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A new type of double-diffusive instability?

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ABSTRACT

A new type of double-diffusive instability of the oscillatory type is predicted for a statically stable fluid layer with two components which have practically the same diffusivity (for example, two suitable types of salts in a molecular case, heat and salt in a case where mixing is due to small-scale turbulence). It is necessary that one component has the concentration value and the other the flux value fixed at the boundaries. A simple example is worked out in detail. The phenomenon should appear in a properly arranged laboratory experiment, and possibly in the oceans, when warmer, saltier water underlies a surface layer of colder, fresher water, and sufficient vertical turbulent mixing is present.

1. Background, and physical arguments for the instability

Known double-diffusive instabilities (see Turner (1973) for a review) hang critically on the difference of diffusivities for the two components (such as heat and salt, sugar and salt, etc.). In a case where the diffusivities of the two components are nearly equal, and the stratification is statically stable, the classical theory gives stability. This result is anticipated for a Boussinesq-type fluid, since the two components having the same diffusivity can be combined into a single-density variable, making the problem formally look the same as the classical problem of thermal convection (Benard convection). In this case, no perturbations can survive when the fluid is statically stable (heated from above).

In a double-diffusive experiment, one can fix either the concentration values or the flux values at the boundaries, and the results differ little (there is a modest change in the critical Rayleigh numbers, etc.). However, it is also possible to introduce "mixed boundary conditions", i.e., that the concentration value is fixed for one component, and the flux value for the other component, at the boundaries. This situation has not, to this author's knowledge, been studied, although it is a far from trivial extension: new instabilities and double-diffusive phenomena seem now possible.

Consider a layer of water, and choose as the stabilizing component salt, the destabilizing component heat. We will assume that these components have the same diffusivity, which is not realistic in a molecular case, but should apply if we have turbulent small-scale mixing. In a proper molecular-type experiment representing this regime, one would use two substances, for example, two salts, with nearly identical diffusivities.

The situation we consider is one of warmer and saltier water below fresher and colder water, with the salinity effect dominating, to give overall static stability. The two temperatures and the salinity flux are fixed at the boundaries. The same salinity flux must, of course, be applied at both boundaries; further, the mean salinity (or the initial salinity if we look at the problem in a time-developing situation) must be fixed to obtain uniqueness. Starting from the diffusive steady state (Fig. 1a), we now make a displacement of a parcel of water from the interior toward the upper boundary. The diffusion will cause the parcel to become colder and fresher, but this does not occur in the proportions determined by the basic diffusive fields: the cooling will be, relatively speaking, more

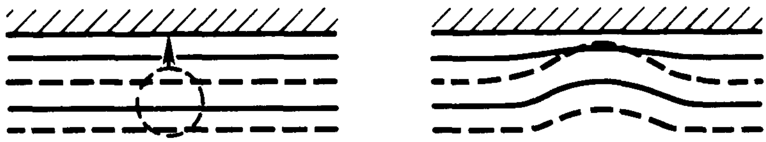

Fig. 1. Schematic picture of isotherms (full lines) and isohalines (dashed lines) in the predicted double-diffusive oscillation, when a parcel of water is moved toward the upper boundary. In the present case, it is assumed that the diffusivities for heat and salt (which can represent two other components) are the same. If a parcel of water (a) is displaced toward the boundary, the isotherms and isohalines are changed differently (b). As the parcel approaches the upper cold boundary, the (negative) temperature gradient gets larger next to the boundary, and extra heat is lost, while the flux of salinity is unchanged at the boundary. This makes the parcel heavy, with extra potential energy as it falls back.

effective (Fig. 1b). This can be explained by restriction of the isotherms near the boundary: they cannot deviate much from the horizontal, when the boundary temperature is fixed and constant, while the isohalines can deform, and intersect the boundary. We can also express it in the following way: as the warmer parcel is placed close to the cold boundary, we will locally increase the (negative) temperature gradient, causing added heat loss through the upper boundary. There is no similar added flux for the salinity, for its value is fixed at the boundary; thus, the parcel gets cooled more effectively than it is made fresher, by the displacement to this new surroundings.

The parcel will become denser than would occur in a similar experiment were the boundary specifications for heat and salt are the same, and the excess density provides potential energy when the parcel falls back. If the effect can be made strong enough, and the phase-relations are right (we must have the maximum density appear after the parcel has arrived at its highest point, at least in harmonic oscillations; such a delay may come from the diffusive process), growing oscillations may be seen.

To test this idea, a stability analysis was carried out for one specific simple example, as described in Section 3. The viscous dissipation is neglected, which is a drastic simplification, but since the diffusive damping is retained, which alone leads to decay of all perturbations in the case with equal-type boundary conditions, we still have a useful test case for the proposed new instability. Moreover, the theoretical result is structurally stable in the sense that the introduction of a small viscosity only gives a small shift in the neutral curve (in the case of free boundaries). As will be seen, the example worked out actually produces a marginally stable solution, which is of the oscillatory type. The present investigation stops there, but it may be of interest to extend the study to find complete neutral curves as functions of Rayleigh and Prandtl numbers, values of the growth rates, etc.

It should be mentioned in this connection that mixed boundary conditions already have been found to produce new phenomena in thermohaline ocean circulation models. Mixed conditions of the type which makes the temperature fixed (or at least controlled by a strong Rayleigh condition) and the salinity flux fixed (or at least controlled by a weak Rayleigh condition) work in the right direction in describing what goes on in the oceans: the surface temperature is rigidly controlled by the atmosphere, since a thermal feedback exists (back radiation, etc.). For the salt, there is no feedback (surface salinity does not affect the flux), and the flux condition is thus more appropriate. The "Rayleigh condition" commonly used in ocean modeling is of the form $Q = k(T^{a} - T)$ for the surface heat flux and $Q_s = k(S^A - S)$ for the surface salinity flux, where T^A, S^A are given forcing values. If $k \to \infty$, we have the limiting case $T = T^A$; if $k \to 0$, $T^A \to \infty$, with kT^A finite, we have the limiting case of prescribed Q. To be realistic, we must at least use different values k_T and k_S for temperature and salinity, with $k_T \gg k_S$, giving what we call a "strong" Rayleigh condition for the heat and a "weak" Rayleigh condition for the salinity.

The phenomena mentioned are instabilities and transitions from thermohaline cells into new forms, in situations where the initial cells normally represent stable solutions. For example, the classical double cell in an equatorially symmetric model ocean, with water sinking near the poles and rising near the equator, can become unstable under mixed boundary conditions and transform to a single pole-to-pole cell. The reader is referred to papers by Stommel (1961), Rooth (1982), Bryan (1986), Walin (1985) and Marotzke et al. (1988).

2. Perturbation equations, and some integral relations

We consider, as mentioned in Section 1, a situation where a water layer is stratified by salt in a stabilizing and heat in a destabilizing way, with overall static stability, and where the two diffusivities take on the same value. In the basic diffusive state, the temperature and salinity vary linearly, and we assume values at the lower boundary ($z=-D/2$): $T_0+\Delta T$ and $S_0+\Delta S$; with ΔT, ΔS positive; at the upper boundary ($z=+D/2$), the values are T_0 and S_0, respectively. If we introduce small perturbations from this state, we have the following perturbation equations, in the standard Boussinesq form:

$$\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} = -\nabla p' + \mu \nabla^2 \boldsymbol{v}' - \hat{\boldsymbol{z}} g \rho', \tag{1}$$

$$\rho' = -\rho_0 \alpha T' + \rho_0 \beta S', \quad \nabla \cdot \boldsymbol{v}' = 0, \tag{2, 3}$$

$$\frac{\partial T'}{\partial t} = \frac{\Delta T}{D} w' + \kappa \nabla^2 T', \tag{4}$$

$$\frac{\partial S'}{\partial t} = \frac{\Delta S}{D} w' + \kappa \nabla^2 S'. \tag{5}$$

We have the dynamic boundary conditions, in a free surface case,

$$w' = \frac{\partial u'}{\partial z} = \frac{\partial v'}{\partial z} = 0, \quad \text{at } z = \tfrac{1}{2} D. \tag{6}$$

For the temperature and salinity, we assume the mixed boundary conditions

$$T' = 0, \quad \frac{\partial S'}{\partial z} = 0, \quad \text{at} \quad z = \pm D/2. \tag{7}$$

If we multiply eq. (1) by $\boldsymbol{v}'$, multiply (4) by T', and (5) by S', and then average in space (horizontally and vertically), as well as in time, for a statistically steady case, we derive the relations

$$\overline{g w' \rho'} = -\overline{\mu |\nabla \boldsymbol{v}'|^2}, \tag{8}$$

$$\frac{\Delta T}{D} \overline{w' T'} = \overline{\kappa |\nabla T'|^2}, \tag{9}$$

$$\frac{\Delta S}{D} \overline{w' S'} = \overline{\kappa |\nabla S'|^2}. \tag{10}$$

In the derivation, use has been made of eq. (2), and the boundary conditions. These conditions remove the boundary terms which occur when we change from $\boldsymbol{v}' \nabla^2 \boldsymbol{v}'$ to $|\nabla \boldsymbol{v}'|^2$, from $T' \nabla^2 T'$ to $|\nabla T'|^2$, etc.

If we multiply eq. (4) by $-\rho_0 \alpha$ and eq. (5) by $\rho_0 \beta$, and add, we have

$$\frac{\partial \rho'}{\partial t} = \frac{\Delta \rho}{D} w' + \kappa \nabla^2 \rho', \tag{11}$$

where $\Delta \rho = -\rho_0 \alpha \Delta T + g_0 \beta \Delta S$ is the top-to-bottom density change. Multiplying by ρ', and averaging as described above gives

$$\frac{\Delta \rho}{D} \overline{w' \rho'} = \overline{\kappa |\nabla \rho'|^2} - \frac{1}{D} \left\{ \overline{\left(\rho' \frac{\partial \rho'}{\partial z} \right)}^{\mathrm{h}}_{z = D/2} - \overline{\left(\rho' \frac{\partial \rho'}{\partial z} \right)}^{\mathrm{h}}_{z = -D/2} \right\}, \tag{12}$$

where h denotes a horizontal-time average. The boundary terms which appear on the right-hand side of eq. (12) vanish if we set either $T' = S' = 0$, or $\partial T'/\partial z = \partial S'/\partial z = 0$ at the boundaries, and no perturbations can survive: eqs. (8) and (12) give opposite signs for $\overline{w' \rho'}$. With the mixed boundary conditions that we want to consider, the boundary terms remain, and may act as a source of potential energy, reversing the sign of $\overline{w' \rho'}$ to a negative one.

In the example worked out in Section 3, viscosity is neglected, $\mu = 0$. Accordingly, we require $\overline{w' \rho'} = 0$. Our task is to find an example of a solution under the boundary condition (7), where the boundary terms can balance $\overline{\kappa |\nabla \rho'|^2}$, and thus make $\overline{w' \rho'} = 0$.

3. The stability problem, and an example of a marginally stable solution

We non-dimensionalize the perturbation eqs. (1–5) with $\mu = 0$, by choosing D as unit of length and D^2/κ as unit of time (κ/D as unit of velocity), ΔT and ΔS as units for T and S, respectively. We then introduce a standard wave-form, with all dependent variables proportional to

$$\exp[i(k_x \cdot x + k_y \cdot y) + rt],$$

where r is a complex growth rate. After elimination of p', u', v', and expressing ρ' in terms of T' and S', we have three equations for the amplitudes of w', T', and S' (we use the notation w, T, S, for simplicity, for these dependent functions):

$$r\left(\frac{d^2}{dz^2} - a^2\right) w = -Ra^2 T + R^* a^2 S, \tag{13}$$

$$r\begin{Bmatrix} T \\ S \end{Bmatrix} = w + \left(\frac{d^2}{dz^2} - a^2\right)\begin{Bmatrix} T \\ S \end{Bmatrix} \tag{14, 15}$$

where $a = D \cdot \sqrt{k_x^2 + k_y^2}$ is a non-dimensional wave number and

$$R = \frac{g\alpha \Delta T D^3}{\kappa^2}, \quad R^* = \frac{g\beta \Delta S D^3}{\kappa^2}$$

are two Rayleigh numbers (with κ^2 replacing $\nu\kappa$). The boundary conditions are

$$w = 0, \quad T = 0, \quad \frac{dS}{dz} = 0 \quad \text{at} \quad z = \pm 1/2. \tag{16a, b, c}$$

The problem has a degenerate form, since T and S satisfy the same form of equations, and the discussion is better carried out in terms of two density-type variables:

$$\begin{aligned} \sigma &= -RT + R^* S, \\ \mu &= -T + S. \end{aligned} \tag{17}$$

The transformed equations and boundary conditions are

$$r\left(\frac{d^2}{dz^2} - a^2\right) w = \sigma a^2, \tag{18}$$

$$r\sigma = (R^* - R) w + \left(\frac{d^2}{dz^2} - a^2\right)\sigma, \tag{19}$$

$$r\mu = \left(\frac{d^2}{dz^2} - a^2\right)\mu, \tag{20}$$

$$w = 0, \quad R^*\mu - \sigma = 0, \quad \frac{d}{dz}(R\mu - \sigma) = 0 \quad \text{at} \quad z = \pm 1/2. \tag{21a, b, c}$$

Note that σ represents the "ordinary density", which gives the buoyancy effects, and which is advected and diffused. The "second density" μ, on the other hand, is only diffused, and only activated by the mixed-boundary conditions (21b, c), which couple it to σ.

The above equations produce two characteristic equations determining the vertical modes of the form $\varepsilon^{\lambda_i z}$:

$$r - \lambda^2 + a^2 = 0, \tag{22}$$

$$r(\lambda^2 - a^2)^2 - r^2(\lambda^2 - a^2) + a^2(R^* - R) = 0, \tag{23}$$

with the first equation associated with the variable μ. We have 6 roots, $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_3$, and use these to construct an even solution (with forms $\cosh \lambda z$). Such a solution should exist, considering the symmetry in z of equations and boundary conditions, and it should represent a more unstable solution than the odd one, which must have more zeros, and thus a smaller z-scale (this assumption is verified by the analysis of the solutions).

Setting

$$w = C_2 \cosh \lambda_2 z + C_3 \cosh \lambda_3 z, \tag{24}$$

$$\sigma = \frac{r}{a^2}\{(\lambda_2^2 - a^2) C_2 \cosh \lambda_2 z + (\lambda_3^2 - a^2) C_3 \cosh \lambda_3 z\}, \tag{25}$$

$$\mu = C_1 \cosh \lambda_1 z, \tag{26}$$

and applying the boundary conditions (21), we obtain a characteristic equation (the condition that the determinant of the linear equations for C_i vanish), which determines the complex growth rate r (appearing in the λ_i:s). To find expected neutral, oscillatory solutions, we set $r = i\omega$, and then solve a complex equation:

$$\begin{aligned} F(\omega) = \frac{R^*}{R} \cosh \frac{\lambda_1}{2} &\left\{\lambda_3(\lambda_3^2 - a^2) \cosh \frac{\lambda_2}{2} \sinh \frac{\lambda_3}{2} - \lambda_2(\lambda_2^2 - a^2) \cosh \frac{\lambda_3}{2} \sinh \frac{\lambda_2}{2}\right\} \\ &- \lambda_1(\lambda_3^2 - \lambda_2^2) \sinh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} \cosh \frac{\lambda_3}{2} \\ = 0. \end{aligned} \tag{27}$$

The equation also involves the parameters R^*, R, and a. We fix the parameter a at a value $\pi/\sqrt{2}$, which is the most unstable wave-number in the classical free surface Benard problem. Since friction is neglected, we rather expect that cells with larger horizontal scale are the most unstable. However, we are satisfied here to give one example, and the chosen cell will do.

In this example, eq. (27) was solved by "shooting", for one given R^* value, which was assumed large enough to produce the instability ($R^* a^2 = 1000$, or $R^* = 202.6$). For one value of R essentially smaller than R^* ($Ra^2 = 500$, or $R = 101.3$) and for another value of R close to R^* ($Ra^2 = 900$, or $R = 182.4$), the curve $F = F(\omega)$ was calculated in the complex plane, as a function of ω (see Fig. 2). These two curves straddle the origin, and one expects a neutral solution to lie in between. Changing R successively, it was possible to find a curve ($Ra^2 = 672.9$, or $R = 136.4$) which passed through the origin, and thus represented a neutral solution, at this point. The value of the frequency was $\omega = 12.37$ which may be compared with the buoyancy frequency, which, in the non-dimensional time unit, is $\sqrt{(R^* - R)} = 8.14$. These calculations were made by use of a calculator (HP 41C).

The profiles $w(z)$, $T(z)$, and $S(z)$ were finally calculated for some phases during one period of the oscillation; the results for four such phases are shown in Fig. 3. It is seen that the boundary terms appearing in the right-hand side of eq. (12) take on a negative sign, allowing them to balance the thermohaline dissipation term $\overline{\kappa|\nabla\rho'|^2}$. At the upper boundary, S' and $\partial T'/\partial z$ take on opposite signs; at the lower boundary, they take on the same sign, which transforms into a positive product $\rho' \partial\rho/\partial z$ at the upper boundary, and a negative corresponding product at the lower boundary.

In this energy budget, there is no buoyancy work. If we bring in viscosity, we must have a

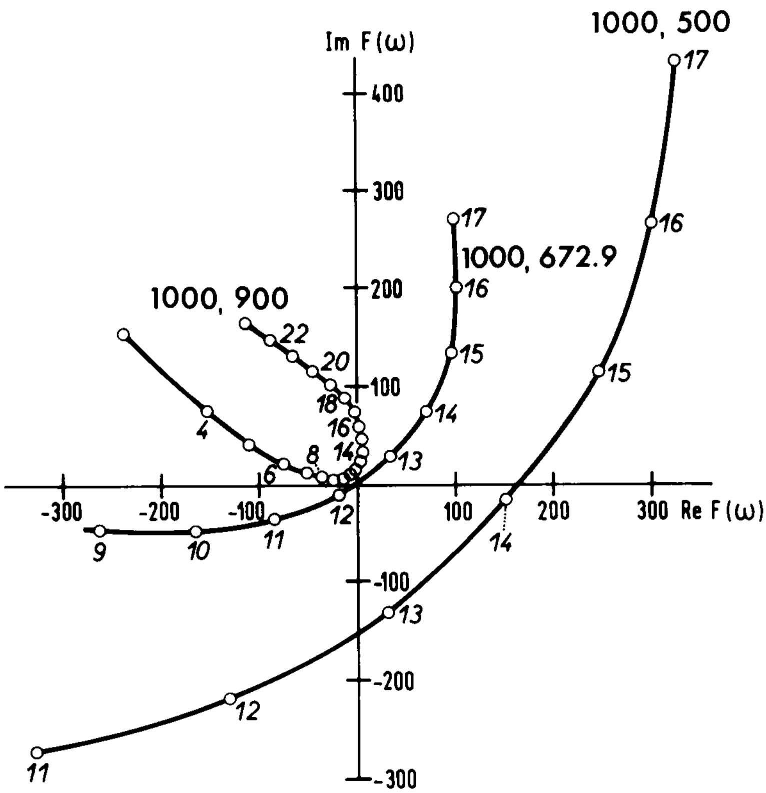

Fig. 2. The complex function F, as function of ω for two selected pairs of Rayleigh numbers R and R^* ($R^* a^2 = 1000$, $Ra^2 = 500$ and $R^* a^2 = 1000$, $Ra^2 = 900$, where $a = \pi/\sqrt{2}$), and a neutral solution ($R^* a^2 = 1000$, $Ra^2 = 672.9$) which gives a solution Re F = Im $F = 0$. The calculation was done by "shooting", with successively changed values of Ra^2.

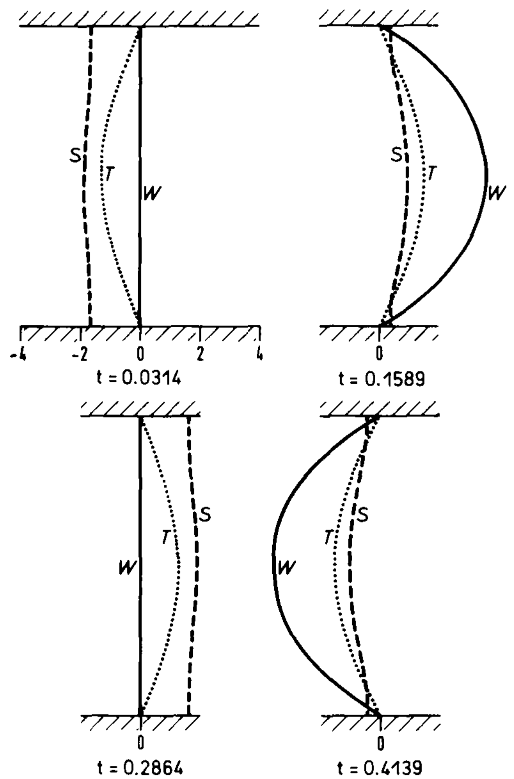

Fig. 3. The vertical structure for w, T, and S for the calculated neutral oscillation ($R^* a^2 = 1000$, $Ra^2 = 672.9$), in four phases of an oscillation. The period of the oscillation is 12.37, the buoyancy frequency is $(R^* - R)^{1/2} = 8.14$, in the non-dimensional time unit.

negative value for $\overline{w'\rho'}$, and the boundary terms must provide more energy. A lowering of the critical value for $R^* - R$ presumably would work, since this will lower the diffusive dissipation associated with the basic density field. However, if the viscous effects are weak, the solution should not be critically changed (this is only true if we apply free surface conditions; with conditions of a solid boundary, there is always a critical change when viscosity is introduced).

4. Can the instability be seen experimentally?

It should be possible to set up an experiment using two components such as two salts, which only differ slightly. However, it will then be necessary to have the components differ in some other respect, which allows the separation in two different boundary conditions; for example, use some electrochemical effect to force the flux. If the two components are practically identical, the separation of the two boundary conditions will become impossible (for a single component, we cannot obviously control a concentration value and a flux at the same time).

If we use two components with large differences in molecular diffusivities, such as heat and salt, it should also be possible to generate the instability, but one may in such cases see disturbing effects from the standard double-diffusive "layer instability", which also starts out as an oscillation. A turbulent heat–salt experiment, with small-scale turbulent mixing generated by, say, a vibrating grid, seems feasible, but in this case, we do not have a constant diffusivity, but one which varies with stratification, etc. This regime has not been looked at theoretically, and it is not known whether the instability can appear, although the general physical arguments given in Section 1 makes it likely.

Finally, it is even more uncertain whether the instability can appear in real oceans. In this case, we have a deep layer, and the phenomenon should be confined to a surface layer with a thickness of order of the diffusive depth $(\kappa/\omega)^{1/2}$, where ω is the frequency of the oscillation. We must of course have a situation where warm, salty water underlies colder, fresher surface water, and there must be sufficient turbulent mixing. If the instability occurs, it would appear either as a standing oscillation or as a propagating internal wave. An oscillation of such a type has possibly been traced in a numerical ocean experiment with mixed boundary conditions (Marotzke et al., 1988), but this needs to be verified in special experiments which can resolve the phenomenon.

5. Acknowledgements

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REFERENCES

Bryan, F. 1986. High-latitude salinity effects and interhemispheric thermohaline circulations. *Nature 323*, 301–304.

Marotzke, J., Welander, P. and Willebrand, J. 1987. Instability and multiple steady states in a meridional-plane model of the thermohaline circulation. *Tellus 40A*, 162–172.

Rooth, C. 1982. Hydrology and ocean circulation. *Progr. Oceanogr. 11*, 131–149.

Stommel, H. 1961. Thermohaline convection with two stable regimes of flow. *Tellus 13*, 224–230.

Turner, J. S. 1973. *Buoyancy effects in fluids* (ch. 8). Cambridge: University Press, 308 pp.

Walin, G. 1985. The thermohaline circulation and the control of ice ages. *Palaeogeogr., Palaeoc. 50*, 323–332.