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Validation study of scale selection in low-order models of Rayleigh–Bénard convection

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ABSTRACT

The scale selection properties of some low-order models of Rayleigh–Bénard convection are compared with the results of some high-order models. The results of the high-order models are inconsistent with those of two previously proposed low-order models in some selected experiments. A new low-order model is offered that is consistent with the high-order models.

1. Introduction

Low-order, Galerkin approximations to the Boussinesq equations for two-dimensional, free-slip, Rayleigh–Bénard convection have recently been used by Van Delden (1984) (hereafter, VD) and Chang and Shirer (1984) (hereafter, CS) to study the evolution and non-linear interaction of two competing horizontal scales. At low Prandtl number, both of these studies find a strong tendency for evolution to steady solutions of exclusively the larger horizontal scale. In CS this tendency leads to a stable solution with a horizontal wavenumber very near the small-wavenumber cut-off to the linear instability. For example, stable solutions are shown in CS with a convective heat transport less than 15% of that of the steady, but unstable, solution at the critical horizontal wavenumber. VD and CS cite Kuo and Platzman (1961), Saltzman (1962) and Shirer and Dutton (1979) as precedents for the level of truncation in their Galerkin models. However, neither VD nor CS produces evidence for cell broadening of such extreme propensity either in experiments or in higher resolution numerical models.

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In this work, we compare the scale selection properties of the 10-coefficient model of VD and the 7-coefficient model of CS with higher resolution spectral models and a finite-difference model. The cell broadening at low Prandtl number that occurs in the models of VD and CS does not occur in the higher resolution models. The purpose of this paper is merely to point out this fact and not to discredit the achievement or promise of low-order modelling. If solutions to a low-order model are to be interpreted as reproductions or rough analogs of natural observations rather than as approximate solutions to the actual partial differential equations from which the model is derived, then this fact is of no great consequence. In my view though, low-order models are most useful when they produce good approximations to the actual solutions of the mathematically defined physical laws thought to govern a system. In this study, it is tacitly assumed that "actual solutions" are uniformly approached as the discretization error in a numerical solution is decreased.

As a small positive contribution to low-order modelling, I am able to offer an 11-coefficient model that has the same scale-selection behavior as the higher resolution models in a few selected experiments. It is certainly easier to discredit a model than it is to credit one, and the 11-coefficient model is offered only as possibly being useful. No thorough investigation is made into the bounds of parameter space for which the 11-coefficient model is valid.

2. The Galerkin model

The two-dimensional, Boussinesq equations are non-dimensionalized as in VD and CS (Saltzman, 1962)

$$\frac{\partial \theta}{\partial t} + J(\psi, \theta) = R\frac{\partial \psi}{\partial x} + \nabla^2 \theta, \tag{1}$$

$$\frac{\partial}{\partial t}\nabla^2 \psi + J(\psi, \nabla^2 \psi) = P\frac{\partial \theta}{\partial x} + P\nabla^4 \psi. \tag{2}$$

Here θ and ψ are the dimensionless temperature fluctuation and stream function, J is the Jacobian operator in x and z, and R and P denote the Rayleigh and Prandtl numbers. In this non-dimensionalization, the depth of the fluid layer is π and the Rayleigh number is the conventional Rayleigh number divided by π^4. The boundary conditions are free-slip and constant temperature in the vertical and periodic in the horizontal.

A spectral form of (1) and (2) is obtained by expanding θ and ψ in terms of orthogonal functions that are consistent with the boundary conditions (Saltzman, 1962):

$$\theta = \sum_i \Theta_i(t)\eta_i(x, z), \tag{3}$$

$$\psi = \sum_i \Psi_i(t)\phi_i(x, z), \tag{4}$$

where

$$\eta_i = \cos p_i \alpha x \sin q_i z, \tag{5}$$

$$\phi_i = \sin m_i \alpha x \sin n_i z. \tag{6}$$

Here m, n and q are positive integers and p is a non-negative integer. The aspect ratio or period of the solution is expressed by α. Substitution of (3) and (4) into (1) and (2) gives a non-linear, autonomous system of ordinary differential equations for the Fourier amplitudes:

$$\dot{\Theta}_i = -\sum_j \sum_k a_{ijk}\Psi_j\Theta_k + R\sum_j b_{ij}\Psi_j - r_i^2\Theta_i, \tag{7}$$

$$\dot{\Psi}_i = -\frac{1}{s_i^2}\sum_j \sum_k c_{ijk}\Psi_j\Psi_k s_k^2 - Ps_i^2\Psi_i - \frac{P}{s_i^2}\sum_j d_{ij}\Theta_j, \tag{8}$$

where

$$\nabla^2 \eta_i = -r_i^2 \eta_i, \tag{9}$$

$$\nabla^2 \phi_i = -s_i^2 \phi_i, \tag{10}$$

$$a_{ijk}\int_0^{\pi}\int_0^{2\pi} \eta_i^2 \, \mathrm{d}x \, \mathrm{d}z = \int_0^{\pi}\int_0^{2\pi} \eta_i J(\phi_j, \eta_k) \, \mathrm{d}x \, \mathrm{d}z, \tag{11}$$

$$c_{ijk}\int_0^{\pi}\int_0^{2\pi} \phi_i^2 \, \mathrm{d}x \, \mathrm{d}z = \int_0^{\pi}\int_0^{2\pi} \phi_i J(\phi_j, \phi_k) \, \mathrm{d}x \, \mathrm{d}z, \tag{12}$$

$$b_{ij}\int_0^{\pi}\int_0^{2\pi} \eta_i^2 \, \mathrm{d}x \, \mathrm{d}z = \int_0^{\pi}\int_0^{2\pi} \eta_i \frac{\partial \phi_j}{\partial x} \, \mathrm{d}x \, \mathrm{d}z, \tag{13}$$

$$d_{ij}\int_0^{\pi}\int_0^{2\pi} \phi_i^2 \, \mathrm{d}x \, \mathrm{d}z = \int_0^{\pi}\int_0^{2\pi} \phi_i \frac{\partial \eta_j}{\partial x} \, \mathrm{d}x \, \mathrm{d}z. \tag{14}$$

As is well known, if R is slowly increased, the complete system (7) and (8) would first become unstable from a state of rest (all amplitudes zero) to a mode with $(\alpha p, q) = (\alpha m, n) = (1/\sqrt{2}, 1)$ at the critical Rayleigh number $R_c = 27/4$. For $R > R_c$, a continuous spectrum of linearly unstable modes could be possible in a state of rest, yet it is a fundamental property of non-linear solutions to (7) and (8) that one fundamental mode and its harmonics tend to dominate and give the solution a cellular character just as in real convection.

In a Galerkin solution of (7) and (8), the set of retained modes is truncated, usually at high wavenumbers so that a certain minimal spatial scale is resolved. For conditions very near the onset of convection, or $R/R_c \to 1^+$, a Galerkin expansion with two temperature modes $(\alpha p, q) = (1/\sqrt{2}, 1)$ and $(0, 2)$ and one streamfunction mode $(\alpha m, n) = (1/\sqrt{2}, 1)$ in fact is an asymptotic solution (Schlüter et al., 1965). This severe truncation has been studied far outside the limit where it formally applies to Rayleigh–Bénard convection; the resulting equations, known as the Lorenz equations, have been studied primarily because of their chaotic solutions. In order to accurately model all aspects of convection for $R > R_c$, an increasing number of modes with larger wavenumber must be retained. For example, the 3-coefficient Lorenz model gives a Nusselt number $\mathrm{Nu} = 2.80$ at $R = 10R_c$ and $P = 6.8$. In the same conditions, a 21-coefficient model which retains all modes with $m + n \leqslant 6$ and $p + q \leqslant 6$ where $m + n$ and $p + q$ are even is required to model the high resolution limit $\mathrm{Nu} = 4.24$ to within 1% (Veronis, 1966). Furthermore, the chaotic temporal behavior of the

Lorenz model is not consistent with higher resolution models (Curry et al., 1984).

That is not to say that severely truncated models are necessarily imprecise when applied to the stability of non-linear Rayleigh–Bénard convection; they can in fact give useful results at least in the range $R_c < R < 10R_c$. For example, both the Lorenz model and higher resolution models are known to predict attraction to steady solutions for single-mode convection in that range. Furthermore, in the next section we see examples of both low-order and higher resolution models generally predicting intransitivity of the wavenumber when the wavenumber is near the critical wavenumber. However, most of the interesting or "non-trivial" loss of stability that occurs in the models of VD and CS does not occur in the higher resolution models.

3. Two cells versus one cell

Consider a periodic box with dimensions such that two cells with the critical horizontal wavenumber $1/\sqrt{2}$ exactly fit in the box. Non-linear, steady-state solutions for one or two convection cells can exist in such a box. For what values of R and P (if any) would one such steady solution lose stability to the other solution and for what values would the other solution then grow and completely dominate the original solution?

This described scenario of cell-size adjustment by doubling or halving the wavenumber is certainly not the only one possible. Two-dimensional numerical simulations in a very long box show a cell-size adjustment process that is more gradual (Ogura, 1971; Rothermel and Agee, 1986). This scenario is also only one particular case of the two-mode interactions modelled in VD and CS. Only this particular two-mode interaction is considered here because the two fundamental modes and their primary interaction via the sum and difference wavenumber modes nicely fit into the corner of wavenumber space (Fig. 1) and therefore can easily be tested against a higher resolution control experiment. The goal here is not to exhaustively study cell-size adjustment, but rather to investigate the applicability of Galerkin models to convection in narrow boxes.

Six truncations and a finite-difference model are investigated with $\alpha = 1/(2\sqrt{2})$. All the truncations contain at least the fundamental mode $(m,n) = (p,q) = (2,1)$, which represents the critical instability, and $(m,n) = (p,q) = (1,1)$, which represents a linear convective roll of double that size. After first calculating the interaction coefficients, (7) and (8) are integrated using a fourth-order Runge-Kutta scheme. The program was checked by successfully reproducing results from Veronis (1966). In this study, the equations are initialized with either the steady one-cell or two-cell solution, or a close approximation to it, for selected values of R and P in parameter space $6.75 < R < 67.5$ and $0.1 < P < 10$. The steady solutions are then perturbed with the other fundamental temperature mode at very small amplitude. The evolution of the system and its

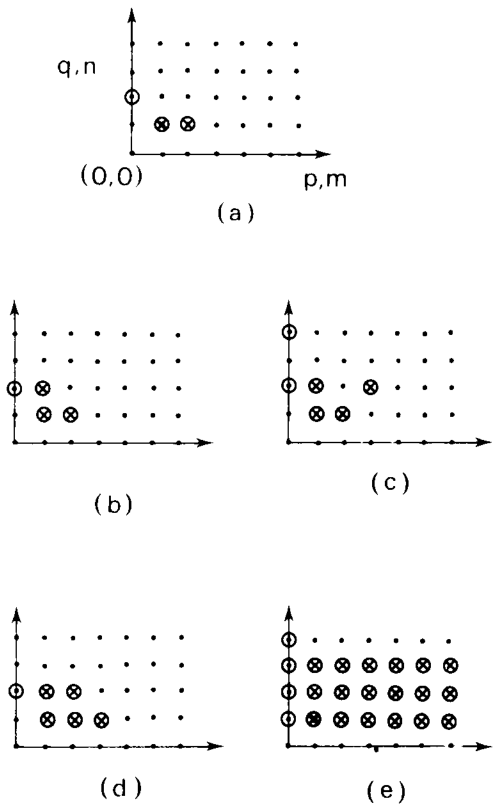

Fig. 1. The wavenumber combinations (p,q) for temperature ○ and (m,n) for streamfunction × retained in the Galerkin models. (a) The union of two Lorenz models; (b) the model of CS; (c) the model of VD; (d) a new 11-coefficient model; (e) the high-resolution 40-coefficient model.

attraction to either the two-cell or one-cell solution is then monitored.

The first truncation investigated is simply the union of two Lorenz models with non-linear interaction between the competing scales occurring only via the mean temperature stratification (0, 2). The one-cell solution is unstable throughout the investigated parameter space and the steady two-cell solution is a global attractor. This result is certainly not consistent with the numerical simulations of Ogura (1971), which indicate, for example, that the one-cell solution is stable at $R = 4R_c$.

The second model is one of the ones used by CS and pioneered by Saltzman (1962). This model is like the first except that it also contains one "catalytic harmonic" (1, 2) which forms an interacting triad with the two fundamental modes. For sufficiently large R and P, both the one-cell and two-cell solutions are stable attractors (Fig. 2). As could be expected from the solutions of Ogura (1971), only the two-cell solution is stable for sufficiently small R, where the Nusselt number of the one-cell solution is significantly smaller than that for the two-cell solution. (The one-cell solution becomes unstable from a state of rest at $R = 11.39$.) For sufficiently small P and large R, only the one-cell state is a stable attractor. Steady or oscillating mixed states are possible along the boundary of this region.

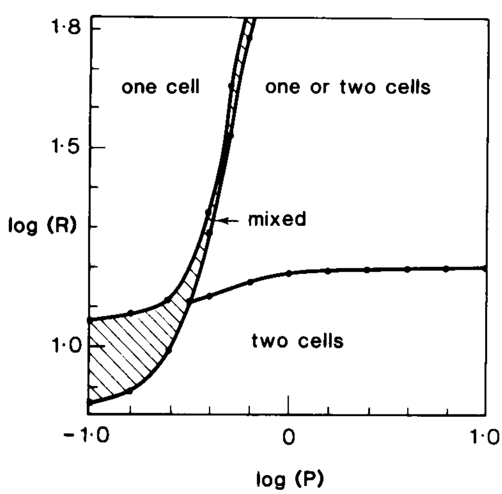

Fig. 2. The possible stable states in the model of CS.

The third model is that used by VD, which is similar to that of CS, but contains in addition a higher wavenumber member (3, 2) of an interacting triad with the two fundamental modes. Higher resolution of the mean stratification is also modelled with the retention of the (0, 4) temperature mode. The results for this model (Fig. 3) are very similar to those of the CS model. They are also consistent with (Fig. 11) of VD.

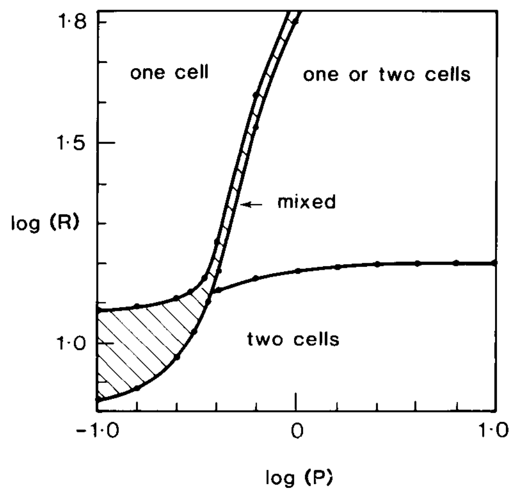

Fig. 3. The possible stable states in the model of VD.

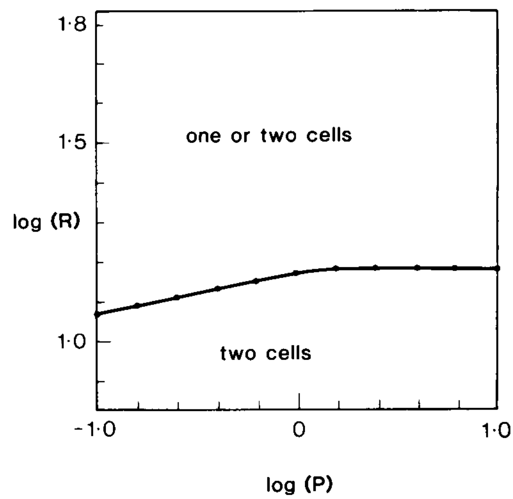

Fig. 4. The possible stable states in the 11-coefficient model and the 40-coefficient model.

The fourth model is a new model which contains a number of overlapping triads with each triad containing two fundamental modes. The interaction of the catalytic harmonic with (2, 1) and (3, 1) is modelled as well as its interaction with (1, 1) and (2, 1). The results of this fourth model (Fig. 4) are profoundly different from those of the second and third models; the two-cell solution does not lose stability to the one-cell solution at low Prandtl number. However, the boundary of the exclusively two-cell region is still very similar. Presumably the stability of the two-cell solution at low Prandtl number is a result of the coupling of the (1, 2) catalytic harmonic to a higher wavenumber mode. This coupling prevents energy from being transferred exclusively between (2, 1) and (1, 1) via (1, 2).

If only the one-cell and two-cell solutions are considered, then the results of the fourth model compare very well with a 40-coefficient, higher resolution model which retains all modes with $m \leqslant 6$, $p \leqslant 6$, $n \leqslant 3$ and $q \leqslant 3$ as well as the (0, 4) temperature mode. The elongated rectangle shape of the retained modes is chosen so that both the fundamental one-cell and two-cell solutions would be resolved as equally as possible. In this case both fundamental solutions are modelled by at least the $K = 4$ triangle of modes of Veronis (1966). The results from this fifth model are the same as those of the fourth model to within the width of the line drawn in Fig. 4. Evidently, increasingly accurate coupling of the fundamental modes and the catalytic harmonic to higher wavenumber modes does not change the stability of the solutions in this case.

As a final comparison between the various models we present some specific time evolutions at $R = 30$ and $P = 0.1$. Fig. 5 shows evolution from steady two-cell solutions perturbed at $t = 0$ with the (1, 1) temperature mode with amplitude

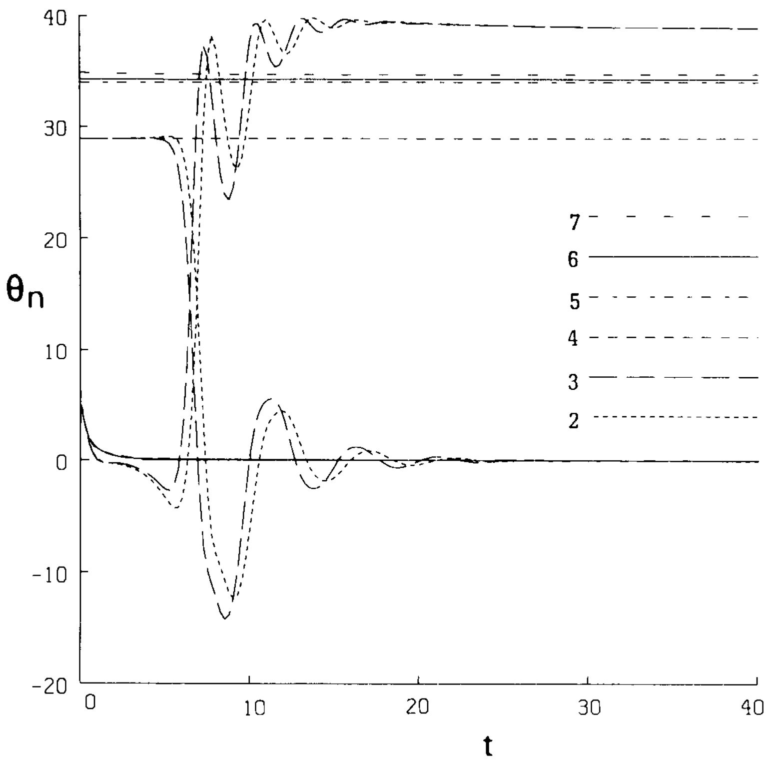

Fig. 5. Time integrations of the various models from the nearly steady two-cell solution at $R = 30$ and $P = 0.1$ perturbed by the (1, 1) temperature mode of amplitude 5.0 at $t = 0$. The amplitude of the (2, 1) temperature mode is shown beginning with amplitude greater than 25.0. The amplitude of the (1, 1) temperature mode is shown beginning from 5.0.

$\Theta = 5.0$. Results from two additional high-resolution models (six and seven) are also shown. All models were run at a time step which was set increasingly small until there was no difference apparent in the plots and no change in the steady solutions to four significant figures. The temporal evolution of models four through seven are obviously very similar; only in the model of VD and CS does a transition to a one-cell solution occur.

Model six is a 106-coefficient spectral model which retains all modes with $m \leqslant 10$, $p \leqslant 10$, $n \leqslant 5$ and $q \leqslant 5$ as well as the $(0, 6)$ temperature mode. This model allows both the fundamental one-cell and two-cell solutions to be represented by at least the Veronis $K = 6$ triangle of modes. The steady two-cell solution has $\mathrm{Nu} = 3.201$ and the amplitude of the $(2, 1)$ temperature mode is $\Theta = 34.20$, a result also given by the Veronis $K = 8$ triangle. Apparently model six is, for all practical purposes, the "actual solution".

Model seven is a finite-difference model with 64 horizontal and 15 internal vertical grid points of uniform spacing. The model uses a second-order leapfrog scheme with Dufort-Frankel time differencing and Arakawa Jacobians (e.g., see Moore and Weiss (1973)). The steady two-cell solution has $\mathrm{Nu} = 3.186$ and the Fourier-analyzed amplitude of the $(2, 1)$ temperature mode is $\Theta = 34.63$, in excellent agreement with model six.

CS also compared their 7-coefficient model with a finite-difference model. The one integration presented by CS begins with a mixture of roughly equal magnitudes of a three-cell solution of critical wavenumber and a two-cell solution. The integration then evolves to a steady two-cell solution. No evidence is presented of a transition from a steady three-cell solution to a two-cell solution occurring as a result of a small two-cell perturbation.

4. Conclusion

This paper has shown that the models of CS and VD are inconsistent with the high-order models for the case of wavelength-doubling at low Prandtl number with $\alpha = 1/(2\sqrt{2})$. This paper has not shown evidence that the CS and VD models are necessarily invalid in all cell-broadening scenarios. Indeed there is theoretical justification (Kidachi, 1982) for those models if α and $R - R_c$ are sufficiently small so that the amplitudes of the competing modes are sufficiently small. So the models could be valid for a transition from $l + 1$ cells of critical wavenumber to l cells if l is sufficiently large. Whether the models have any usefulness when α is small and R is large enough to allow for a broad spectrum (meaning factor of two or greater) of unstable modes remains unconfirmed.

As in Curry et al. (1984), Yoden (1985), Tung and Rosenthal (1985) and Cehelsky and Tung (1987), this paper points out yet another example of a low-order model that does not fully live up to its promise of being accurate outside the limits where it is asymptotically justified. Hopefully, this short study offers more than just a disappointment to those who wish to develop and use low-order models as a mathematical tool; a simple wavenumber-doubling bifurcation is shown to be faithfully represented by an 11-coefficient low-order model. However, I will avoid formulating a general rule about under what circumstances this model will remain faithful to the behavior of the high-order system.

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