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SHORT CONTRIBUTIONS

A note on similarity theory for atmospheric boundary layers in the presence of background stable stratification

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ABSTRACT

Simple but rather general arguments permit the construction of a new version of similarity theory for turbulent boundary layers in stratified rotating fluid. The failure of the Ekman boundary layer model in the presence of imposed background stable stratification is explained by introducing the scaling length $L_N = u_*/N$, where u_* is the friction velocity and N the buoyancy frequency, which is chosen to characterize the background stratification (can also be considered as initial stratification). The expressions for the maximum height of turbulent boundary layer under different external conditions are derived and compared with observations. Also some remarks are made on resistance laws for stratified boundary layers.

1. The scaling of ABL height

The purpose of this note is to show that rather a simple generalization of the concept of equilibrium boundary layer in the presence of imposed stable stratification can give an explanation for the observed variability of the heights of the atmospheric boundary layer (ABL). Let us start with a classical example of an equilibrium boundary layer above the flat homogeneous plate, whose characteristics change in the downstream direction (see Fig. 1a). The free stream velocity U_∞ is constant, which corresponds to the situation when motion is stationary and quasi-plane parallel. However, the characteristics of the boundary layer depend not only on z but also on the downstream coordinate x and is rather accurately described by the relationship

$$h \sim \frac{u_*}{U_\infty} x. \tag{1}$$

Replacing x by $U_\infty t$ shows that this corresponds to the effective entrainment rate u_e

$$u_e = \frac{\partial h}{\partial t} \sim u_*, \tag{2}$$

where u_* can be a slowly varying function of time (as well as of x in (1)). The latter result is also applicable for surface stress-induced entrainment in homogeneous fluid (Phillips, 1977). Now a simple generalization of such a picture is when

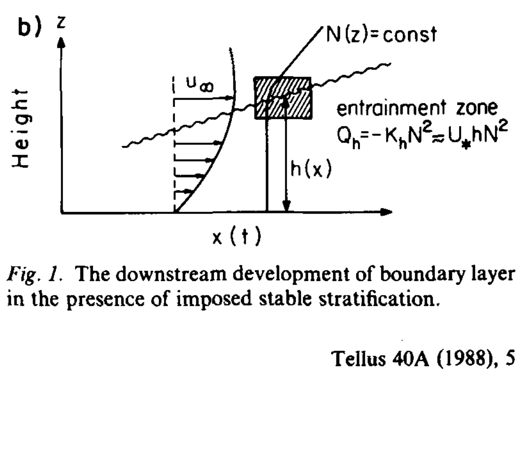

Fig. 1. The downstream development of boundary layer in the presence of imposed stable stratification.

the growth of the turbulent boundary layer occurs in the presence of imposed stable stratification $N(z,x)$ (Fig. 1b). For simplicity, let us consider that $N(z) = \text{const.} = N$, where N is the Brunt–Väisälä frequency, which in a general case can be a slowly varying function of the downstream coordinate x. We will assume that the growth of the boundary layer cannot be unlimited in this case, since the flux Richardson number Rf_h in the vicinity of the entrainment zone (see Fig. 1b) must be less than its limiting value Rf_{lim} to maintain the entrainment. The basis for my estimate of flux Richardson number Rf_h will be the following simple formula for the entrainment buoyancy flux Q_h

$$Q_h = -K_h N^2, \tag{3}$$

where K_h is an effective turbulent diffusivity responsible for the overall entrainment process, which can be written as

$$K_h \simeq u_* h. \tag{4}$$

Here we avoid a numerical constant of proportionality being of order unity. The above-mentioned condition for Rf_h can be written as

$$Rf_h = \frac{-Q_h}{u_*^2 \, du/dz} \leqslant Rf_{lim}, \tag{5}$$

which with $du/dz \sim u_\infty/h$ leads to the estimate of the maximum height of the boundary layer

$$h_{max} = bL_N, \quad L_N = \frac{u_*}{N}, \tag{6}$$

where the scaling length L_N corresponds to the scale of a well-mixed region developed by upward diffusion of turbulent energy in the presence of a background stable stratification, and b is a numerical coefficient. It was shown (Kitaigorodskii and Joffre, 1988) that the reasonable range for values of the coefficient b in (6) is 4–14 (they depend on ratio u_∞/u_*), which was found to be in good agreement with observations, at least for situations with small values of surface heat flux (compare to the entrainment flux). Thus L_N is a good measure of the height of the turbulent boundary layer in initially stably stratified fluid. Of course, it is obvious that the entrainment zone has its own structure (thickness, shear, temperature gradient, typical dissipation rate . . .). However, the purpose of my note was to convince the reader that if the quasi-equilibrium height of the ABL can exist, it must still depend on the overall external and internal parameters (the entrainment zone itself does not have it, because we consider the growth of ABL in the presence of a *given* imposed stratification). What is physically meant is that the rate of growth of the boundary layer up to its equilibrium scale is much larger (it has its own time-scale) than the evolution of the entrainment zone due to small-scale processes (like local shear instability, breaking of internal waves and so on) *after* the quasi-equilibrium height is reached. Recent numerical calculations demonstrate this fact rather convincingly, and also direct observations of turbulence in the benthic boundary layer agree with the conclusion that L_N is a proper scale for the depth of the benthic BL in the presence of imposed stratification.

The application of the scale L_N to the estimates of the heights of planetary boundary layers needs first of all its comparison with the internal Ekman scale $L_e = u_*/\Omega$ (Ω-Coriolis parameter), which is a good measure of the height of the neutrally stratified Ekman layer. Since as a rule

$$\frac{L_e}{L_N} = \frac{N}{\Omega} \gg 1, \tag{7}$$

we can conclude that in the presence of imposed stable stratification, the height of the boundary layer never reaches the Ekman value L_e, and thus rotation in such a situation is not important.

Here, I do not claim that rotation is neglected on the physical grounds. More than that, in the presence of a possible rotation of the wind vector, what my arguments demonstrate is the important fact: in the presence of a non-negligible imposed stable stratification N, the turbulent boundary layer in a stratified rotating fluid never reaches equilibrium on the typical Ekman scale, but rather achieves a quasi-equilibrium thickness characterized by the scale $h_N \cong bu_*/N$. Even though experimentally determined values of b are around 5–10, it does not change the basic conclusion that there can exist situations where $L_N/L_e \ll 1$, since Ω/N can be as small as 5×10^{-2}–10^{-3}.

The additional proof of this, based on a rather arbitrary parameterization of turbulent energy balance in mixed layers, in particular in the parameterization of the viscous dissipation, can be found in Kitaigorodskii and Joffre (1988). The most important generalization of the arguments presented above for estimates of the heights of atmospheric boundary layers must be done by taking into account the influence of the surface buoyancy flux Q_s. Because the diurnal cycle has a period of the same order as Ω^{-1}, we can with the same justification as in (7) neglect the effect of non-stationarity due to variation in Q_s and consider the typical times for growth of ABL in the presence of imposed stable stratification to be much less than Ω^{-1}. Thus, for simplicity, we can consider Q_s either constant or a very slowly varying function of time (or downstream coordinate). The application of the scale L_N to the estimates of the heights of stratified boundary layers needs its comparison with the Monin-Obukhov scale $L_* = -u_*^3/\kappa Q_s$, where $\kappa = 0.4$. We will define a new stratification parameter μ_N as

$$\mu_N = \frac{L_N}{L_*} = -\frac{Q_s}{N u_*^2}, \tag{8}$$

so that in terms of *classical* classification of stratification conditions (without imposed stratification N), $\mu_N = 0$ corresponds to the *neutral* case, and $\mu_N \gtrless 0$ will define stable and unstable conditions. The equilibrium height of the boundary layer in such conditions must be described by the relationship

$$h = L_N Y(\mu_N), \tag{9}$$

where the behavior of the function $Y(\mu_N)$ for the cases of unstable and stable stratification can be different. Thus it is convenient to introduce two functions $Y_+(\mu_N)$ for $\mu_N \geqslant 0$ and $Y_-(\mu_N)$ for $\mu_N \leqslant 0$. Let us consider first the case of stable stratification when the underlying surface acts as a storage for heat, so that $Q_s < 0$. We will have two asymptotic forms for $Y_+(\mu_N)$ for small and large absolute values of μ_N, namely

$$\text{as} \quad \mu_N \to 0, \qquad Y_+(\mu_N) = \text{const} = b, \tag{10}$$

$$\text{as} \quad \mu_N \to +\infty, \quad Y_+(\mu_N) = c\mu_N^{-1}. \tag{11}$$

The latter result corresponds to the well-known formula for mixed layers

$$h = cL_*, \tag{12}$$

where typical values of c are 2.0–4.0 (Kitaigorodskii, 1973). However, Fig. 2 indicates rather a value around 30 in the region where $Y_+ = c\mu_N$. My explanation for this disagreement with oceanic data can only be based on the fact that the range of values of μ_N observed in the atmospheric and oceanic boundary layers can be very different, so that when the predictions of the theory have an asymptotic character, the experimental data cover the different range of values of the nondimensional parameter μ_N, and thus neither oceanic data, nor ABL data presented in Kitaigorodskii and Joffre (1988) provide a final

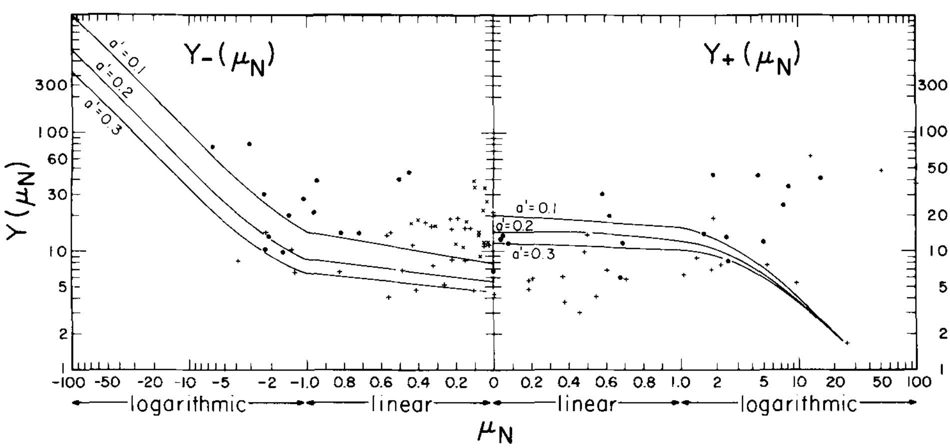

Fig. 2. Empirical representation of the dimensionless function $Y(\mu_N)$. The lines correspond to the solutions of the turbulent energy balance equation (Kitaigorodskii and Joffre, 1988). The experimental points represent the results of 3 expeditions: ICE-77, WANGARA and JASIN (for details, see Kitaigorodskii and Joffre, 1988).

answer yet. Actually, the values 2–4 was found in Kitaigorodskii (1960) and later confirmed by Kraus and Turner (1967), but there were no further experimental investigations, and the accuracy of determining surface fluxes across the ocean surface is not comparable with the direct or profile estimates of the heat flux in the ABL.

For unstable stratification, the surface buoyancy flux is an additional source of turbulent energy and therefore we can consider the regime of free convection, when friction velocity u_* must be excluded from the set of governing parameters. This leads to the following result

$$\text{as} \quad \mu_N \to -\infty, \quad Y_-(\mu_N) = \mathrm{d}[\mu_N]^{1/2}. \tag{13}$$

Eq. (13) gives the following estimate for h:

$$h = \mathrm{d}L_c, \quad L_c = \left\{\frac{|Q_s|}{N^3}\right\}^{1/2}. \tag{14}$$

The physical meaning of the convective scale L_c is clear: it corresponds to the scale of the mixed layer developed under convective mixing in an initially linearly stratified fluid. The numerical constant d must be of the order unity (Kitaigorodskii and Joffre, 1988). The results (13) and (14) must be considered with certain caution. Because Q_s is a new source of generation turbulence, we must also require that rotation is not important in this case, as it was in (7). Thus (13) and (14) must be valid when

$$\frac{L_e}{L_c} = \frac{u_*}{\Omega}\left\{\frac{N^3}{|Q_s|}\right\}^{1/2} \gg 1. \tag{15}$$

The simplest way to be convinced that the inequality (15) can be fulfilled is to rewrite it in the form

$$|\mu_0|^{-1/2}\left(\frac{N}{\Omega}\right)^{3/2} \gg 1, \tag{16}$$

where $\mu_0 = L_e/L_*$ is a stratification parameter for the stationary Ekman layer (the so-called Monin–Kazanski parameter). For $(N/\Omega) \sim 100$ and realistic values of μ_0 (even as large as 200–400), (16) is a good approximation.

Thus we come to a very interesting conclusion, that both for negative and positive surface heat flux values, when the underlying surface is either a source of atmospheric heat or a sink, the growth of the boundary layer in the presence of an *imposed stable stratification* is not influenced very significantly by rotation. The calculations and data analysis in Kitaigorodskii and Joffre (1988) can be considered as some confirmation of the above conclusion.

Here, we present in Fig. 2a their calculations of the functions $Y_+(\mu_N)$ and $Y_-(\mu_N)$ together with the data of three field experiments: (1) ICE-77, carried out over the sea ice fields of the Bothnian Bay between Finland and Sweden (64°30′N, 23°E) during a period of about 3 weeks; (2) Wangara experiment; (3) North sea and Atlantic ocean data together with the JASIN-data over the North Atlantic (for details see Kitaigorodskii and Joffre, 1988). Since the unstable case has an additional source of energy, there must be an asymmetrical behavior of $Y_+(\mu_N)$ and $Y_-(\mu_N)$ at least at moderate and large values of $|\mu_N|$. The calculations (Kitaigorodskii and Joffre, 1988) confirm this fact, though the scatter in the observations is too big to make any definite conclusions.

2. On the resistance laws for ABL in the presence of imposed stable stratification

We will consider now the effect of imposed stable stratification on the structure of a horizontal boundary layer over a flat plate in a non-rotating case (since rotation is not important here (see Section 1)). In the simplest case ($N = 0$), the relevant external parameters for description of the equilibrium boundary layer are U_∞, h and roughness parameter z_0. Now we must add to this the buoyancy difference $\Delta B = B(h) - B(z_0)$ across the boundary layer, so that the ratio h/z_0 and the bulk Richardson number $\mathrm{Ri}_h = h\Delta B/u_\infty^2$ must define variations of the drag coefficient C_f and Stanton number C_{fB}. We arrive at

$$\left(\frac{C_f}{2}\right)^{1/2} = \frac{u_*}{U_\infty} = F_f\left(\frac{h}{z_0}, \mathrm{Ri}_h\right), \tag{17}$$

$$\left(\frac{C_{fB}}{2}\right) = \frac{-Q_s}{\Delta B U_\infty} = \frac{B_*}{\Delta B}\frac{u_*}{U_\infty} F_{fB}\left(\frac{h}{z_0}, \mathrm{Ri}_h\right). \tag{18}$$

The velocity and buoyancy distributions in the ABL can be written in terms of internal set of parameters h/z_0, z/h, $\zeta_h = h/L_*$, so that

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{u_*}{h} G_u\left(\frac{h}{z_0}, \zeta_h, \frac{z}{h}\right), \tag{19}$$

$$\frac{\mathrm{d}B}{\mathrm{d}z} = \frac{B_*}{h} G_{\mathrm{B}}\left(\frac{h}{z_0}, \zeta_h, \frac{z}{h}\right). \tag{20}$$

Traditional assumptions about the asymptotic behavior of G_u, G_B for $h/z_0 \gg 1$, $z/z_0 \gg 1$ and z/h finite, and for $h/z_0 \gg 1$, $z/h \ll 1$ and $\zeta_h z/h \ll 1$ together with the existence of overlapping region, leads to the well-known results

$$C_f^{-1/2} = \frac{1}{\kappa\sqrt{2}} \left\{\ln \frac{h}{z_0} - b_0(\zeta_h)\right\}, \tag{21}$$

$$\frac{B(h) - B(z_0)}{B_*} = \frac{1}{\alpha_b(0)\kappa} \left\{\ln \frac{h}{z_0} - C_0(\zeta_h)\right\}, \tag{22}$$

where $\alpha_h(0) = \text{const.} \sim 1$, and $b_0(\zeta_h)$ and $C_b(\zeta_h)$ are universal functions. To detail the expressions (21) and (22), we can use now (9)–(15). Then when we take

$$\zeta_h = \mu_N Y(\mu_N), \tag{23}$$

the expressions (21) and (22) are reduced to the form

$$C_f^{-1/2} = \frac{1}{\kappa\sqrt{2}} \left\{\ln \frac{u_*}{Nz_0} - b_0'(\mu_N)\right\}, \tag{24}$$

$$\frac{B(h) - B(z_0)}{B_*} = \frac{1}{\alpha_b(0)\kappa} \left\{\ln \frac{u_*}{Nz_0} - C_0'(\mu_N)\right\}. \tag{25}$$

There were no attempts to construct the functions $b_0'(\mu_N)$ and $C_0'(\mu_N)$ on the basis of observations. However, it is worthwhile to mention that if we replace μ_N by the expression

$$\mu_N = \mu_0 \frac{\Omega}{N}, \tag{26}$$

the formulae (21) and (23) can be presented in the well-known traditional form of Ekman boundary layer theory

$$C_f^{-1/2} = \frac{1}{\kappa\sqrt{2}} \left\{\ln R_0 \frac{u_*}{U_\infty} - B_0\left(\mu_0, \frac{\Omega}{N}\right)\right\},$$

$$B_0 = \ln \frac{\Omega}{N} + M\left(\mu_0 \frac{\Omega}{N}\right), \tag{27}$$

$$\frac{B(h) - B(z_0)}{B_*} = \frac{1}{\alpha_b(0)} \left\{\ln R_0 \frac{u_*}{U_\infty} - C_0\left(\mu_0, \frac{\Omega}{N}\right)\right\},$$

$$C_0 = \ln \frac{\Omega}{N} + N\left(\mu_0 \frac{\Omega}{N}\right), \tag{28}$$

where $R_0 = u_\infty/\Omega z_0$. The dimensionless functions $B_0(\mu_0)$, $C_0(\mu_0)$ were constructed by many authors (see, for example, Clarke (1970), Melgarejo and Deardorff (1974). The scatter in experimental data are usually so big that one of the explanations of it can be variations in Ω/N (there are some systematic differences between morning and evening periods in the observed data, which can be attributed to the difference in N). However, the most important remark on the expressions (27)–(29) is that the variations of drag coefficient and Stanton number with Ekman scale L_e and Rossby number Ro are fictitious in presence of imposed stable stratification, since as it was shown in Section 1 that rotation is probably not important here at all. To check these conclusions will require more data and their careful analysis. The other applications of the above-mentioned similarity model can be found in the response of a Benthic Boundary Layer in initially continuously stratified fluid, subject to a suddenly imposed barotropic pressure gradient. The formation of a well-mixed Benthic Boundary Layer (BBL) is what is observed in such a case, and recent numerical calculations show that BBL rapidly reaches a quasi-steady state after only one pendulum day.

REFERENCES

Clarke, R. H. 1970. Observational studies in the atmospheric boundary layer. *Quart. J. Roy. Met. Soc. 96*, 19–114.

Kitaigorodskii, S. A. 1960. Calculating the thickness of the oceanic upper mixed layer. *Izv. Akad Nauk USSR, Ser. Geof. 3*, 425–431 (English ed. 284–287).

Kitaigorodskii, S. A. 1973. *The physics of air-sea interaction*. Israel Prog. Scient. Translation, Jerusalem, 236.

Kitaigorodskii, S. A. and Joffre, S. M. 1988. In search of simple scaling for the heights of the stratified atmospheric boundary layer. *Tellus 40A*, 419–433.

Kraus, E. B. and Turner, J. S. 1967. A one-dimensional model of the seasonal thermocline. II. The general theory and its consequences. *Tellus 19*, 98–106.

Melgarego, J. W. and Deardorff, J. W. 1974. Stability functions for the boundary layer resistance laws based upon observed boundary layer heights. *J. Atmos. Science 31*, 1324–1333.

Phillips, O. M. 1977. *Dynamics of the upper ocean* (2nd edition). Cambridge: University Press.