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The fractal dimension of relative lagrangian motion

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ABSTRACT

14 clusters of drifters are studied to obtain the fractal dimension of relative trajectories between pairs of drifters. Averaging over all experiments, the relative trajectories have a fractal dimension of 1.3 over separation scales ranging from 10 m to 4000 m. The clusters were deployed in Lake Erie, the Atlantic Equatorial Undercurrent, and in coastal waters off the south shore of Long Island. The average fractal dimension was quite similar for groups of experiments in each of these 3 areas, but it fluctuates over a greater range (1.59 to 1.12) when we consider individual experiments. This and consideration of variation of the generalized fractal dimension for higher moments, raise the possibility that trajectories are multifractal. The estimates we obtain for fractal dimension of relative lagrangian motion should be regarded as first estimates, pending analysis of more substantial data sets.

1. Introduction

We will study the dispersive motion of a cluster of drifters that are constrained to move in the horizontal plane. Thus we consider all pairs of drifters in a cluster and consider the trajectory of one drifter relative to another. The relative trajectories are irregular over a wide range of scales. One might expect, therefore, that such curves are fractal, having a fractal dimension greater than the topological dimension of the relative trajectory but less than the topological dimension of the space that relative trajectories are found in.

Osborne et al. (1986, 1989) have previously studied the fractal behaviour of drifter trajectories at scales typical of GFD turbulence, (20 to 150 km). Their measurements were made in the Kuroshio current and yielded a fractal dimension of approximately 1.3 for the trajectories of 3 drifters with the average drifter displacement removed from each trajectory. We on the other hand will be looking at relative motion of drifters separated at scales where internal wave dynamics are most relevant.

Oceanographers usually interpret relative dispersion using statistical properties such as variance, transition probability density functions and diffusivities (Davis, 1983; Okubo, 1962). Sometimes these statistical properties can be linked to dynamical processes by analytic solutions to the equations of motion (Herterich and Hasselmann, 1982; Sanderson and Okubo, 1988), but more often they are linked to dynamics by separating the flow into mean and turbulent components and considering the form of the averaged momentum and energy equations (Osborn, 1980). Some very useful descriptions of relative dispersion are entirely empirical (Stommel, 1949; Ichiye and Olson, 1960; Okubo, 1971) or are based on dimensional arguments and the existence of an inertial subrange.

In the above context, we consider yet another way of describing relative dispersion, *viz.*, in terms of its fractal geometry. The mathematics of fractal geometry and their links to dynamical processes are relatively unexplored compared to more classical approaches. However relative trajectories do have a fractal nature and therefore studies from the fractal viewpoint are likely to lead to new insight and a better description of this process. For example Hertschel and Procaccia (1984) use observations of the fractal dimension of clouds (Lovejoy, 1982) to develop a theory for relative diffusion. Furthermore, knowledge of the fractal dimension for stochastic models, such as fractional Brownian motion (Mandelbrot and Van Ness, 1968) or Lévy walks (Shlesinger and Klafter, 1986). Fractals also provide a measure of the patchy distribution of material dispersing in the ocean, since the fractal co-dimension indicates the fraction of space occupied by trajectories.

2. The data

Drifter cluster data from three geographic regions will be used in the present analysis. Basic information about each cluster such as the number of drifters in a cluster and period tracked over, is summarized in Table 1. The scale of each cluster is indicated by $\sigma_{rc} = [2\sigma_x \sigma_y]^{1/2}$, where σ_x^2, σ_y^2 are the variance of x, y components of drifter position relative to the centroid. The cluster dimensions span a range of scales from 45 m to 6000 m.

The first group of clusters CIPREA, D1, D2, D3 were deployed in the salinity core of the Atlantic Equatorial Undercurrent (depth 70 m to 90 m). These experiments have been reported previously by Fahrbach et al. (1986). Clusters were tracked using the ships radar. The CIPREA cluster was deployed in the Gulf of Guinea (longitude 03°43′W), whereas D1, D2, D3 were deployed further west at an approximate longitude of 21°W.

The second group of experiments were conducted as part of the Lagrangian Eulerian Diffusion Study (LEDS). Cluster deployments were approximately 5 km off the south shore of Long Island in the vicinity of Shinnecock Inlet. The water depth was about 30 m and the experimental site was chosen for its linear shoreline and gently sloping bathymetry. Wind speeds were less than 6 m/s throughout the experiments. Drifters were tracked using two small boats equipped with Motorola Mini-Ranger systems (accuracy ± 1 m). In the present analysis we will only consider the three clusters with the longest time series. The LEDS3 cluster was deployed in the surface layer during the summer when the water column was stratified (buoyancy frequency 0.037 s^{-1}). The LEDS10 and LEDS11 clusters were deployed during winter when the bouoyancy frequency was 0.013 s^{-1}.

The final group of experiments, CCIW1–7, were conducted by the Canadian Centre for Inland Waters (CCIW). One of these experiments, CCIW1, has been previously reported by

Table 1. *Information about drifter cluster experiments*

Cluster name	Number of drifters in the cluster	Number of position fixes/ drifter	Time interval between fix's (min)	Drogue depth (m)	σ_{rc} (m)	Location	Date (Start)
CIPREA	6	88	30		2600	Atlantic	15 Aug. 1978
D1	9	100	30	See	3300	Equatorial	14 Feb. 1979
D2	8	88	30	text	5300	Undercurrent	18 May 1979
D3	5	77	30		6000		14 Jun. 1979
LEDS3	14	99	6	3	60	South Coast	
LEDS10	20	32	10	4.6	45	of	
LEDS11	20	37	10	4.6	55	Long Island	
CCIW1	9	444	20	3	1200		11 Jul. 1979
CCIW2	8	113	20	3	1900	Central	16 Aug. 1979
CCIW3	9	56	30	3	860	Basin	05 Sep. 1979
CCIW4	7	102	30	3	850	of	08 Sep. 1979
CCIW5	10	48	30	3	690	Lake	13 Aug. 1980
CCIW6	10	81	30	10	800	Erie	15 Aug. 1980
CCIW7	10	69	30	3	800		18 Aug. 1980

Sanderson (1987). Drifter positions were determined using the ships radar. The experimental site was in the Central Basin of Lake Erie, in an area with a flat bottom. The water column consisted of a well mixed surface layer separated from a deep unstratified layer by a sharp thermocline. Drogues were set at depths within the surface layer. Drifter trajectories indicated inertial oscillations and wind driven motion.

3. Estimation of fractal dimension

The fractal dimension D of a set of points in a two-dimensional space can be defined

$$D = \lim_{\varepsilon \to 0} \frac{\log n(\varepsilon)}{\log 1/\varepsilon}, \qquad (1)$$

where $n(\varepsilon)$ is the number of disks, of radius ε, needed to completely cover all points in the set. The above definition can be extended to higher dimensional spaces, but throughout the following work we will only be considering motion on a horizontal plane. In particular we are interested in the fractal properties of the motion that acts to disperse a cluster of particles. Thus for each cluster experiment, we consider pairs of drifters and study the relative trajectory of one drifter to the other. In this way a cluster of N drifters yields $N(N-1)/2$ possible pairs. We will use the "yardstick" method to obtain a "divider dimension" D_L which approximates the Hausdorff dimension D. The algorithm described by Rapaport (1985) was used to obtain the length $L(\eta)$ of individual drifter trajectories as a function of the length of the yardstick η. Average values of the trajectory length $\bar{L}(\eta)$ were then obtained by averaging over values of $L(\eta)$ obtained from all pairs of drifters in the patch. Log–log plots of $\bar{L}(\eta)$ versus η are shown for each experiment in Fig. 1. The dashed lines represent the best fit linear regression.

Some discussion of the range of values of η is required. The minimum length of η was chosen to be the root mean square step length between consecutive points on the relative trajectories. Rapaport (1985) observed that true linear behaviour of constant step random walks require η to be approximately 6 times the step length. The plots in Fig. 1 are in general quite linear at the smaller values of η, implying that this condition is not so critical for our data. However we conducted tests on artificially generated data, which indicated that it was necessary to consider larger minimum values of η when the fractal dimension of the generated data approached 2.

The maximum length of η was chosen to be 1/5 the net displacement of the longest trajectory. At large values of η, we occasionally have fewer trajectories sufficiently long to give an estimate of $L(\eta)$. In this circumstance the value of $\bar{L}(\eta)$ was calculated by averaging over the reduced number of trajectories. Alternatively, we could have considered these shorter trajectories to have zero length for the larger values of η and include these in the calculation of $\bar{L}(\eta)$. In this circumstance the slope of a few of the lines in Fig. 1 increases slightly, but not significantly in view of the statistical uncertainty to be discussed later.

Divider fractal dimensions D_L are calculated from the slope of the linear best fits in Fig. 1 using

$$\bar{L}(\eta) = \eta^{1-D_L},$$

and are presented in Table 2. Averaging over all the experiments gives a mean divider fractal dimension of $\bar{D}_L = 1.34$. This fractal behaviour seems to extend over a range of scales from 10 m to 4000 m, based upon the linearity of plots in Fig. 1. However fractal dimensions for individual experiments fluctuate over a range from 1.59 ± 0.02 for D1 to 1.12 ± 0.01 for CCIW7. (The uncertainties are the standard deviation about the least square fit.) These are both many standard deviations from 1.34, and are suggestive of multifracticallity.

We expect the finite length of relative trajectories to lead to errors in our estimation of D_L beyond the standard deviation about the least squares fit. We therefore divided our longest time series CCIW1 into shorter segments from which we calculate D_L. Table 3 presents the mean and standard deviation for the values of D_L obtained from independent segments of a given trajectory length. It also presents the average standard deviation about the least squares fit. Long trajectories (200 position fixes per drifter) have a standard deviation of the estimates of D_L that is very similar to that about the least squares regression fit. However shorter trajectories (55 to 27 position fixes per drifter) have a standard deviation of estimates of D_L that is 3 to 5 times

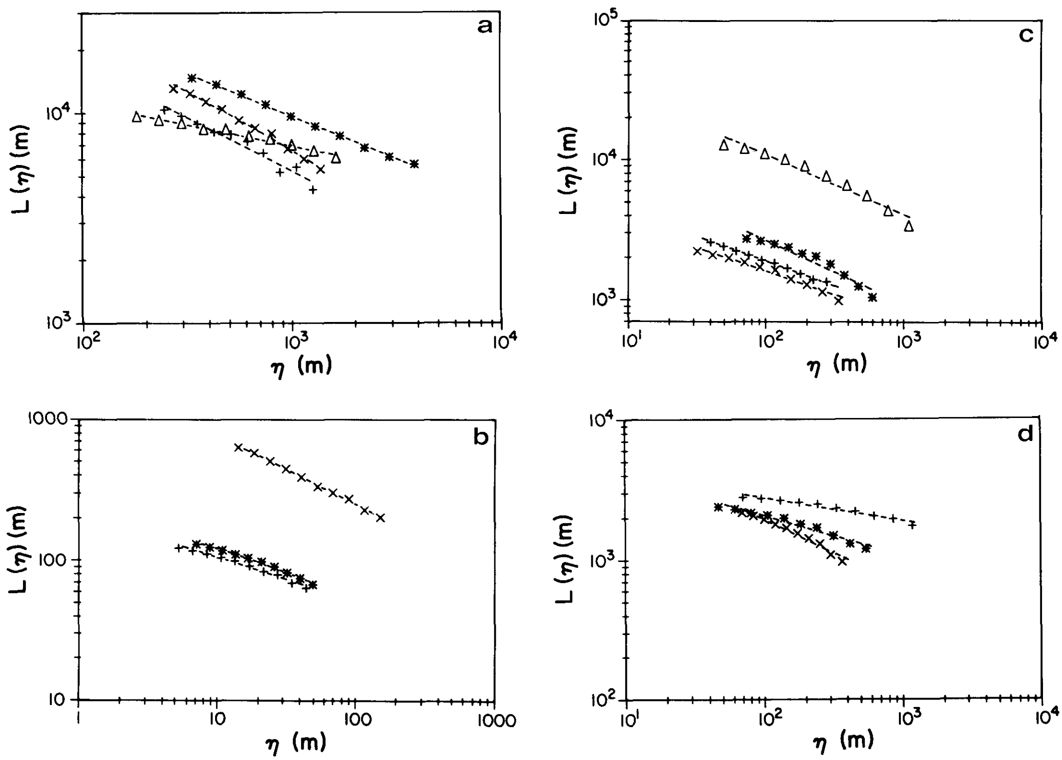

Fig. 1. Log–log plots of $\overline{L}(\eta)$ versus η for relative trajectories of pairs of particles with various clusters. The least squares best linear fit is shown by a dashed line. In (a), the symbols represent different cluster experiments as follows: △ CIPREA, × D_1, ∗ D_2, + D_3. In (b), the symbols represent cluster experiments as follows: × LEDS3, ∗ LEDS10, + LEDS11. In (c), the symbols represent cluster experiments as follows: △ CCIW1, × CCIW2, ∗ CCIW3, + CCIW4. In (d), the symbols represent: × CCIW5, ∗ CCIW6, + CCIW7.

larger than that about the least squares regression fit. Thus the true uncertainty in the estimates of D_L is often quite a bit greater than suggested by the standard deviation in Table 2 especially for shorter trajectories. Nevertheless our previous conclusion that D_L fluctuates significantly between experiments is still valid. We will therefore proceed to investigate the multifracticallity of the relative trajectories.

Multifractal properties require more than a single fractal dimension to describe the behaviour of statistical moments at different scales (Schertzer and Lovejoy, 1987). We will follow Osborne et al. (1989) and compute generalized fractal dimensions D_q using the box counting algorithm. If $D_q = D_{q'}$ for $q \neq q'$, then this indicates that the curve is monofractal, whereas for multifractals $D_q < D_{q'}$ for $q > q'$.

Our trajectories are finite, and this will result in statistical uncertainties in estimates of $D_{q'}$–D_q. To investigate this matter, we divide our longest set of trajectories CCIW1 into independent segments. These segments are then analyzed to determine how the mean value and standard deviation of $D_{1.5}$–D_5 vary as a function of trajectory length. The results are presented in Table 4. Again statistical uncertainties grow significantly as the length of the time series is reduced. In view of this we will only consider data for which the trajectories have at least 90 steps per drifter when calculating D_q in the range $q = 1.5$ to 5.

Values of D_q are presented as a function of q in Table 5. There is a general decrease in D_q as q is increased, beyond that which might be expected to arise from statistical fluctuations. The differ-

Table 2. *Values of divider fractal dimension D_L and generalized fractal dimension $D_{q=2}$, for pair trajectories within cluster experiments; uncertainties are the standard deviation about the least squares fit*

Experiment	Number of pair trajectories	D_L	$D_{q=2}$	Number of position fixes per drifter
CIPREA	14	1.20 ± 0.01	1.24 ± 0.02	88
D1	21	1.59 ± 0.02	1.54 ± 0.03	100
D2	21	1.32 ± 0.01	1.47 ± 0.07	88
D3	10	1.51 ± 0.04	1.45 ± 0.06	77
LEDS3	68	1.42 ± 0.01	1.26 ± 0.02	99
LEDS10	148	1.25 ± 0.01	1.25 ± 0.04	32
LEDS11	140	1.24 ± 0.01	1.27 ± 0.04	37
CCIW1	35	1.42 ± 0.03	1.41 ± 0.04	444
CCIW2	26	1.31 ± 0.02	1.31 ± 0.02	113
CCIW3	28	1.37 ± 0.03	1.21 ± 0.05	56
CCIW4	21	1.34 ± 0.01	1.49 ± 0.04	102
CCIW5	30	1.45 ± 0.03	1.43 ± 0.03	48
CCIW6	43	1.28 ± 0.02	1.23 ± 0.02	81
CCIW7	31	1.12 ± 0.01	1.09 ± 0.04	69

Table 3. *Average values for estimates of fractal dimension from different length segments of the CCIW1 trajectories. We present the average standard error of the regression σ_r and the standard deviation of the estimates of fractal dimension σ_e*

Trajectory segment length (steps)	Average of the estimates of fractal dimension	σ_r	σ_e
444	1.42	± 0.03	—
222	1.43	± 0.04	—
111	1.39	± 0.03	± 0.03
55	1.34	± 0.02	± 0.07
27	1.36	± 0.02	± 0.10

Table 4. *Average values of $D_{1.5}$–D_5 computed from segments of CCIW1 trajectories; we present the standard deviation about the least squares fits σ ($D_{1.5}$–D_5), and the standard deviation of $D_{1.5}$–D_5 obtained from the different segments σ_e ($D_{1.5}$–D_5)*

Trajectory segment length (steps)	Average of ($D_{1.5}$–D_5)	σ ($D_{1.5}$–D_5)	σ_e ($D_{1.5}$–D_5)
444	0.13	0.03	—
222	0.19	0.03	—
111	0.22	0.04	0.03
55	0.24	0.05	0.06
27	0.22	0.07	0.12

ence between $D_{1.5}$ and D_5 indicates the possibility of at least some multi-fractal behaviour. However this is a very tentative result since our estimates of the fractal dimension of the higher order moments are based on a quite limited number of data points. Significant variability of fractal dimension between clusters, indicates that multi-fractal behaviour might be more observable in longer time series. The box counting algorithm gives the correlation dimension (Osborne et al., 1989; Grassberger and Procaccia, 1983) for the case $q = 2$. The correlation dimension $D_{q=2}$ is presented alongside the divider dimension D_L in Table 2. For most experiments $D_{q=2}$ and D_L are very similar.

The average and standard deviation of $D_{q=2}$ and D_L for all the experiments in Table 2 is $D = 1.34 \pm 0.12$. This fractal behaviour pertains to relative motion for scales ranging from 10 to 4000 m. Osborne et al. (1989) obtained an

Table 5. *Values of the generalized fractal dimension D_q for those experiments in which the drifter time series consisted of at least 88 fixes per drifter; the errors D_q are the standard deviation about the least squares fit; the bottom row shows $\Delta D_q = D_{1.5} - D_5$ for each experiment*

	Generalized fractal dimension D_q						
q	CIPREA	D1	D2	LEDS3	CCIW1	CCIW2	CCIW4
1.5	1.25 ± 0.02	1.56 ± 0.03	1.50 ± 0.07	1.27 ± 0.02	1.43 ± 0.04	1.32 ± 0.02	1.40 ± 0.04
2.0	1.24 ± 0.02	1.54 ± 0.03	1.47 ± 0.07	1.26 ± 0.02	1.41 ± 0.04	1.31 ± 0.02	1.40 ± 0.04
2.5	1.22 ± 0.02	1.51 ± 0.04	1.43 ± 0.07	1.23 ± 0.03	1.39 ± 0.03	1.29 ± 0.02	1.38 ± 0.04
3.0	1.19 ± 0.02	1.49 ± 0.04	1.39 ± 0.08	1.20 ± 0.03	1.37 ± 0.03	1.27 ± 0.03	1.37 ± 0.04
3.5	1.16 ± 0.03	1.47 ± 0.04	1.35 ± 0.08	1.17 ± 0.04	1.35 ± 0.03	1.25 ± 0.03	1.35 ± 0.05
4.0	1.14 ± 0.03	1.45 ± 0.05	1.31 ± 0.08	1.14 ± 0.04	1.33 ± 0.03	1.23 ± 0.04	1.34 ± 0.05
4.5	1.11 ± 0.04	1.44 ± 0.05	1.28 ± 0.08	1.12 ± 0.05	1.32 ± 0.03	1.21 ± 0.04	1.32 ± 0.05
5.0	1.09 ± 0.04	1.42 ± 0.05	1.24 ± 0.09	1.10 ± 0.05	1.30 ± 0.03	1.20 ± 0.04	1.31 ± 0.06
ΔD_q	0.16	0.14	0.26	0.17	0.13	0.12	0.09

average fractal dimension of 1.27 ± 0.06. However their single particle trajectories were at larger scales (20 km to 150 km) and had the time average velocity removed. Buoyancy and rotational dynamics clearly contribute to the relative motion in our data set, whereas geophysical fluid dynamical turbulent dynamics pertain to the trajectories analyzed by Osborne et al. (1989). The similarity in fractal dimensions obtained here with those of Osborne et al (1989) is therefore surprising. It suggests that fractal scaling of relative drifter trajectories might extend over a much broader range of scales than considered here, or expected from dynamical considerations.

However Okubo (personal communication) analyzed a satellite image (chlorophyll concentration) of the mid-Pacific and obtained different fractal dimensions. The original image contains a large patch of phytoplankton located between 2°N and 2°S latitude and 135° and 141°W longitude. A method similar to this drifter study was used to evaluate the fractal dimension of concentration contours. They obtained the fractal dimension D_L of 1.46 at scales of 2 to 22 km and 1.18 at larger scales of 24–50 km. They argue that at scales less than 50 km, biological processes play a minor role compared with physical processes, and hence the patch boundaries are considered a manifestation of physical processes at these scales. Clearly there is a need for further analysis of tracer distributions and drifter trajectories in order to better determine the range and nature of their fractal scaling properties.

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