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Deep circulations under simple classes of stratification

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ABSTRACT

Deep circulations, where the motion field is vertically aligned over one or more scale heights, are considered within the framework of the Primitive Equations in three dimensions. Important in both tropospheric and stratospheric applications, such motions are admitted by special classes of stratification, where two families of thermodynamic surfaces are not completely independent, as is true under generally baroclinic conditions. Under barotropic and equivalent barotropic stratifications, the number of degrees of freedom are reduced by one, so that the attending circulations are governed by a two-dimensional reduction of the full primitive equations.

The reduced two-dimensional equations governing these motions are derived from the three-dimensional primitive equations in spherical geometry. In so doing, a mapping is established between the full primitive equations and a reduced two-dimensional system, under these special classes of stratification. Very little approximation is required in the derivation of this system, particularly within the framework of isentropic coordinates. Although not strictly equal to the shallow water equations, the reduced set is of the same general form and therefore is amenable to the same methods of solution. Moreover, while the shallow water equations serve as a heuristic analogue of atmospheric behavior, the equivalent barotropic system derived here explicitly represents atmospheric variables. The equivalent depth, an arbitrary parameter in the shallow water equations, emerges implicitly in the derivation of this system, which obeys a conservation principle analogous to Ertel's theorem for three-dimensional flow.

Owing to its correspondence to the full primitive equations, the equivalent barotropic system provides a physical basis for carrying out two-dimensional integrations (e.g., detailed transport calculations of very high resolution) to investigate deep atmospheric motions. The reduced system also provides a natural framework for investigating the behavior of total column abundances of atmospheric constituents, e.g., total ozone, measured by satellite. Although the immediate application of this system is to study barotropic circulations, because temperature variations along a lower bounding material surface are fully accounted for, the reduced system may be attractive for investigating a wider class of motions, e.g., baroclinic disturbances where the heat flux is confined to the surface.

1. Introduction

A great deal of attention has been drawn recently to deep motions of the atmosphere, in particular to circulations which are aligned vertically over one or more scale heights. For example, tropical–extratropical teleconnection patterns (Horel and Wallace, 1981) exhibit little or no tilt in the vertical, at least over the depth of the troposphere. Two-dimensional calculations, capitalizing on the barotropic structure of these disturbances, have been reasonably successful in reproducing observed behavior (Hoskins and Karoly, 1981). Calculations in three-dimensions (Garcia and Salby, 1987) produce similar features, but reveal the barotropic structure as prevailing well above the tropopause.

In the stratosphere, much of the interest in barotropic behavior surrounds transport considerations. There, a good deal of attention has been stimulated by McIntyre and Palmer's (1984)

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identification of global-scale disturbances in terms of the breaking of planetary waves, and from supporting analyses of tracer transport (Leovy et al., 1985) revealing deep material motions accompanying these circulations. The now widely recognized signature of this behavior in the winter stratosphere is a displacement of the vortex off the pole and a deformation of the circulation into a comma-like shape, its axis being drawn anticyclonically about the Aleutian Ridge which drives the flow out of zonal symmetry.

Tracer movements such as those of ozone, oberved directly by satellite (Leovy et al., 1985), and potential vorticity, a dynamical tracer derived from satellite retrieved temperatures (Clough et al., 1985), reveal material being drawn away from the body of the vortex towards tropical latitudes. In the vertical, these motions occur almost in unison (i.e., with little phase tilt) over a layer of several scale heights. Horizontally, on isentropic surfaces, a tongue of polar (high-vorticity, ozone-poor) air is exchanged with a complementary tongue of tropical (low-vorticity, ozone-rich) air, resulting in a net entrainment of tropical and extratropical fluid. This exchange of material between high and low latitudes reflects a horizontal mixing or "stirring" of the layer in which this type of circulation prevails.

Irreversible fluid motions accompanying this behavior have been suggested by McIntyre and Palmer as acting to destroy the vorticity of the circumpolar flow and thereby the vortex itself. Whether the entrainment represented by these global-scale tongues is strictly a signature of the breaking of planetary waves or applies to a wider class of motions (Rood, 1985; McIntyre and Palmer, 1986) is but one of a host of important issues surrounding the phenomenon, though the differences in this case may be largely semantic. Quasi-linear descriptions, not including strong nonlinearity of the form usually associated with wavebreaking and mixing, have successfully reproduced the salient aspects of the behavior thus far resolved in satellite measurements (Salby and Garcia, 1987). Nonetheless, the fundamental importance of such motions to maintaining the global-scale circulation and to transport considerations is widely accepted. That these motions occur, to a large degree, coherently over a deep layer of the stratosphere is also well recognized.

Many of the important questions surrounding the aforementioned behavior have to do with the creation of small horizontal scales in distributions of potential vorticity and chemical trace species. Implicitly, the notion of wavebreaking carries with it the generation of many spatial scales, whereby variance of conserved quantities (e.g., long-lived species) "cascades" to higher wavenumbers. Kinematically, this suggests a scattering of fluid elements about their mean position in the latitude–height plane, i.e., parcel dispersion. Even quasi-linear descriptions (Salby and Garcia, 1987) suggest the generation of small scales in distributions of passive scalars through the shearing of fluid elements, material tongues eventually collapsing into narrow filaments.

How efficiently such scales are generated, how their development affects the large-scale circulation, and how such anomalies are eventually destroyed are key issues. In order that these and other questions be addressed properly, it will be necessary to perform transport calculations of very high resolution, so that numerical truncation does not mask genuine physical processes relating to parcel dispersion. Only then will it be possible to faithfully address questions surrounding the cascade of variance to small scales and its role in the large-scale circulation and in homogenizing trace species distributions.

It would be desirable to conduct such calculations in three dimensions, as vertical shear may also play an important role in cascading variance to small scales. However, current computational capabilities and even projections for the near future do not admit full three-dimensional calculations with the resolution necessary to properly address many of the important issues. An attractive alternative, capitalizing on the deep nature of these motions, is to focus on the horizontal behavior by conducting two-dimensional integrations.

Barotropic integrations have served as a paradigm of atmospheric behavior for some time, as they have in more recent applications (Juckes and McIntyre, 1987). The shallow water equations are used frequently as a surrogate for the full primitive equations when the motion is approximately two-dimensional. Although in widespread use, their validity in this context is more heuristic than it is formal. While a correspondence does exist between motions in a barotropic fluid (where a law of barotropy is used in place of other constitutive relations such as ideal gas behavior) and shallow water motion, a well defined relationship between circulations in the atmosphere and shallow water motion has not been established. In particular, how variables describing atmospheric motions carry over to those in shallow water integrations is largely qualitative. Another complication in identifying such calculations with atmospheric behavior is the choice and significance of the equivalent depth, which must be prescribed in the barotropic system. Although intuitively representing the mean height of the fluid column, the equivalent depth is really an arbitrary parameter of the shallow water equations, whose specification is more or less ad hoc. Unfortunately, the nature of the shallow water circulations is known to depend critically on its value.

An alternative approach, pioneered by Charney (1949), is to derive approximate equations governing equivalent barotropic motion, where the atmospheric flow field is vertically aligned. However, as Charney and others (e.g., Charney and Eliassen, 1949; Eliassen and Kleinschmidt, 1957) have noted, reducing the primitive equations to a two-dimensional subsystem may also be ad hoc, requiring in these formulations the neglect of ageostrophic advection, nonlinearity, or static stability. The framework perhaps most amenable to this sort of development is the quasi-geostrophic system, in which a single potential vorticity equation can be derived when the pressure tendency is approximated by vertical advection alone (Holton, 1972). Although serving well in mid-latitudes, this framework becomes tenuous for global calculations when displacements in the tropics are important.

In this paper, we consider three-dimensional atmospheric motion which is deep in the sense of lining up vertically over several scale heights. For weak vertical tilt (e.g., disturbances of long vertical wavelength), the flow field is similar at all levels, horizontal motion being parallel and varying with altitude only in magnitude. Such motion is admitted by special classes of stratification where two families of thermodynamic surfaces are not completely independent, as is true under general baroclinic conditions. It may therefore be regarded as a subset of the motions possible under general three-dimensional stratification. The simplest such class is "barotropic stratification", where isopycnal surfaces coincide with isobaric surfaces. In this case, the motion is purely columnar, fluid elements initially vertical remaining so. It will be seen that this stratification does not admit divergent motion for atmospheric circulations. A wider class of motions is admitted by "equivalent barotropic stratification", where isopycnal and isobaric surfaces, though not coinciding, share a common horizontal structure, so that two families of thermodynamic surfaces (e.g., pressure and specific volume) are similar to one another and line up vertically. As a result, the vertical shear and hence the horizontal velocity are parallel at all levels. Both classes of stratification admit motions whose vertical structures are in some sense prescribed, in accord with the stratification. (It will be seen that the vertical structure of the flow, in fact, determines the stratification.) Consequently, the flow is governed essentially by a two-dimensional reduction of the complete equations, amenable to very accurate integration.

Our focus here is to derive rigorously the reduced two-dimensional equations, governing these restricted classes of motion, from the three-dimensional primitive equations in spherical geometry. In so doing, a mapping is established between the full primitive equations and general shallow water behavior, for these special classes of stratification. Although not strictly equal to the shallow water equations, the reduced two-dimensional system is of the same general form and is therefore amenable to the same methods of solution. Moreover, while the shallow water equations serve as a heuristic analogue of atmospheric behavior, the equivalent barotropic system derived here explicitly represents atmospheric motions. The correspondence between variables describing deep atmospheric motion and those of shallow water behavior is established here with far less approximation than has been resorted to previously, thus constituting a more rigorous and well-defined relationship between the two systems. Furthermore, the equivalent depth, an arbitrary parameter in the shallow water equations, emerges implicitly in the derivation of the reduced two-dimensional set. In addition to facilitating detailed transport calculations, the equivalent barotropic system provides a natural framework for investigating the behavior of total column abundances of atmospheric constituents, e.g., total ozone, monitored by satellite.

We consider an infinite atmospheric layer, bounded below by a material surface. This may be thought to represent a deep but finite layer in which the stratification falls into one of the classes considered and for which influence from overlying layers is negligible. The lower bounding material surface may be in contact with orography, in which case its elevation is fixed and the atmospheric layer extends down to the earth's surface, or it may simply be a fluid interface internal to the atmosphere (e.g., near the tropopause for stratospheric applications), in which case its elevation is variable as in contact with an elastic membrane. In Section 2, the simplest case of barotropic stratification is considered. A conservation principle, analogous to Ertel's theorem, is derived and properties of the restricted class of solutions are examined. In Section 3, the wider class of equivalent barotropic stratification is considered. Reduced two-dimensional equations governing the motion are derived in both isobaric and isentropic coordinates, in the presence of friction and diabatic effects. A conservation principle for each of the cases is derived. Conclusions are presented in Section 4.

2. Barotropic stratification

Under barotropic stratification, the usual two thermodynamic degrees of freedom for an ideal gas are reduced to only one. There is only one independent thermodynamic variable, all others being determined from this one through constitutive relations. Choosing a thermodynamic variable as the vertical coordinate therefore simplifies the problem greatly.

For isobaric coordinates, the following expression of barotropy is implied

$$\alpha = \alpha(p) \tag{1}$$

relating pressure and specific volume. Two formalisms are used in connection with this stratification. When a "barotropic fluid" is invoked, the law of barotropy (1) is used in place of other constitutive relations. On the other hand, for a gas, the pressure in combination with (1) and the ideal gas law completely specifies the thermodynamic state, viz. all other thermodynamic variables. As a result, all thermodynamic surfaces coincide, and therefore isobaric coordinates are equivalent to isentropic coordinates.

We consider an atmospheric layer, unbounded above and bounded below by a material surface (Fig. 1), whose elevation at any instant is given by $z_0(\boldsymbol{x}, t)$,

$$\boldsymbol{x} = (x, y) \tag{2}$$

representing the horizontal coordinate. The lower boundary may be regarded as either an orographic surface, in which case it is fixed in time, or a material interface such as the tropopause which may be displaced as through a flexible membrane.

Letting

$$\boldsymbol{v} = (u, v) \tag{3}$$

denote the horizontal velocity leads to the following form of the primitive equations in pressure coordinates

$$\frac{d\boldsymbol{v}}{dt} + \omega\frac{\partial \boldsymbol{v}}{\partial p} + f\boldsymbol{k} \times \boldsymbol{v} = -\nabla_p \Phi, \tag{4a}$$

$$\frac{\partial \Phi}{\partial p} = -\alpha, \tag{4b}$$

$$\frac{\partial \omega}{\partial p} + \nabla_p \cdot \boldsymbol{v} = 0 \tag{4c}$$

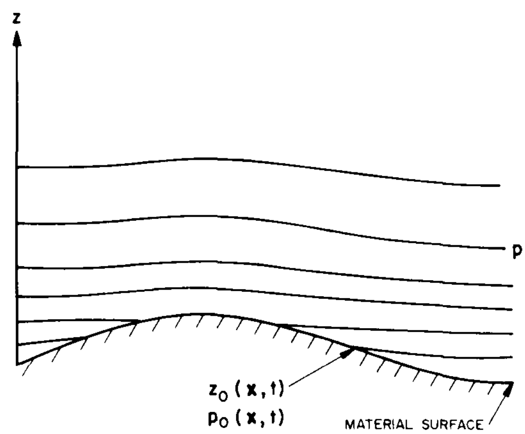

Fig. 1. Infinite layer, bounded below by a material surface of elevation $z_0(\boldsymbol{x}, t)$, for a barotropic fluid. The surface pressure $p_0(\boldsymbol{x}, t)$ is in general variable due to horizontal convergence.

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_p \tag{4d}$$

denotes the horizontal component of the advection operator on isobaric surfaces.

The geopotential field in the interior may be evaluated by integrating the hydrostatic equation (4b) vertically between the lower bounding material surface and some arbitrary isobaric surface p. In general, the pressure on the lower boundary $p_o(\boldsymbol{x}, t)$, reflecting the column weight, will vary in space and time due to horizontal convergence. Consequently, under generally divergent motion, isobaric surfaces will intersect the lower boundary. The surface pressure $p_o(\boldsymbol{x}, t)$ is thus an unknown of the problem, its departure from static conditions representing the incremental column weight induced through horizontal convergence.

Integrating (4b) up to the isobaric surface p gives

$$\Phi(\boldsymbol{x}, p, t) = \Phi_o(\boldsymbol{x}, t) - \int_{p_o(\boldsymbol{x}, t)}^{p} \alpha(p)\,\mathrm{d}p, \tag{5}$$

where Φ_o reflects the geopotential of the lower boundary, and the law of barotropy (1) has been introduced. Then with Leibnitz' Rule, the horizontal gradient on pressure surfaces is

$$\nabla_p\Phi = \nabla\Phi_o - \left\{\int_{p_o(\boldsymbol{x}, t)}^{p} \nabla_p\alpha(p)\,\mathrm{d}p + \alpha(p)\nabla_p p - \alpha(p_o)\nabla p_o\right\},$$

which reduces to

$$\nabla_p\Phi = \nabla\Phi_o + \alpha_o\nabla p_o. \tag{6}$$

There are two contributions: one due to elevation of the lower boundary and the other to pressure variations along this bounding surface. In particular, neither term depends on p; the motion is forced by displacements of the lower boundary. With the exception of inertial motion (where the pressure force is not involved), this together with (4a) implies that $\boldsymbol{v}$ is not a function of p, i.e., the motion is *columnar*.

Then the vertical advection term in (4a) vanishes and conservation of momentum reduces to

$$\frac{d\boldsymbol{v}}{dt} + f\boldsymbol{k}\times\boldsymbol{v} = -(\nabla_p\Phi)_o \tag{7}$$

where

$$(\nabla_p\Phi)_o = \nabla\Phi_o + \alpha_o\nabla p_o \tag{8}$$

represents the horizontal geopotential gradient on pressure surfaces, evaluated at the lower boundary.

To close the system, we integrate the continuity equation (4c) vertically from the lower boundary to the top of the atmosphere, giving

$$\omega(\mathrm{o}) - \omega(p_o) + \int_{p_o(\boldsymbol{x}, t)}^{0} \nabla_p\cdot\boldsymbol{v}\,\mathrm{d}p = 0. \tag{9}$$

Since $v \neq v(p)$, this leads to

$$\omega_o + p_o\nabla\cdot\boldsymbol{v} = 0, \tag{10}$$

where

$$\omega_o = (\dot{p})_o = \left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)_o + w_o\left(\frac{\partial p}{\partial z}\right)_o \tag{11}$$

is the total pressure tendency evaluated at the lower boundary.

To evaluate this term, we consider the relationship between the interior pressure field and the variation along the lower boundary. By definition,

$$p_o(\boldsymbol{x}, t) = p(\boldsymbol{x}, z_o(\boldsymbol{x}, t), t), \tag{12}$$

so

$$\frac{\partial p_o}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_{z_o} + \left(\frac{\partial p}{\partial z}\right)_{z_o}\frac{\partial z_o}{\partial x} \tag{13}$$

and similar relations hold for derivatives with respect to other variables. Therefore, it follows that

$$\frac{\mathrm{d}p_o}{\mathrm{d}t} = \left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)_o + \left(\frac{\partial p}{\partial z}\right)_o\frac{\mathrm{d}z_o}{\mathrm{d}t}. \tag{14}$$

Since the lower boundary is defined as a material aggregate, fluid initially in contact with this surface remains in contact with it,

$$(\dot{z})_o = \frac{\mathrm{d}z_o}{\mathrm{d}t} = w_o, \tag{15}$$

so (14) gives finally

$$\frac{\mathrm{d}p_o}{\mathrm{d}t} = (\dot{p})_o = \omega_o, \tag{16}$$

and (10) becomes

$$\frac{\mathrm{d}p_o}{\mathrm{d}t} + p_o\nabla\cdot\boldsymbol{v} = 0. \tag{17}$$

Then the equations

$$\frac{d\boldsymbol{v}}{dt} + f\boldsymbol{k} \times \boldsymbol{v} = -g\nabla z_o - \alpha_o \nabla p_o \tag{18a}$$

$$\frac{dp_o}{dt} + p_o \nabla \cdot \boldsymbol{v} = 0 \tag{18b}$$

together with the law of barotropy (1) form a closed system in the unknowns $\boldsymbol{v}$ and p_o.*

Equations (18) are analogous to the shallow water system forced by an arbitrary lower boundary (Appendix). In particular, deflections of the lower boundary in combination with the surface pressure gradient in (18a) play the role of the free surface displacement in the shallow water momentum equation (A9a). The surface pressure in the integrated continuity equation (18b) mirrors the column height in the shallow water system (A9b). While the reduced equations (18) are of the form of the shallow water operator, it is to be noted that the equivalent depth is implicit to this system, its value dictated by the mean pressure at the lower bounding surface.

From the reduced system (18) we may derive a conservation principle analogous to Ertel's Theorem. Applying the curl to (18a) leads to

$$\frac{d\zeta}{dt}\boldsymbol{k} + \boldsymbol{v}\cdot\nabla f\boldsymbol{k} = -(\zeta + f)\boldsymbol{k}\nabla\cdot\boldsymbol{v} - \nabla\alpha_o \times \nabla p_o \tag{19}$$

where $\zeta = \nabla \times \boldsymbol{v}$ and $\boldsymbol{k}$ is the unit normal. With the law of barotropy (1)

$$\nabla\alpha_o \times \nabla p_o = \left(\frac{d\alpha}{dp}\right)_o \nabla p_o \times \nabla p_o \equiv 0, \tag{20}$$

and the baroclinic term vanishes. Then (19) may be written

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\nabla\cdot\boldsymbol{v}. \tag{21}$$

With (18b) this becomes

$$\frac{1}{(\zeta + f)}\frac{d}{dt}(\zeta + f) - \frac{1}{p_o}\frac{dp_o}{dt} = 0, \tag{22}$$

or

$$\frac{d}{dt}\left[\ln\left(\frac{\zeta + f}{p_o}\right)\right] = 0, \tag{23}$$

leading finally to the conservation principle

$$\frac{d}{dt}\left(\frac{\zeta + f}{p_o}\right) = 0 \tag{24a}$$

where the conserved quantity

$$Q = \left(\frac{\zeta + f}{p_o}\right) \tag{24b}$$

plays the role of potential vorticity. This is a direct analogue of Ertel's theorem in three dimensions and of conservation of potential vorticity in the shallow water system (A16). Vertical stretching, which is the primary means of changing absolute vorticity under barotropic stratification, is embodied in the surface pressure anomaly p_o, reflecting the incremental weight of a fluid column due to horizontal convergence.

As mentioned previously, there are two contexts in which the barotropic system may be viewed. It may be regarded as governing the motion of a barotropic fluid, in which case the law of barotropy (1) is used in place of other constitutive relationships. In this application, the system (18) and the corresponding conservation principle (24) form a complete analogue of shallow water motion. The barotropic system may also be applied to a gas, in which case the law of barotropy can only be used in a manner consistent with other relationships which hold, such as the ideal gas law. Our focus here is on applying the reduced system to deep atmospheric motions, ideally the entire atmospheric column above specified level, say the tropopause. As such, properties of the gas cannot be neglected. It is therefore the latter possibility which must now be explored.

As noted earlier, for an ideal gas under barotropic stratification, there is really only one independent thermodynamic variable, and consequently all thermodynamic surfaces coincide. In particular, isentropic surfaces are coincident with isobaric surfaces. But for adiabatic motion, the first law reduces to an expression of conservation of potential temperature. Hence, an isentropic surface is also a material surface. Since the lower boundary is defined to be a material surface, it follows that potential temperature and therefore pressure, if initially constant on this boundary (e.g., static initial conditions), must remain constant. But this implies that p_o is constant, in which case the integrated continuity equation

* Derivatives in (18) do not involve pressure, as all of the variables are height-independent.

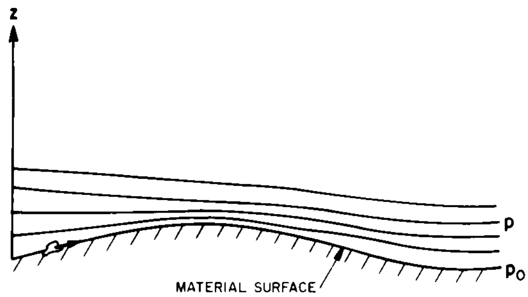

Fig. 2. As in Fig. 1, but for an atmospheric layer under barotropic stratification. The lower bounding material surface coincides with an isobaric surface, and horizontal divergence is excluded.

(18b) reduces to a statement of simple nondivergence—there is no column stretching. It follows that the motion is purely geostrophic, fully captured by a simple vorticity balance. Thus the picture for barotropic stratification, in the case of atmospheric motion, has an isobaric surface always coincident with the lower bounding material surface, as shown in Fig. 2. Consistent with the reduced class of motions possible under this stratification is the conservation principle (24), which may likewise be simplified to just an expression of conservation of absolute vorticity, having no contribution from vortex stretching.

Thus, for purposes of describing deep atmospheric motions, barotropic stratification does not afford an attractive means of mapping the three-dimensional primitive equations into a reduced two-dimensional system. The class of motions possible under this stratification is too narrow to accommodate many circulations of interest. In particular, horizontal divergence may play an important role, especially in tropical regions where material displacements are large or in regions where strong flow curvature is established. In order to accommodate divergent motion, a wider class of stratification is necessary.

3. Equivalent barotropic stratification

We consider now the situation where two thermodynamic variables, though not completely determined by one another, share a common horizontal structure (Charney, 1949). Then the two scalar fields are vertically aligned, and on any level surface isopleths of one variable are everywhere parallel to isopleths of the other. It follows from the hydrostatic relation that the pressure gradient force and thus the horizontal velocity are parallel at all levels. We consider motion under such stratification in both isobaric and isentropic coordinates.

3.1. Isobaric coordinates

The domain is again unbounded above and bounded below by a material surface whose elevation is given by $z_0(\boldsymbol{x}, t)$. However, in this case, the stratification cannot be described by a single set of thermodynamic surfaces, as was true under barotropic stratification. Instead, two families of thermodynamic surfaces are displaced relative to one another, although sharing symmetry about a common vertical axis along which they are mutually aligned, as indicated in Fig. 3. Under this configuration, isopleths, e.g., of specific volume, are parallel at all levels. Therefore, the hydrostatic relation implies that the change in horizontal pressure force across some layer points in the same direction as that acting at the base of the layer.

This situation is equivalent to the conditions that the horizontal pressure force is parallel at all levels

$$\nabla_p \Phi \parallel (\nabla_p \Phi)_0, \tag{25a}$$

where $(\nabla_p \Phi)_0$ denotes the gradient at the lower boundary, and likewise, the velocity varies with pressure only in magnitude

$$\boldsymbol{v}(\boldsymbol{x}, p, t) = A(p) \cdot \langle \boldsymbol{v} \rangle (\boldsymbol{x}, t) \tag{25b}$$

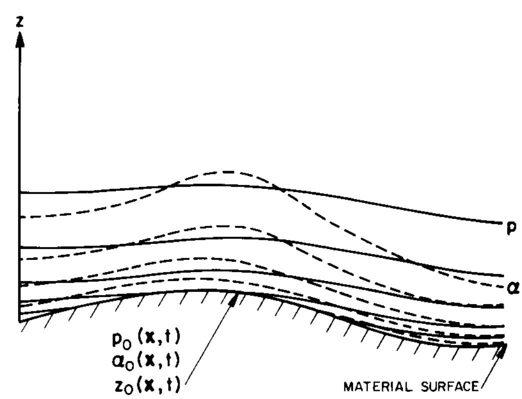

Fig. 3. Infinite atmospheric layer, bounded below by a material surface, under equivalent barotropic stratification, showing coordinate surfaces for isobaric coordinates.

(Charney, 1949). Here

$$\langle \bullet \rangle = \frac{1}{p_o} \int_0^{p_o(x,t)} (\bullet)\,dp \tag{26}$$

denotes the column average and $A(p)$ is the equivalent barotropic structure, normalized to unity,

$$\langle A \rangle = 1, \tag{27}$$

which determines the stratification. It will be seen shortly that the local departure of $A(p)$ from unity reflects the degree of baroclinicity, or equivalently the departure from barotropic stratification.

The primitive equations are as in (4). Integrating the hydrostatic equation (4b) from the base of the layer to some pressure surface p again gives (5). Then with Leibnitz' Rule, the horizontal pressure force becomes

$$\nabla_p \Phi = \nabla \Phi_o + \alpha_o \nabla p_o - \int_{p_o(x,t)}^{p} \nabla_p \alpha \, dp. \tag{28}$$

There are two contributions: a surface gradient equivalent to (8) and an altitude dependent term arising here out of the variable stacking of isopycnal surfaces. Now in order that the condition of equivalent barotropy (25a) hold, it follows from (28) that the gradient of specific volume must be parallel at all levels to the surface geopotential gradient

$$\nabla_p \alpha \parallel (\nabla_p \Phi)_o, \tag{29}$$

where $(\nabla_p \Phi)_o$ is given by (8). Equivalently, we may write

$$\nabla_p \alpha = -\frac{\partial B}{\partial p}(x, p, t)(\nabla_p \Phi)_o \tag{30a}$$

for some scalar function $B(x, p, t)$. Then integrating (30a) with respect to p, along with the condition

$$B(x, p_o(x, t), t) = 0 \tag{30b}$$

required for consistency, leads to the following form for the horizontal pressure force

$$\nabla_p \Phi = [1 + B(x, p, t)](\nabla_p \Phi)_o. \tag{31}$$

In terms of the column averaged motion $\langle v \rangle$ and the equivalent barotropic structure $A(p)$, the momentum equation (4a) becomes

$$\left(A(p)\frac{\partial}{\partial t} + A^2(p)\langle v \rangle \cdot \nabla\right)\langle v \rangle + \omega \langle v \rangle A'(p) + A(p) f k \times \langle v \rangle = -[1 + B(x, p, t)](\nabla_p \Phi)_o. \tag{32}$$

As in previous formulations (e.g., Charney, 1949; Eliassen and Kleinschmidt, 1952), the vertical advection term must be discarded, usually argued as small. Once this is done, the momentum equation simplifies to

$$\frac{d\langle v \rangle}{dt} + f k \times \langle v \rangle = -\left[\frac{1 + B(x, p, t)}{A(p)}\right](\nabla_p \Phi)_o \tag{33}$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + A(p)\langle v \rangle \cdot \nabla \tag{34}$$

and is applicable at any level. With the ideal gas law, the surface gradient appearing on the right hand side of (33) may be cast into a form analogous to the pressure force in the Shallow Water Equations (A11a), in terms of the elevation and pressure at the lower boundary,

$$(\nabla_p \Phi)_o = g(\nabla z_o + H_o \nabla \ln(p_o/p_{oo})) \tag{35}$$

where H_o denotes the surface scale eight (taken as constant) and p_{oo} a reference pressure.

The vertical structures represented in $B(x, p, t)$ and $A(p)$ are not independent. In fact, they are both reflections of the stratification, which is prescribed. For geostrophic scaling, the two may be related directly as

$$B(x, p, t) = \frac{A(p)}{A(p_o(x, t))} - 1, \tag{36}$$

in which case the horizontal pressure force is simply

$$\nabla_p \Phi = \frac{A(p)}{A_o}(\nabla_p \Phi)_o, \tag{37}$$

where

$$A_o = A(p_o(x, t)) \tag{38}$$

is the equivalent barotropic structure evaluated at the surface.

Under more general considerations, there is also only one independent vertical structure. That is, if one is specified, the other is determined, e.g., through a column integral. The exact relationship between $B(x, p, t)$ and $A(p)$ is not essential, as each is dictated by the stratification which is prescribed and arbitrary. However, as an observational fact (certainly for the motions considered here), the velocity and streamfield have ostensibly the same vertical structure. This property suggests that the large-scale motion field is not far removed from geostrophy, and therefore (36) should be reasonably accurate in relating the two structures. In any event, as will be seen shortly, the resulting two-dimensional system is of the same general form, irrespective of whether this approximation is introduced.

From (31), it can be seen that $B(\boldsymbol{x}, p, t)$ represents the departure from barotropic stratification. In particular, when $B(\boldsymbol{x}, p, t)$ is identically zero, the horizontal pressure force is independent of altitude, and (36) implies a constant value for $A(p)$ and thus columnar motion. This, of course, is just the situation considered in Section 2, where isopycnal and isobaric surfaces coincide. For $B(\boldsymbol{x}, p, t)$ not identically zero, these two families of surfaces deviate from one another, but in a manner preserving a single horizontal structure. Then, there exists a temperature variation along isobaric surfaces, but one whose gradient points always in the same direction as the surface gradient $(\nabla_p \Phi)_o$. By thermal wind, this leads to a reinforcement of the velocity $\boldsymbol{v}$ with height according to (25b) and the vertical structure $A(p)$.

Turning now to the continuity equation, applying the column average to (4c) results in

$$\omega_o + p_o \nabla \cdot \langle \boldsymbol{v} \rangle = 0, \tag{39}$$

which with (16) implies

$$\frac{d_o p_o}{dt} + p_o \nabla \cdot \langle \boldsymbol{v} \rangle = 0, \tag{40}$$

where

$$\frac{d_o}{dt} = \frac{\partial}{\partial t} + A_o \langle \boldsymbol{v} \rangle \cdot \nabla \tag{41}$$

represents horizontal advection at the lower boundary.

Then, evaluating (33) also at the surface with (36) leads to the closed system

$$\boxed{\begin{aligned} &\frac{d_o \langle \boldsymbol{v} \rangle}{dt} + f\boldsymbol{k} \times \langle \boldsymbol{v} \rangle = -\frac{g}{A_o} \nabla \left(z_o + H_o \ln\left(\frac{p_o}{p_{oo}}\right)\right) && \text{(42a)} \\ &\frac{d_o p_o}{dt} + p_o \nabla \cdot \langle \boldsymbol{v} \rangle = 0 && \text{(42b)} \end{aligned}}$$

in the unknowns $\langle \boldsymbol{v} \rangle$, representing the column integrated motion, and p_o, reflecting the column height.*

The system (42) is again an analogue of the Shallow Water Equations. As in the case of barotropic stratification, the equivalent depth is implicit in the reduced system. Furthermore, approximations requiring linear behavior or geostrophy are not essential in its derivation. Without the relationship (36), suggested by geostrophic scaling, evaluating (33) at the surface together with (30b) recovers the same result. In contrast to the case of barotropic stratification, the reduced system here admits divergent motion. Quasi-geostrophic formulations (e.g., Charney, 1949; Eliassen and Kleinschmidt, 1957; Holton, 1972) approximate the surface pressure tendency, represented by ω_o in (39), by the prevailing contribution from vertical advection alone (11), leading to a simple vorticity balance. When the shallow water equations are used to govern this motion, this approximation is unnecessary.

A conservation principle analogous to Ertel's theorem can again be determined for the system (42). Its derivation parallels that of the barotropic case (24). However, it is slightly more involved because of the need to evaluate advection terms at the surface. Applying the curl to (42a) leads to

$$\frac{d_o \langle \zeta \rangle}{dt} \boldsymbol{k} + \langle \boldsymbol{v} \rangle \cdot \nabla f \boldsymbol{k} = -(A_o \langle \zeta \rangle + f) \boldsymbol{k} \nabla \cdot \langle \boldsymbol{v} \rangle, \tag{43}$$

$\langle \zeta \rangle = \nabla \times \langle \boldsymbol{v} \rangle$ representing the column averaged vorticity. Then writing (43) as

$$\begin{aligned} &\frac{d_o \langle \zeta \rangle}{dt} + A_o \langle \boldsymbol{v} \rangle \cdot \nabla \left(\frac{f}{A_o}\right) \\ &\quad = -A_o \left[\langle \zeta \rangle + \left(\frac{f}{A_o}\right) \right] \nabla \cdot \langle \boldsymbol{v} \rangle \end{aligned} \tag{44}$$

allows consolidation of the relative and planetary vorticity, giving

$$\frac{d_o}{dt} \left[\langle \zeta \rangle + \left(\frac{f}{A_o}\right) \right] = -A_o \left[\langle \zeta \rangle + \left(\frac{f}{A_o}\right) \right] \nabla \cdot \langle \boldsymbol{v} \rangle. \tag{45}$$

* The momentum equation can also be evaluated at an "equivalent barotropic level" (Charney, 1949; Holton, 1972), e.g., at which $A(p) = A^2(p)$, which is convenient in the quasi-geostrophic framework. However, for the primitive equations this offers no particular advantage as it leads to the advection operator being defined differently in the two equations, (40) necessarily being evaluated at the surface. Again, since $A(p)$ is prescribed, the level(s) chosen are arbitrary.

With the integrated continuity equation (42b), this may be written

$$\frac{1}{[\langle\zeta\rangle+(f/A_o)]}\frac{\mathrm{d}_o}{\mathrm{d}t}\left[\langle\zeta\rangle+\left(\frac{f}{A_o}\right)\right] - A_o\frac{1}{p_o}\frac{\mathrm{d}_o p_o}{\mathrm{d}t}=0, \tag{46}$$

or

$$\frac{\mathrm{d}_o}{\mathrm{d}t}\left\{\ln\left[\langle\zeta\rangle+\left(\frac{f}{A_o}\right)\right]\right\}-\frac{\mathrm{d}_o}{\mathrm{d}t}\{\ln p_o^{A_o}\}=0, \tag{47}$$

giving finally the conservation principle

$$\frac{\mathrm{d}_o}{\mathrm{d}t}\left[\frac{\langle\zeta\rangle+(f/A_o)}{p_o^{A_o}}\right]=0 \tag{48a}$$

where

$$Q=\left[\frac{\langle\zeta\rangle+(f/A_o)}{p_o^{A_o}}\right] \tag{48b}$$

is identified as the potential vorticity. This form is similar to that emerging under barotropic stratification (reducing to (24a) when $A_o = 1$), but accommodates horizontal divergence, i.e., p_o need not be constant. The factor A_o appears because of the advection operator in (42) being evaluated at the surface. However, since $A(p)$ is specified, another level could just as easily have been chosen, leading to a somewhat different conservation expression consistent with the resulting system. It is worth noting that approximation of the motion by geostrophy in order to eliminate the advection terms in (42b) (Charney, 1949), is not necessary in deriving the conservation principle (48).

The system (42) and the attending conservation principle (48) thus form an analogue of the shallow water equations subjected to arbitrary lower boundary forcing. In particular, the right hand side of (42a), involving elevation and pressure at the lower bounding material surface, plays the role of the free surface displacement in the shallow water momentum equation (A11a), and the surface pressure in (42b) plays the role of the column height in the shallow water continuity equation (A11b). Only one approximation has been essential in deriving this analogue: neglect of vertical advection of momentum.

3.2. *Isentropic coordinates*

For adiabatic motion, isentropic coordinates have the unique advantage that material does not cross coordinate surfaces, so the term reflecting vertical advection of momentum automatically vanishes. When the motion is not adiabatic, this system is still attractive, because the vertical velocity is determined directly by the heating. Therefore, if the heating is prescribed, or if it follows in a specified manner from other variables, no new unknowns are introduced into the problem. The motion is thus describable by a similar set of equations. We consider first the simpler case of adiabatic motion.

The geometry is the same as before, $z_o(\boldsymbol{x},t)$ describing the lower bounding material surface of an unbounded layer. On this material surface $p_o(\boldsymbol{x},t)$ and in general $\theta_o(\boldsymbol{x},t)$ are variable (Fig. 4). However, in the event that the motion is adiabatic and begins from an initially static level distribution with θ constant at the lower boundary, θ_o remains constant as fluid elements in contact with the boundary remain so, each preserving its initial potential temperature. Thus, the lower bounding material surface remains coincident with an isentropic surface (cf. Fig. 2).

Under equivalent barotropic stratification, we have the conditions on streamfunction

$$\nabla_\theta\Psi \parallel (\nabla_\theta\Psi)_o. \tag{49a}$$

the horizontal pressure force being parallel at all levels, and

$$v = A(\theta)\cdot\langle v\rangle(\boldsymbol{x},t), \tag{49b}$$

the velocity varying with altitude only in magnitude. In (49b)

$$\langle\bullet\rangle=\frac{1}{p_o}\int_\infty^{\theta_o}(\bullet)\left(\frac{\partial p}{\partial\theta}\right)\mathrm{d}\theta \tag{50}$$

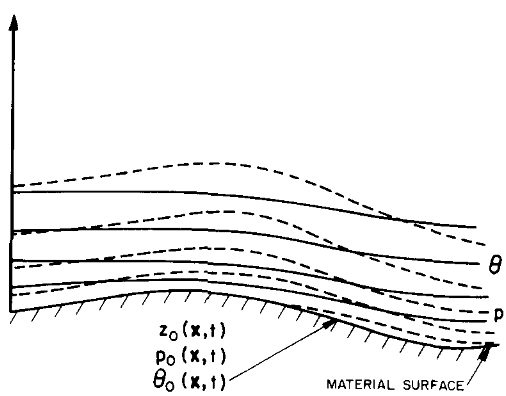

Fig. 4. As for Fig. 3, but for isentropic coordinates.

still represents the column average, and $A(\theta)$ is normalized,

$$\langle A \rangle = 1. \tag{51}$$

The primitive equations, in combination with the ideal gas law, become for adiabatic motion

$$\begin{cases} \dfrac{d_\theta \boldsymbol{v}}{dt} + f\boldsymbol{k} \times \boldsymbol{v} = -\nabla_\theta \Psi & \text{(52a)} \\ \dfrac{\partial \Psi}{\partial \theta} = c_p \left(\dfrac{p}{p_{oo}}\right)^\kappa = c_p \left(\dfrac{T}{\theta}\right) & \text{(52b)} \\ \dfrac{d_\theta}{dt}\left(\dfrac{\partial p}{\partial \theta}\right) + \left(\dfrac{\partial p}{\partial \theta}\right) \nabla_\theta \cdot \boldsymbol{v} = 0 & \text{(52c)} \end{cases}$$

$$\frac{d_\theta}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla_\theta \tag{52d}$$

denoting horizontal advection on isentopic surfaces, and

$$\Psi = c_p T + g z_\theta \tag{52e}$$

is the Montgomery streamfunction.

Integrating the hydrostatic equation (52b) from the lower boundary to the isentropic surface θ in the interior results in

$$\Psi = \Psi_o(\boldsymbol{x}, t) + c_p \int_{\theta_o}^{\theta} T \, d\ln\theta. \tag{53}$$

Then the horizontal pressure force is just

$$\nabla_\theta \Psi = \nabla \Psi_o + c_p \int_{\theta_o}^{\theta} \nabla_\theta T \, d\ln\theta, \tag{54}$$

where $\nabla\Psi_o$ is identified with $(\nabla_\theta \Psi)_o$ in (49a).

Again, in order that the condition of equivalent barotropy (49a) hold, the contribution from variable stacking of isothermal layers must be parallel at all levels to the gradient at the lower boundary,

$$\nabla_\theta T \parallel \nabla \Psi_o. \tag{55}$$

Then, proceeding as in Subsect. 3.1, we arrive at the following form for the horizontal pressure force

$$\nabla_\theta \Psi = [1 + B(\theta)] \nabla \Psi_o, \tag{56}$$

where the vertical structure $B(\theta)$ is determined by the stratification, i.e., $A(\theta)$.

The momentum equation (52a) then becomes

$$\left(\frac{\partial}{\partial t} + A(\theta)\langle \boldsymbol{v} \rangle \cdot \nabla\right) \langle \boldsymbol{v} \rangle + f\boldsymbol{k} \times \langle \boldsymbol{v} \rangle = -\left[\frac{1 + B(\theta)}{A(\theta)}\right] \nabla \Psi_o. \tag{57}$$

With (52b) and (52e), this may be written, in terms of the surface elevation and pressure (as in the shallow water equations), as

$$\frac{d\langle \boldsymbol{v} \rangle}{dt} + f\boldsymbol{k} \times \langle \boldsymbol{v} \rangle = -\left[\frac{1 + B(\theta)}{A(\theta)}\right] g \nabla \left[z_o + \frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^\kappa\right] \tag{58a}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + A(\theta)\langle \boldsymbol{v} \rangle \cdot \nabla \tag{58b}$$

and

$$\Gamma = g/c_p \tag{58c}$$

is the dry adiabatic lapse rate.

For geostrophic scaling, $B(\theta)$ may again be related directly to the equivalent barotropic structure,

$$B(\theta) = \frac{A(\theta)}{A_o} - 1, \tag{59}$$

and hence

$$\nabla_\theta \Psi = \frac{A(\theta)}{A_o} \nabla \Psi_o. \tag{60}$$

Turning now to the continuity equation, (52c) is

$$\left(\frac{\partial}{\partial t} + A(\theta)\langle \boldsymbol{v} \rangle \cdot \nabla\right)\left(\frac{\partial p}{\partial \theta}\right) + \left(\frac{\partial p}{\partial \theta}\right) A(\theta) \nabla \cdot \langle \boldsymbol{v} \rangle = 0. \tag{61}$$

Integrating with respect to θ from the surface to the top of the atmosphere leads to

$$\frac{\partial p_o}{\partial t} + \langle \boldsymbol{v} \rangle \cdot \nabla p_o \langle A \rangle + p_o \langle A \rangle \nabla \cdot \langle \boldsymbol{v} \rangle = 0 \tag{62}$$

or

$$\frac{\hat{d} p_o}{dt} + p_o \nabla \cdot \langle \boldsymbol{v} \rangle = 0, \tag{63a}$$

where

$$\frac{\hat{d}}{dt} = \frac{\partial}{\partial t} + \langle \boldsymbol{v} \rangle \cdot \nabla. \tag{63b}$$

Then taking the column average $\langle\ \rangle$ of (58a) and using (59) results in the closed system

$$\boxed{\begin{aligned} &\frac{\hat{d}\langle \boldsymbol{v} \rangle}{dt} + f\boldsymbol{k} \times \langle \boldsymbol{v} \rangle = -\frac{g}{A_o} \nabla \left[z_o + \frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^\kappa\right], && \text{(64a)} \\ &\frac{\hat{d} p_o}{dt} + p_o \nabla \cdot \langle \boldsymbol{v} \rangle = 0, && \text{(64b)} \end{aligned}}$$

in the unknowns $\langle \boldsymbol{v} \rangle$ and p_o. The system (64) is very similar to (42), except that the surface value A_o does not appear in the advection operator (63b). No approximation involving the neglect of vertical advection of momentum is required in its derivation. Without introducing the relationship (59), inspired by geostrophic scaling, the resulting system would be virtually identical, with $1/A_o$ replaced by the constant $\langle (1+B)/A \rangle$. However, again, the similarity between the vertical structures of velocity and streamfield is an observed feature of the motions being considered, and therefore this form should be adequate for the system (64).

A conservation principle may be derived as in previous cases. Owing to the absence of A_o in the advection operator appearing in (64), all terms involving $\langle \boldsymbol{v} \rangle$ are of the same form. Consequently, a series of operations similar to those leading to (48) results in the simpler expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\langle \zeta \rangle + f}{p_o}\right] = 0 \tag{65a}$$

with

$$Q = \left[\frac{\langle \zeta \rangle + f}{p_o}\right] \tag{65b}$$

representing the potential vorticity.

The system (64) together with its conservation principle (65) constitutes the purest analogue of shallow water motion. Horizontal convergence is fully accommodated, with approximations involving the neglect of either ideal gas behavior, ageostrophic motion, or vertical advection of momentum being unnecessary.

We move lastly to the situation when heating and friction are present. Under diabatic flow, vertical advection of momentum appears due to motion across isentropic surfaces. This is introduced through the time rate of change of potential temperature, ω_θ, which no longer vanishes, as was true under adiabatic conditions. To close the system, an additional equation must be added. With the first law, the primitive equations become

$$\frac{\mathrm{d}_\theta \boldsymbol{v}}{\mathrm{d}t} + \omega_\theta \frac{\partial \boldsymbol{v}}{\partial \theta} + f\boldsymbol{k} \times \boldsymbol{v} = -\nabla_\theta \Psi + \boldsymbol{F}, \tag{66a}$$

$$\frac{\partial \Psi}{\partial \theta} = c_p\left(\frac{p}{p_{oo}}\right)^{\kappa} = c_p\left(\frac{T}{\theta}\right), \tag{66b}$$

$$\frac{\mathrm{d}_\theta}{\mathrm{d}t}\left(\frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\omega_\theta \frac{\partial p}{\partial \theta}\right) + \left(\frac{\partial p}{\partial \theta}\right)\nabla_\theta \cdot \boldsymbol{v} = 0, \tag{66c}$$

$$\omega_\theta = \frac{\theta}{c_p T}\dot{q}, \tag{66d}$$

$\dot{q}$ representing the heating rate per unit mass and $\boldsymbol{F}$ any frictional force.

The geometry is similar to that of adiabatic motion, except that the lower boundary no longer remains coincident with an isentropic surface (Fig. 4), and therefore in general $\theta_o(\boldsymbol{x}, t)$ is variable. Beginning from an initially static condition with level surfaces, the $\theta_o = \text{const.}$ surface, initially coincident with the lower boundary, will be driven in some generally irregular manner away from this material surface.

Proceeding as for adiabatic motion, leads again to general forms (56) and (60) for the horizontal pressure force. Then, the momentum equation becomes

$$\left(\frac{\partial}{\partial t} + A(\theta)\langle \boldsymbol{v} \rangle \cdot \nabla\right)\langle \boldsymbol{v} \rangle + \frac{\mathrm{d}\ln A}{\mathrm{d}\theta}\omega_\theta \langle \boldsymbol{v} \rangle + f\boldsymbol{k} \times \langle \boldsymbol{v} \rangle = -\frac{1}{A_o}(\nabla_\theta \Psi)_o + \boldsymbol{F} \tag{67}$$

with $1/A_o$ representing the more general form $[(1+B)/A]$ as in (57) and

$$(\nabla_\theta \Psi)_o = \nabla \Psi_o - c_p\left(\frac{p_o}{p_{oo}}\right)^{\kappa} \nabla \theta_o, \tag{68}$$

now involving the potential temperature gradient along the lower boundary. With (66b) and (52e), this becomes

$$(\nabla_\theta \Psi)_o = g\nabla\left[z_o + \frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\right] - c_p\left(\frac{p_o}{p_{oo}}\right)^{k} \nabla \theta_o. \tag{69}$$

Then the momentum equation is

$$\frac{\mathrm{d}\langle \boldsymbol{v} \rangle}{\mathrm{d}t} + \frac{\mathrm{d}\ln A}{\mathrm{d}\theta}\omega_\theta \langle \boldsymbol{v} \rangle + f\boldsymbol{k} \times \langle \boldsymbol{v} \rangle = \frac{-g}{A_o}\nabla\left[z_o + \frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\right] + \frac{c_p}{A_o}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\nabla \theta_o + \boldsymbol{F}, \tag{70a}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + A(\theta)\langle \boldsymbol{v} \rangle \cdot \nabla. \tag{70b}$$

The continuity equation (66c) may be integrated between the surface and the top of the atmosphere, giving

$$\int_{\infty}^{\theta_o(x,t)} \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right) d\theta + \int_{\infty}^{\theta_o(x,t)} A(\theta)\langle \boldsymbol{v}\rangle \cdot \nabla_\theta \left(\frac{\partial p}{\partial \theta}\right) d\theta$$

$$+ \int_{\infty}^{\theta_o(x,t)} \frac{\partial}{\partial \theta}\left(\omega_\theta \frac{\partial p}{\partial \theta}\right) d\theta$$

$$+ \int_{\infty}^{\theta_o(x,t)} A(\theta)\left(\frac{\partial p}{\partial \theta}\right) \nabla \cdot \langle \boldsymbol{v}\rangle \, d\theta = 0. \tag{71}$$

Together with the relationship

$$\int_{\infty}^{\theta_o(x,t)} \frac{\partial f}{\partial x} d\theta = \frac{\partial}{\partial x}\int_{\infty}^{\theta_o} f \, d\theta - f_o \frac{\partial \theta_o}{\partial x}, \tag{72}$$

and similar expressions which follow from Leibnitz' rule, (71) may be written

$$\frac{\partial}{\partial t}\int_{\infty}^{\theta_o(x,t)} \left(\frac{\partial p}{\partial \theta}\right) d\theta - \left(\frac{\partial p}{\partial \theta}\right)_o \frac{\partial \theta_o}{\partial t}$$

$$+ \langle \boldsymbol{v}\rangle \cdot \nabla \int_{\infty}^{\theta_o(x,t)} A(\theta)\left(\frac{\partial p}{\partial \theta}\right) d\theta$$

$$- A_o\left(\frac{\partial p}{\partial \theta}\right)_o \langle \boldsymbol{v}\rangle \cdot \nabla \theta_o$$

$$+ \left(\omega_\theta \frac{\partial p}{\partial \theta}\right)_o + \nabla \cdot \langle \boldsymbol{v}\rangle \int_{\infty}^{\theta_o(x,t)} A(\theta)\left(\frac{\partial p}{\partial \theta}\right) d\theta = 0, \tag{73}$$

which with (50) reduces to

$$\frac{\partial p_o}{\partial t} + \langle \boldsymbol{v}\rangle \cdot \nabla \langle A\rangle p_o - \left(\frac{\partial p}{\partial \theta}\right)_o \left[\frac{\partial \theta_o}{\partial t} + A_o \langle \boldsymbol{v}\rangle \cdot \nabla \theta_o\right]$$

$$+ \omega_\theta|_o \left(\frac{\partial p}{\partial \theta}\right)_o + \langle A\rangle p_o \nabla \cdot \langle \boldsymbol{v}\rangle = 0, \tag{74}$$

or with (51)

$$\frac{\mathfrak{d} p_o}{dt} + p_o \nabla \cdot \langle \boldsymbol{v}\rangle = \left(\frac{\partial p}{\partial \theta}\right)_o \left[\frac{d_o \theta_o}{dt} - \omega_\theta|_o\right], \tag{75}$$

$\mathfrak{d}/dt$ being given by (63b) and d_o/dt by (41).

The right hand side of (75), connected ultimately to the heating, must now be evaluated. By definition,

$$\theta_o(x,t) = \theta(x, z_o(x,t), t) \tag{76}$$

so

$$\frac{\partial \theta_o}{\partial x} = \left(\frac{\partial \theta}{\partial x}\right)_{z_o} + \left(\frac{\partial \theta}{\partial z}\right)_{z_o} \frac{\partial z_o}{\partial x} \tag{77}$$

together with similar expressions lead to

$$\frac{d\theta_o}{dt} = \left(\frac{d\theta}{dt}\right)_o + \left(\frac{\partial \theta}{\partial z}\right)_o \frac{dz_o}{dt}. \tag{78}$$

With (15), this reduces to

$$\frac{d\theta_o}{dt} = (\dot{\theta})_o = \omega_\theta|_o. \tag{79}$$

Therefore, the right-hand side of (75) vanishes identically, leaving (63a), as for adiabatic motion and a similar expression (40) resulting in isobaric coordinates. That all three of these frameworks yield essentially the same relation is a consequence of the continuity equation being integrated between a lower bounding material surface and the top of the atmosphere. The resulting expression, governing the same material volume in all three cases, represents a balance between column weight (surface pressure) and horizontal convergence, this balance being independent of coordinate system and of heating.

Lastly, we must consider the vertical momentum flux term in (70a), which does follow from internal heating. Using (66d) and (66b) for the cross isentropic motion, ω_θ gives

$$\frac{d \ln A}{d\theta}\, \omega_\theta \langle \boldsymbol{v}\rangle = \frac{d \ln A}{d\theta} \frac{1}{c_p}\left(\frac{p}{p_{oo}}\right)^{-\kappa} \dot{q}\langle \boldsymbol{v}\rangle. \tag{80}$$

Then taking

$$\dot{q} = Q(\theta)\langle \dot{q}\rangle, \tag{81}$$

gives

$$\frac{d \ln A}{d\theta}\, \omega_\theta \langle \boldsymbol{v}\rangle = \frac{\langle \boldsymbol{v}\rangle \langle \dot{q}\rangle}{c_p} \frac{d \ln A}{d\theta} Q\left(\frac{p}{p_{oo}}\right)^{-\kappa}. \tag{82}$$

Now $Q(\theta)$ is in general not independent of $A(\theta)$. It may be fully prescribed, as in the case of forcing, or it may be specified as a function of $A(\theta)$, as in the case of dissipation. Since $A(\theta)$ is also specified and arbitrary, it is really only the horizontal component $\langle \dot{q}\rangle$ which enters prognostically. Because of this arbitrariness, $Q(\theta)$ may be chosen more or less out of convenience, the reduced system which results being essentially the same irrespective of the choice of heating profile.

If we take

$$Q(\theta) = \left(\theta_{oo} \frac{d \ln A}{d\theta}\right)^{-1}, \tag{83}$$

θ_{oo} representing a mean reference value at the lower boundary surface, the vertical flux term simplifies

$$\frac{d \ln A}{d\theta}\, \omega_\theta \langle \boldsymbol{v}\rangle = \frac{\langle \boldsymbol{v}\rangle \langle \dot{q}\rangle}{c_p \theta_{oo}} \left(\frac{p}{p_{oo}}\right)^{-\kappa},$$

so (70a) becomes

$$\frac{\mathrm{d}\langle \boldsymbol{v}\rangle}{\mathrm{d}t}+\frac{\langle \boldsymbol{v}\rangle\langle \dot{q}\rangle}{c_p\theta_{oo}}\left(\frac{p}{p_{oo}}\right)^{-\kappa}+f\boldsymbol{k}\times\langle \boldsymbol{v}\rangle = \frac{-g}{A_o}\nabla\left[z_o+\frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\right] + \frac{c_p}{A_o}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\nabla\theta_o+\boldsymbol{F}. \tag{84}$$

Then, taking the column average gives

$$\frac{\hat{\mathrm{d}}\langle \boldsymbol{v}\rangle}{\mathrm{d}t}+\frac{\langle \boldsymbol{v}\rangle\langle \dot{q}\rangle}{c_p\theta_{oo}p_o}\int_0^{p_o}\left(\frac{p}{p_{oo}}\right)^{-\kappa}\mathrm{d}p+f\boldsymbol{k}\times\langle \boldsymbol{v}\rangle = \frac{-g}{A_o}\nabla\left[z_o+\frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\right] + \frac{c_p}{A_o}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\nabla\theta_o+\langle\boldsymbol{F}\rangle, \tag{85}$$

which may be reduced to*

$$\frac{\hat{\mathrm{d}}\langle \boldsymbol{v}\rangle}{\mathrm{d}t}+f\boldsymbol{k}\times\langle \boldsymbol{v}\rangle = \frac{-g}{A_o}\nabla\left[z_o+\frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\right] + \frac{c_p}{A_o}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\nabla\theta_o+\frac{\langle \boldsymbol{v}\rangle}{c_v\theta_{oo}}\left(\frac{p_o}{p_{oo}}\right)^{-\kappa}\langle\dot{q}\rangle. \tag{86}$$

Eqs. (86) and the continuity equation (63a) are similar to the set (64) governing adiabatic motion. Only here, θ_o is an additional unknown due to heating. It may be determined in terms of the other variables through the first law (66d), applied at the surface. Then with $\langle\dot{q}\rangle$ and $\langle\boldsymbol{F}\rangle$ specified, e.g., in terms of prognostic variables, the equations

$$\frac{\hat{\mathrm{d}}\langle \boldsymbol{v}\rangle}{\mathrm{d}t}+f\boldsymbol{k}\times\langle \boldsymbol{v}\rangle = \frac{-g}{A_o}\nabla\left[z_o+\frac{\theta_o}{\Gamma}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\right] + \frac{c_p}{A_o}\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\nabla\theta_o+\frac{\langle \boldsymbol{v}\rangle}{c_v\theta_{oo}}\left(\frac{p_o}{p_{oo}}\right)^{-\kappa}\langle\dot{q}\rangle+\langle\boldsymbol{F}\rangle, \tag{87a}$$

$$\frac{\hat{\mathrm{d}}p_o}{\mathrm{d}t}+p_o\nabla\cdot\langle\boldsymbol{v}\rangle=0, \tag{87b}$$

$$\frac{\mathrm{d}_o\theta_o}{\mathrm{d}t}=\frac{A_o}{A_o'}\left(\frac{p_o}{p_{oo}}\right)^{-\kappa}\frac{\langle\dot{q}\rangle}{c_p\theta_{oo}}, \tag{87c}$$

constitute a closed system in the unknowns $\langle\boldsymbol{v}\rangle$, p_o, and θ_o, operators $\hat{\mathrm{d}}/\mathrm{d}t$ and $\mathrm{d}_o/\mathrm{d}t$ being given by (63b) and (41), respectively.

A conservation principle may also be developed for this system. Its derivation is similar to those already given, so it is not reproduced here. The final result is

$$\frac{\hat{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\langle\zeta\rangle+f}{p_o}\right]=\frac{1}{p_o}\left\{\frac{1}{A_o}\nabla\left(\frac{p_o}{p_{oo}}\right)^{\kappa}\times\nabla\theta_o + \frac{1}{c_v\theta_{oo}}\nabla\times\left[\left(\frac{p_o}{p_{oo}}\right)^{-\kappa}\langle\boldsymbol{v}\rangle\langle\dot{q}\rangle\right]+\nabla\times\langle\boldsymbol{F}\rangle\right\}, \tag{88}$$

potential vorticity being created/destroyed directly through heating and indirectly through potential temperature gradients at the lower boundary and also through the solenoidal component of the frictional force $\boldsymbol{F}$.

Before concluding, it should be noted that the heating appearing in (87), e.g., its specification in terms of other prognostic variables, must be consistent with the formalism. In particular, the heating must be such that it does not drive the stratification away from equivalent barotropy. It would appear that so long as $\dot{q}$ is of a separable form (81), with the horizontal structure mirroring that of the streamfield, this should be satisfied.

4. Conclusions

From the three-dimensional primitive equations in spherical geometry, we have derived the reduced two-dimensional system governing deep atmospheric motions under simple classes of stratification. In doing so, a mapping has been established between the full primitive equations and general shallow water behavior, for these special classes of motion. Although not strictly equal to the shallow water equations, the reduced two-dimensional system is of the same general form and is therefore amenable to the same methods of solution. Moreover, while the shallow water equations serve as a heuristic analogue of atmospheric behavior, the equivalent barotropic system derived here explicitly represents atmospheric motions.

The correspondence between variables describing deep atmospheric motion and those of shallow water behavior is established here with very little approximation, especially within the

* A different choice of heating profile $Q(\theta)$ would have led to essentially the same result, the last term on the right hand side involving the surface pressure p_o in general through a vertical integral.

framework of isentropic coordinates. The vertical alignment of the circulation is analogous to behavior involving a single vertical mode (Gent and McWilliams, 1984). Furthermore, the equivalent depth, an arbitrary parameter of the shallow water equations, is implicit in the reduced two-dimensional system, entering through the weight of the column lying above the lower bounding surface.

The direct correspondence between these two systems constitutes a physical basis for carrying out two-dimensional integrations (e.g., transport calculations of very high resolution) to describe atmospheric behavior. In addition, the reduced set provides a natural framework for investigating the behavior of total column abundances of atmospheric constituents, such as total ozone, measured by satellite. Towards this end, the equivalent barotropic system can be augmented by a column integrated continuity equation, governing, in general, a photochemically-active species, the resulting equation being of the same form as (87c).

Under the simplest class of barotropic stratification, where isopycnal and isobaric surfaces coincide, the shallow water equations result directly. The motion in this case is purely columnar. Fluid elements initially vertical remain so. However, within the confines of ideal gas behavior, this stratification is restricted to a fairly narrow class of circulations, because it excludes divergent motions. More generally, under equivalent barotropic stratification, where isopycnal and isobaric surfaces though not necessarily coinciding share a common vertical axis of symmetry, the shallow water operator again results, but with the forcing terms modified to account for the baroclinic nature of the stratification. In this case, divergent motions are admitted. Horizontal divergence under this stratification is determined by the departure of isopycnal surfaces from isobaric surfaces, i.e., the degree of baroclinicity. The latter is reflected in the equivalent barotropic structure A. Specifically, under barotropic stratification $A \equiv 1$, whereas under more general conditions the local departure of A from unity (e.g., over a scale height) represents the degree of baroclinicity.

In each of the cases, the reduced two-dimensional system governing atmospheric motion is of the form of the general shallow water equations subjected to arbitrary lower boundary forcing. Isentropic coordinates afford the simplest vehicle for achieving this result, coordinate surfaces in this framework (at least under adiabatic motion) coinciding with material surfaces. Even under diabatic conditions, isentropic coordinates still have the virtue of relating the component of flow across coordinate surfaces directly to the heating.

A conservation principle analogous to Ertel's theorem for three-dimensional flow holds for each case. The potential vorticity emerges in the form of the shallow water potential vorticity, only with the column height (reflecting horizontal convergence) replaced by the surface pressure. Under barotropic stratification, this expression reduces to simply a statement of conservation of absolute vorticity.

Because the shallow water system can be extracted from first principles, viz., the full primitive equations, an explicit correspondence between variables in shallow water motion and those in atmospheric circulations emerges under these special classes of stratification. Horizontal motion in the shallow water system corresponds to the column averaged velocity in the atmospheric circulation, weighted according to pressure. Similarly, the column height in the shallow water equations corresponds to, in the atmospheric case, the pressure at the lower bounding material surface. One distinction between these two sets of equations is that in the atmospheric system the momentum and integrated continuity equations are coupled nonlinearly (through the pressure gradient term), whereas in the shallow water system this term is linear in both (A9) and (A11). The more complicated interdependence, logarithmic in pressure coordinates and geometric in isentropic coordinates, stems from the nonlinearity of the hydrostatic relation governing atmospheric motions.

The atmospheric system can be cast into the canonical form (A9), for which this term becomes linear, without altering the essential character of the equations. For isentropic coordinates, introducing the transformation

$$h = \frac{1}{A_0}\left[z_0 + \frac{\theta_0}{\Gamma}\left(\frac{p_0}{p_{00}}\right)^{\kappa}\right] \tag{89a}$$

$$h_0 = \frac{z_0}{A_0} \tag{89b}$$

reduces (87) to the system

$$\frac{\hat{d}\langle \boldsymbol{v}\rangle}{dt} + f\boldsymbol{k} \times \langle \boldsymbol{v}\rangle = -g\nabla h + \frac{g}{\theta_o}(h - h_o)\nabla\theta_o + \frac{\gamma\langle \boldsymbol{v}\rangle}{gA_o}\frac{(\theta_o/\theta_{oo})}{(h - h_o)}\langle \dot{q}\rangle + \langle \boldsymbol{F}\rangle, \tag{90a}$$

$$\frac{\hat{d}}{dt}(h - h_o) + \kappa(h - h_o)\nabla\cdot\langle \boldsymbol{v}\rangle = (h - h_o)\frac{1}{\theta_o}\frac{\hat{d}\theta_o}{dt}, \tag{90b}$$

$$\frac{d_o\theta_o}{dt} = \frac{1}{gA_o'}\frac{(\theta_o/\theta_{oo})}{(h - h_o)}\langle \dot{q}\rangle, \tag{90c}$$

where $\gamma = c_p/c_v$. No additional nonlinearity is generated in this process, because the logarithmic relationsip between surface pressure and horizontal divergence (e.g., (64b)) unravels the geometric dependence in (89) following from the hydrostatic equation. The effective column height is seen to be proportional to $(p_o/p_{oo})^\kappa$. Actually, the variable h is proportional to the Montgomery streamfunction at the lower boundary through (52b) and (52e). With the system in this form, forcing, arising out of lower boundary displacements, enters naturally through the integrated continuity, or height, eq. (90b). The tradeoff in making this transformation is that the nonlinearity is passed onto the conserved quantity

$$Q = \left[\frac{\langle\zeta\rangle + f}{(h - h_o)^{1/\kappa}}\right] \tag{91}$$

which is now identified as the potential vorticity, the conservation principle (88) becoming

$$\frac{\hat{d}}{dt}\left[\frac{\langle\zeta\rangle + f}{(h - h_o)^{1/\kappa}}\right] = \frac{1}{(h - h_o)^{1/\kappa}}\left\{g\nabla(h - h_o) \times \frac{1}{\theta_o}\nabla\theta_o + \frac{\gamma}{gA_o}\nabla \times \left[\frac{(\theta_o/\theta_{oo})}{(h - h_o)}\langle \boldsymbol{v}\rangle\langle \dot{q}\rangle\right] + \nabla \times \langle \boldsymbol{F}\rangle - \frac{1}{\kappa}(\langle\zeta\rangle + f)\frac{1}{\theta_o}\frac{\hat{d}\theta_o}{dt}\right\}. \tag{92}$$

Although the inspiration for deriving this two-dimensional reduction of the Primitive Equations was to provide a physical basis for investigating barotropic circulations, the behavior which can be studied with this system may in fact represent a wider class of motions. On first inspection, it would appear that one important type of motion which cannot be accommodated within this framework is baroclinic instability. Usually ascribed to synoptic-scale weather patterns in the troposphere, these motions are well known to derive their energy from baroclinic conversion. However, the picture which emerges both theoretically and from observations has the attending heat flux, associated with this conversion, confined to a shallow layer adjacent to the surface, e.g., Geisler and Garcia (1977), Randall and Stanford (1985), Salby (1982). Wave structure away from this region is largely barotropic.

If, in the spirit of Bretherton (1966), the baroclinicity is represented as an infinitesimal sheet at the surface, the fundamental process responsible for these motions is captured in the reduced two-dimensional set (90). For example, a potential temperature variation can be prescribed initially along the lower boundary, making baroclinic growth possible. Since θ_o is accounted for as one of the prognostic variables, the tendency of baroclinic eddies to homogenize the associated potential vorticity gradient through their heat flux is represented in the reduced system. In particular, the operation used to extract this reduced system, namely vertical integration over a column extending upwards from a material surface, does not exclude such motions. It would therefore appear that this set of equations could provide a framework for investigating, with great accuracy, behavior associated with nonlinearity of the horizontal advection terms, without sacrificing the basic mechanism underlying such motions. An application along these lines would be to study the horizontal dispersion of tropospheric constituents, a problem even more demanding of resolution than stratospheric applications.

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6. Appendix

General shallow water motion

Consider a thin layer of incompressible fluid, bounded above by a free surface of elevation $h(\boldsymbol{x}, t)$ and below by an impermeable but flexible boundary whose elevation is given by $h_0(\boldsymbol{x}, t)$ (Fig. 5). The governing equations are then

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + w\frac{\partial \boldsymbol{v}}{\partial z} + f\boldsymbol{k} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla_z p \tag{A1a}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{A1b}$$

$$\frac{\partial w}{\partial z} + \nabla_z \cdot \boldsymbol{v} = 0, \tag{A1c}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla_z \tag{A1d}$$

and the shallowness of the layer allows the motion to be treated hydrostatically (Gill, 1982).

Integrating the hydrostatic equation (A1b) from some level z, interior to the fluid, up to the free surface gives

$$p = \rho g(h(\boldsymbol{x}, t) - z). \tag{A2}$$

Then the horizontal pressure force

$$\frac{1}{\rho}\nabla_z p = g\nabla h \tag{A3}$$

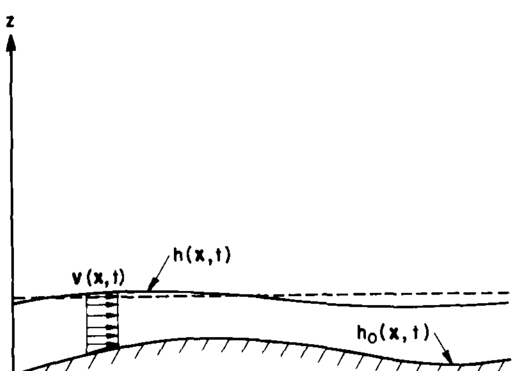

Fig. 5. Finite fluid layer, bounded below by an impermeable but mobile boundary of elevation $h_0(\boldsymbol{x}, t)$ and above by a free surface of elevation $h(\boldsymbol{x}, t)$. Horizontal velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ indicated.

is not a function of height. Consequently, with the exclusion of inertial motion,

$$\boldsymbol{v} \neq \boldsymbol{v}(z), \tag{A4}$$

and the momentum equation reduces to

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + f\boldsymbol{k} \times \boldsymbol{v} = -g\nabla h, \tag{A5}$$

the motion being columnar.

If the continuity equation (A1c) is now integrated between the lower boundary and the free surface, with (A4) we arrive at

$$w(h) - w(h_o) + (h - h_o)\nabla \cdot \boldsymbol{v} = 0, \tag{A6}$$

where the subscript z has been omitted, as $\boldsymbol{v}$ is not a function of elevation. But by definition,

$$w(h) = \dot{h} = \frac{\mathrm{d}h}{\mathrm{d}t} \tag{A7}$$

where the overdot refers to material derivative. Thus (A6) may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(h - h_o) + (h - h_o)\nabla \cdot \boldsymbol{v} = 0, \tag{A8}$$

and the motion is governed by the system

$$\boxed{\begin{aligned} &\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + f\boldsymbol{k} \times \boldsymbol{v} = -g\nabla h, && \text{(A9a)} \\ &\frac{\mathrm{d}}{\mathrm{d}t}(h - h_o) + (h - h_o)\nabla \cdot \boldsymbol{v} = 0, && \text{(A9b)} \end{aligned}}$$

in the variables $\boldsymbol{v}$ and h. Introducing the transformation

$$\Delta h = h = h_o \tag{A10}$$

leads to the equivalent system

$$\boxed{\begin{aligned} &\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + f\boldsymbol{k} \times \boldsymbol{v} = -g\nabla[h_o + \Delta h], && \text{(A11a)} \\ &\frac{\mathrm{d}}{\mathrm{d}t}(\Delta h) + (\Delta h)\nabla \cdot \boldsymbol{v} = 0. && \text{(A11b)} \end{aligned}}$$

Eqs. (A9b) and (A11b) reflect a balance between instantaneous column height and horizontal divergence, accounting for variations in the elevation of the lower bounding surface.

A conservation principle for this system may be derived. Applying the curl to (A9a) and collecting terms leads to

$$\frac{d\zeta}{dt} + \boldsymbol{v}\cdot\nabla f = -(\zeta + f)\nabla\cdot\boldsymbol{v}. \tag{A12}$$

where $\zeta = \nabla \times \boldsymbol{v}$, or

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\nabla\cdot\boldsymbol{v}. \tag{A13}$$

Together with (A9b), this may be rearranged in the form

$$\frac{1}{(\zeta + f)}\frac{d}{dt}(\zeta + f) - \frac{1}{(h - h_o)}\frac{d}{dt}(h - h_o) = 0 \tag{A14}$$

or

$$\frac{d}{dt}[\ln(\zeta + f)] - \frac{d}{dt}[\ln(h - h_o)] = 0 \tag{A15}$$

and finally

$$\frac{d}{dt}\left(\frac{\zeta + f}{\Delta h}\right) = 0 \tag{A16a}$$

where the fluid variant

$$Q = \left(\frac{\zeta + f}{\Delta h}\right) \tag{A16b}$$

represents the potential vorticity.

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