L} \quad \frac{V}{L} \quad \frac{W}{h}\right)\frac{V}{L} \quad \frac{Vf}{a}\right\},$$

$$(\mathrm{e}) \sim \frac{1}{\rho_0}\left(\frac{V}{L} \quad f\right)\left\{\left(\frac{C}{L} \quad \frac{V}{L} \quad \frac{W}{h}\right)\frac{\rho_1}{h\rho_0} \quad \frac{W}{H}\sigma_0\right\},$$

$$(\mathrm{f}) \sim \left(\frac{V}{L} \quad f\right)\frac{\sigma_0}{\rho_0}\left\{\left(\frac{C}{L} \quad \frac{V}{L} \quad \frac{W}{h}\right)\frac{\rho_1}{\rho_0} \quad \frac{W}{H}\right\}.$$

From each of these we now eliminate $W$, $f$ and the state variables by introducing the parameters $A$, $B$ and Ro from (2), (3) and (4), applying the order equalities (5) up to (12), and removing the factor $fV\sigma_0/L\rho_0$. Thus, using the subscript 0 with (a) to (f) to indicate multiplication by $L\rho_0/fV\sigma_0$, we get

$$(a_0) \sim \frac{\mathrm{Ro}}{B}(\mathrm{Ro}+1)$$
$$\times\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro}\left(1 \quad \frac{h}{a} \quad A\frac{h^2}{L^2}\right) \quad \frac{h}{a}\right\}$$
$$\sim A\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro}\left(1 \quad \frac{h}{a} \quad A\frac{h^2}{L^2}\right) \quad \frac{h}{a}\right\},$$

$$(b_0) \sim \left\{\mathrm{Ro}\left(1 \quad \frac{h}{a} \quad A\frac{h^2}{L^2}\right) \quad \frac{h}{L}\right\}$$
$$\times\left(\frac{C}{V} \quad 1 \quad A\right)\frac{\mathrm{Ro}}{B}(\mathrm{Ro}+1)$$
$$\sim \left\{\mathrm{Ro}\left(1 \quad \frac{h}{a} \quad A\frac{h^2}{L^2}\right) \quad \frac{h}{L}\right\}\left(\frac{C}{V} \quad 1 \quad A\right)A,$$

$$(c_0) \sim \left\{\mathrm{Ro}\left(1 \quad \frac{h}{a} \quad A\frac{h^2}{L^2}\right) \quad \frac{h}{L}\right\}\frac{\mathrm{Ro}}{B}(\mathrm{Ro}+1)$$
$$\times\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\frac{fVL}{gh}(\mathrm{Ro}+1) \quad A\frac{h}{H}\right\}$$
$$\sim \left\{\mathrm{Ro}\left(1 \quad \frac{h}{a} \quad A\frac{h^2}{L^2}\right) \quad \frac{h}{L}\right\}A$$
$$\times\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\frac{fVL}{gh}(\mathrm{Ro}+1) \quad A\frac{h}{H}\right\},$$

$$(d_0) \sim \left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro} \quad \frac{L}{a},$$

$$(e_0) \sim (\mathrm{Ro} \quad 1)\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\frac{\mathrm{Ro}}{B}(\mathrm{Ro}+1) \quad A\frac{h}{H}\right\}$$
$$\sim (\mathrm{Ro} \quad 1)\left\{\left(\frac{C}{V} \quad 1 \quad A\right)A \quad A\frac{h}{H}\right\}$$
$$\sim (\mathrm{Ro} \quad 1)\left(\frac{C}{V} \quad 1 \quad A \quad \frac{h}{H}\right)A,$$

$$(f_0) \sim (\mathrm{Ro} \quad 1)$$
$$\times\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\frac{fVL}{gh}(\mathrm{Ro}+1) \quad A\frac{h}{H}\right\}.$$

Note how (9) has been used to simplify all but $(d_0)$ and $(f_0)$.

These relations can now be applied, together with (13) to (20), to compare, for each of the six scale regimes considered, the order of magnitude of all terms in the original potential vorticity equation.

*Planetary scale*:

$(a_0)$, $(b_0)$, $(c_0) \ll 1$,

$(d_0)$, $(e_0)$, $(f_0) \sim 1$, namely:

$$(d_0) \sim \frac{L}{a},$$

$$(e_0) \sim 1\left(\frac{C}{V} \quad 1 \quad A \quad \frac{h}{H}\right)A,$$

$$(f_0) \sim 1\left\{A\frac{h}{H}\right\}.$$

*Synoptic scale*:

$(a_0)$, $(b_0)$, $(c_0) \ll \mathrm{Ro}$,

$(d_0)$, $(e_0)$, $(f_0) \sim \mathrm{Ro}$, namely:

$$(d_0) \sim \left(\frac{C}{V} \quad 1\right)\mathrm{Ro} \quad \frac{L}{a},$$

$$(e_0) \sim (1)\left(\frac{C}{V} \quad 1 \quad \frac{h}{H}\right)A,$$

$$(f_0) \sim (1)\left\{A\frac{h}{H}\right\}.$$

*Deep mesoscale*:

$(a_0)$ to $(f_0) \sim 1$, namely:

$$(a_0) \sim A\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro}(1)\right\},$$

$$(b_0) \sim \{\mathrm{Ro}(1)\}\left(\frac{C}{V} \quad 1 \quad A\right)A,$$

$$(c_0) \sim \{\mathrm{Ro}(1)\}\,A\left\{A\frac{h}{H}\right\},$$

$$(d_0) \sim \left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro},$$

$$(e_0) \sim (\mathrm{Ro} \quad 1)\left(\frac{C}{V} \quad 1 \quad A \quad \frac{h}{H}\right)A,$$

$$(f_0) \sim (\mathrm{Ro} \quad 1)\left\{A\frac{h}{H}\right\}.$$

*Shallow mesoscale*:

$(c_0)$, $(f_0) \ll 1$,

$(a_0)$, $(b_0)$, $(d_0)$, $(e_0) \sim 1$, namely:

$$(a_0) \sim A\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro}(1)\right\},$$

$$(b_0) \sim \{\mathrm{Ro}(1)\}\left(\frac{C}{V} \quad 1 \quad A\right)A,$$

$$(d_0) \sim \left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro},$$

$$(e_0) \sim (\mathrm{Ro} \quad 1)\left(\frac{C}{V} \quad 1 \quad A\right)A.$$

*Deep small scale*:

$(a_0)$ to $(f_0) \sim \mathrm{Ro}$, namely:

$$(a_0) \sim A\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro}\left(1 \quad A\frac{h^2}{L^2}\right)\right\},$$

$$(b_0) \sim \left\{\mathrm{Ro}\left(1 \quad A\frac{h^2}{L^2}\right)\right\}\left(\frac{C}{V} \quad 1 \quad A\right)A,$$

$$(c_0) \sim \left\{\mathrm{Ro}\left(1 \quad A\frac{h^2}{L^2}\right)\right\}A\left\{A\frac{h}{H}\right\},$$

$$(d_0) \sim \left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro},$$

$$(e_0) \sim (\mathrm{Ro})\left(\frac{C}{V} \quad 1 \quad A \quad \frac{h}{H}\right)A,$$

$$(f_0) \sim (\mathrm{Ro})\left\{A\frac{h}{H}\right\}.$$

*Shallow small scale*:

$(c_0)$, $(f_0) \ll \mathrm{Ro}$,

$(a_0)$, $(b_0)$, $(d_0)$, $(e_0) \sim \mathrm{Ro}$, namely:

$$(a_0) \sim A\left\{\left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro}(1)\right\},$$

$$(b_0) \sim \{\mathrm{Ro}(1)\}\left(\frac{C}{V} \quad 1 \quad A\right)A,$$

$$(d_0) \sim \left(\frac{C}{V} \quad 1 \quad A\right)\mathrm{Ro},$$

$$(e_0) \sim (\mathrm{Ro})\left(\frac{C}{V} \quad 1 \quad A\right)A.$$

The above schematics can now be translated into actual terms retained in the equation on each scale. Having done that, we will, however, in each case retain any further (small) terms that may be required to save the conservation property of the potential vorticity equation. The result is as follows.

*Planetary scale*:

$$0 = \frac{\sigma}{\rho}\frac{\mathrm{D}f}{\mathrm{D}t} + \frac{f}{\rho}\frac{\mathrm{D}\sigma}{\mathrm{D}t} - \frac{\sigma f}{\rho^2}w\frac{\partial \rho_0}{\partial z},$$

with conservation form

$$0 = \frac{\mathrm{D}}{\mathrm{D}t}\frac{\sigma f}{\rho_0} \tag{23}$$

by noting that only the dominant part of $\rho$, namely $\rho_0$, is needed throughout, since

$$\frac{\mathrm{D}\rho_0}{\mathrm{D}t} = w\frac{\partial \rho_0}{\partial z}.$$

This remark about $\rho$ turns out to hold for the other scales too.

*Synoptic scale*:

$$0 = \frac{\sigma}{\rho}\frac{\mathrm{D_h}}{\mathrm{D}t}(\zeta + f) + \frac{f}{\rho}\frac{\mathrm{D}\sigma}{\mathrm{D}t} - \frac{\sigma f}{\rho^2}w\frac{\partial \rho_0}{\partial z},$$

with conservation form

$$0 = \frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{\sigma}{\rho_0}(\zeta + f)\right], \tag{24}$$

after reintroduction of some $\zeta$-terms.

*Deep mesoscale*:

$$0=\frac{\nabla \ln \theta}{\rho}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\left(\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\right)+\frac{1}{\rho}\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\nabla \ln \theta$$
$$-\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot(\nabla \ln \theta)\frac{1}{\rho^2}w\frac{\partial \rho_0}{\partial z}$$
$$+\frac{\sigma}{\rho}\frac{\mathrm{D}\zeta}{\mathrm{D}t}+\frac{1}{\rho}(\zeta+f)\frac{\mathrm{D}\sigma}{\mathrm{D}t}-\sigma(\zeta+f)\frac{1}{\rho^2}w\frac{\partial \rho_0}{\partial z},$$

with conservation form

$$0=\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{1}{\rho_0}\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot\nabla \ln \theta+\frac{\sigma}{\rho_0}(\zeta+f)\right], \tag{25}$$

by which a "beta"-term (an individual derivative of the Coriolis parameter $f$) has been reintroduced.

*Shallow mesoscale*:

$$0=\frac{\nabla \ln \theta}{\rho}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}+\frac{1}{\rho}\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\nabla \ln \theta$$
$$+\frac{\sigma}{\rho}\frac{\mathrm{D}\zeta}{\mathrm{D}t}+\frac{1}{\rho}(\zeta+f)\frac{\mathrm{D}\sigma}{\mathrm{D}t},$$

which, after dividing by $\rho$, gives the conservation form

$$0=\frac{\mathrm{D}}{\mathrm{D}t}\left[\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot\nabla \ln \theta+\sigma(\zeta+f)\right], \tag{26}$$

again with a beta-term reintroduced.

*Deep small scale*:

$$0=\frac{\nabla \ln \theta}{\rho}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\left\{\boldsymbol{k}\times\left(\frac{\partial \boldsymbol{V}}{\partial z}-\nabla w\right)\right\}$$
$$+\frac{1}{\rho}\boldsymbol{k}\times\left(\frac{\partial \boldsymbol{V}}{\partial z}-\nabla w\right)\cdot\frac{\mathrm{D}}{\mathrm{D}t}\nabla \ln \theta$$
$$-\boldsymbol{k}\times\left(\frac{\partial \boldsymbol{V}}{\partial z}-\nabla w\right)\cdot(\nabla \ln \theta)\frac{1}{\rho^2}w\frac{\partial \rho_0}{\partial z}$$
$$+\frac{\sigma}{\rho}\frac{\mathrm{D}\zeta}{\mathrm{D}t}+\frac{\zeta}{\rho}\frac{\mathrm{D}\sigma}{\mathrm{D}t}-\sigma\zeta\frac{1}{\rho^2}w\frac{\partial \rho_0}{\partial z},$$

with conservation form

$$0=\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{1}{\rho_0}\boldsymbol{k}\times\left(\frac{\partial \boldsymbol{V}}{\partial z}-\nabla w\right)\cdot\nabla \ln \theta+\frac{\sigma\zeta}{\rho_0}\right]. \tag{27}$$

*Shallow small scale*:

$$0=\frac{\nabla \ln \theta}{\rho}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\left\{\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\right\}+\frac{1}{\rho}\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\nabla \ln \theta$$
$$+\frac{\sigma}{\rho}\frac{\mathrm{D}\zeta}{\mathrm{D}t}+\frac{\zeta}{\rho}\frac{\mathrm{D}\sigma}{\mathrm{D}t},$$

with conservation form, again after dividing by $\rho$,

$$0=\frac{\mathrm{D}}{\mathrm{D}t}\left[\boldsymbol{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\cdot\nabla \ln \theta+\sigma\zeta\right]. \tag{28}$$

The equations (23) to (28) are the approximate forms of the potential vorticity equation we set out to derive. Their derivation, by direct specialization of the general potential vorticity equation, is in contrast to the more indirect way usually followed in modelling, when a conservation equation that crops up is conveniently denoted as potential vorticity equation. The above equations could thus serve as a basis of comparison with the conventional forms occurring in models.

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