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Statistical effect of interactions between resolvable and unresolvable scales of two-dimensional motion

By PHILIP D. THOMPSON, *National Center for Atmospheric Research,*[1] *Boulder, Colorado, USA*

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ABSTRACT

This paper deals with a problem related to that of "subgrid-scale parameterization" of the statistical effects of unresolvable small-scale motions on the evolution of motions on resolvable scales. The approach is based on a second-moment closure in which the variables are an ensemble-averaged vorticity field, the mean transport of vorticity by small-scale deviations from the mean motion, the variance of small-scale deviations from the mean vorticity and their characteristic scale. This closure, which applies specifically to two-dimensional inviscid flow, differs from those based on the equations for the Reynolds stresses, in that it does not involve correlations between pressure and velocity fluctuations. It is found that, in the early stages of evolution, the mean transport of vorticity by small-scale motions is highly dependent on the local scale of the mean vorticity field relative to the characteristic scale of deviations from that mean. In particular, the statistical effect of small-scale motions is *counter*-gradient diffusive transport of mean vorticity if the local scale of the mean vorticity field is large, but *down*-gradient if the scale of the mean vorticity field is only slightly larger than the scale of deviations from that mean. Although the interpretation of our results for long times is not as simple as that described above, it is at least clear that mean vorticity is not universally transported down-gradient by the subgrid-scale motions.

1. Introduction

In numerical weather prediction and in many research applications of numerical methods of integrating the equations of atmospheric hydrodynamics, the smallest scales of motion are unresolved by the finite-difference grid, by finite elements or by the finite number of modes in a spectral representation. In some instances, when very high resolution is technically and economically feasible, the unresolvable scales of motion may be treated as if they were not present. In most cases, however, the scales that can be resolved explicitly are large enough that there are fairly strong interactions between motions on smaller scales and between motions on the resolvable and unresolvable scales. The problem, then, is to simulate the effects of those interactions without knowing the details of the small-scale motions.

There are two well-known approaches to problems of this kind. The first (Lilly, 1962) assumes that the local Reynolds stress (the average transport of momentum by the unresolvable small-scale motions, taken over a mesh-element) is proportional to the corresponding component of the tension and shear stresses of the resolvable velocity field. The factor of proportionality then plays the role of a coefficient of "eddy-viscosity". Under the further assumption that the small-scale motions instantaneously establish a balance beneath the rate at which the kinetic energy of the small-scale motions is generated and the rate at which it is dissipated, the coefficient of "eddy-viscosity" takes on a definite form: from considerations of dimensionality, it is proportional to the absolute value of the total deformation of the resolvable velocity field and the square of the mesh-interval, by a dimensionless factor of order

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unity. The latter is determined empirically.

This approach, which is substantially the same as that discussed by Smagorinsky (1963), has the generally desirable effect of leading to dissipation rates considerably in excess of that associated with molecular viscosity. It does, however, carry the further implication that the transport of large-scale momentum by small-scale eddies is down-gradient. This is somewhat at variance with the observation that the transport of momentum by eddies of synoptic scale is the dominant mechanism by which zonal momentum is transferred counter-gradient to maintain the mean westerly flow of the atmosphere (Starr, 1968). This suggests that the transport of momentum on the resolvable scales by the unresolvable small-scale motions is probably not universally down-gradient, and that the concept of "eddy viscosity" may not be valid locally.

An alternative to the approach outlined above—and one that appears to avoid its most stringent assumptions—is based on a second-moment closure of the evolution equations for the Reynolds stresses and other second-moments of the unresolvable small-scale motions. Closures of this type, which are described at length by Daly and Harlow (1970), Deardorff (1973, 1974) and Launder et al. (1975), are derived from the primitive form of the equations of motion and thus involve the correlation between small-scale fluctuations of velocity and pressure. Although the connection between those fluctuations is fairly simple in incompressible flow, it is still indirect enough that the "pressure-velocity" correlation is expressed in an approximate form, containing an experimentally determined dimensionless constant. Indeed, the appeal of this approach is somewhat lessened by the introduction of an element of empiricism.

One way of deriving a moment-closure that is free of the particular difficulty pointed out above is to proceed from the vorticity equation for a homogeneous fluid, rather than from the primitive equations of motion—simply because the pressure-gradient force is a potential vector, and pressure therefore does not appear in the vorticity equation. This alternative is especially attractive in the case of two-dimensional flow, in which there is only one component of vorticity; moreover, if the flow is incompressible and inviscid, that component of vorticity is conserved locally.

Stated briefly, the purpose of this paper is to examine the interactions between the resolvable and unresolvable scales of two-dimensional motion of an inviscid incompressible fluid and, in particular, to derive a second-moment closure from the vorticity equation.

Our view of the problem is the following: We imagine that the initial velocity-field consists of two parts, one resolvable and the other unresolvable. The resolvable field is continuous and contains only spectral components whose wavenumbers k are smaller than that corresponding to the smallest resolvable scale. Call that wavenumber k_0. For concreteness, we may think of the resolvable part of the initial field as a discrete double Fourier series constructed from observations at discrete points, but regarded as a continuous function of position.

The unresolvable part of the initial velocity field, containing only continuous spectral components whose wavenumbers are greater than k_0, is not known in any detail, but it is assumed that a large ensemble of such fields has certain definite statistical properties. To avoid initial bias, for example, we suppose that the ensemble has zero mean and is statistically isotropic and homogeneous. We suppose further that the power spectrum of the ensemble is known and is a smooth continuation of the spectrum of the resolvable part of the initial velocity-field. We may imagine that each member of the ensemble of unresolvable initial velocity-fields is formed by choosing the amplitudes of the sine and cosine components for each $k > k_0$ at random from a large population, whose frequency distribution matches the known power-spectrum.

We now consider a large ensemble of initial velocity fields, each member of which consists of one member of the ensemble of randomly-generated unresolvable fields, superposed on the same resolvable field. We then imagine that an ensemble of predicted velocity-fields is constructed by integrating forward in time, starting with each member of the ensemble of *initial* velocity fields. None of these predictions may, of course, be "correct". Indeed, we have no way of knowing whether or not any one of them is correct. One can, however, say that the "expected" or most probable state evolving from a given resolvable initial state and an unbiased ensemble of unresolvable initial states is the average over the ensemble of predictions.

One might naturally wonder why an ensemble-average prediction originating in an ensemble of initial states containing unresolvable and statistically isotropic small-scale fields should be any better than that in which the small-scale fields are ignored entirely. The significant point is that the unresolvable fields are systematically deformed by the resolvable motions: as a result, they become anisotropic and interact with the resolvable motion in a highly preferential way, independent of the composition of the initial ensemble.

Needless to say, the construction of a large ensemble of individual predictions is an imaginary experiment, described merely as an expository device. The real aim is to analyze the interactions between the resolvable and unresolvable motions, by deriving a simple closure in which the dependent variables are the ensemble-averaged vorticity and certain quadratic statistics of the unresolvable motions. In doing so, we consider the case of inviscid flow and follow closely the development given in Thompson (1986). In fact, the derivation is almost exactly the same, with one decisive change in interpretation: owing to the way in which we have defined an ensemble of initial velocity-fields, the initial ensemble-average is just the resolvable velocity-field, and the deviation from the ensemble-average is initially the unresolvable part. At later times, we are concerned only with ensemble-averages and the statistics of deviations from those averages.

Starting from the vorticity equation, we first derive the evolution equations for the ensemble-averaged vorticity and the deviation from that average. From these we show that the approximations of the second-moment closure do not destroy the essential conservation properties—namely, that the total kinetic energy and enstrophy are invariant in the ensemble-average.

The average vorticity is not locally conserved following the average motion, since its changes depend on the net average transport of vorticity by the deviations from average velocity. That transport vanishes initially, when the unresolvable motions are statistically isotropic. It changes with time, however, owing to the straining of the initially isotropic small-scale velocity-fields by the large-scale resolvable motions. An equation for the rate of change of the average transport of vorticity by the small-scale motions is derived in some detail in Thompson (1986), hereafter referred to as SDP: we outline here the main intermediate results of that derivation and describe briefly the intervening steps. The changes in transport are found to depend on the gradient of average vorticity, the variance of the small-scale vorticity and its mean-square gradient.

With a simple and plausible hypothesis about changes in the scale of the unresolvable motions, we finally arrive at a closed system of stochastic partial differential equations in which the variables are average vorticity, local variance of small-scale vorticity and average small-scale transport of vorticity. The form of these equations leads to a clear qualitative interpretation of the interactions between the average motion and the smaller-scale deviations from that average.

2. Coupling between resolvable and unresolvable scales of motion

As pointed out in the introduction, the average of the ensemble of initial velocity-fields is the resolvable part, and the small-scale deviations from that average are unresolvable. The quadratic statistics of the deviations, as will appear later, vary on resolvable scales. Thus, our next concern is to see how the ensemble-averaged motion interacts with deviations from that average. We begin with the vorticity equation for two-dimensional motion of an incompressible, inviscid fluid:

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{V} \cdot \nabla \zeta = 0, \tag{1}$$

where $\boldsymbol{V}$ is the velocity vector, $\zeta = \boldsymbol{k} \cdot \nabla \times \boldsymbol{V}$, $\boldsymbol{k}$ is a unit vector normal to the plane of motion and ∇ is the vector gradient in that plane. The equation of continuity reduces to:

$$\nabla \cdot \boldsymbol{V} = 0,$$

so that

$$\boldsymbol{V} = \boldsymbol{k} \times \nabla \psi, \quad \zeta = \nabla^2 \psi.$$

Let us write $\boldsymbol{V}$ as:

$$\boldsymbol{V} = \overline{\boldsymbol{V}} + \boldsymbol{V}'$$

in which the overbar denotes the ensemble-average, and the prime denotes the deviation from that average. (We shall also use angle brackets $\langle \ldots \rangle$ to denote the ensemble average, when the quantity to be averaged contains more than a few symbols.) Substituting the expression above into (1), we obtain:

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{\partial \zeta'}{\partial t} + \bar{V}\cdot\nabla\bar{\zeta} + \bar{V}\cdot\nabla\zeta' + V'\cdot\nabla\bar{\zeta} + V'\cdot\nabla\zeta' = 0. \tag{2}$$

We now average each term of this equation, with the result that:

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{V}\cdot\nabla\bar{\zeta} + \nabla\cdot\overline{\zeta' V'} = 0. \tag{3}$$

The evolution equation for ζ' is found by subtracting (3) from (2); i.e.,

$$\frac{\partial \zeta'}{\partial t} + \bar{V}\cdot\nabla\zeta' + V'\cdot\nabla\bar{\zeta} + \nabla\cdot\zeta' V' - \nabla\cdot\overline{\zeta' V'} = 0. \tag{4}$$

Since our aim is to derive a second-moment closure, we now omit from (4) those terms containing products of deviations, namely, the 4th and 5th terms. Thus, we replace (4) by:

$$\frac{\partial \zeta'}{\partial t} + \bar{V}\cdot\nabla\zeta' + V'\cdot\nabla\bar{\zeta} = 0. \tag{5}$$

It should be noted that (3) and (5) may be regarded as a coupled system of equations in which the dependent variables are $\bar{\psi}$ and ψ'.

Before proceeding further, let us reassure ourselves that the approximations of a second-moment closure do not destroy the fundamental conservation properties of the full nonlinear equations. Multiplying each term of (5) by ζ' and averaging, we may write it in the form:

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{\zeta'^2} + \frac{1}{2}\nabla\cdot\overline{\zeta'^2}\,\bar{V} + \overline{\zeta' V'}\cdot\nabla\bar{\zeta} = 0. \tag{6}$$

Thus, integrating each term of this equation over the area A occupied by the flow, we find that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \overline{\zeta'^2}\,\mathrm{d}A + \frac{1}{2}\int_A \nabla\cdot\overline{\zeta'^2}\,\bar{V}\,\mathrm{d}A + \int_A \overline{\zeta' V'}\cdot\nabla\bar{\zeta}\,\mathrm{d}A = 0,$$

where $\mathrm{d}A$ is an area-element of A. The second term may be transformed (by Gauss' theorem) into a line integral taken around the boundary: that integral vanishes if the boundary conditions are periodic, or if the flow is contained within a closed, rigid boundary. The equation above then reduces to:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \overline{\zeta'^2}\,\mathrm{d}A = -\int_A \overline{\zeta' V'}\cdot\nabla\bar{\zeta}\,\mathrm{d}A. \tag{7}$$

Similarly, multiplying each term of (3) by $\bar{\zeta}$, we may write it as:

$$\frac{1}{2}\frac{\partial}{\partial t}\bar{\zeta}^2 + \frac{1}{2}\nabla\cdot\bar{\zeta}^2\,\bar{V} + \bar{\zeta}\nabla\cdot\overline{\zeta' V'} = 0,$$

whence, integrating each term of this equation over A, we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \bar{\zeta}^2\,\mathrm{d}A + \frac{1}{2}\int_A \nabla\cdot\bar{\zeta}^2\,\bar{V}\,\mathrm{d}A + \int_A \nabla\cdot\bar{\zeta}\overline{\zeta' V'}\,\mathrm{d}A - \int_A \overline{\zeta' V'}\cdot\nabla\bar{\zeta}\,\mathrm{d}A = 0.$$

The 2nd and 3rd terms are both transformable into line integrals around the boundary; again, these vanish, and the equation above becomes:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \bar{\zeta}^2\,\mathrm{d}A = \int_A \overline{\zeta' V'}\cdot\nabla\bar{\zeta}\,\mathrm{d}A. \tag{8}$$

Now, adding (7) and (8), we conclude that

$$\frac{\partial}{\partial t}\int_A (\bar{\zeta}^2 + \overline{\zeta'^2})\,\mathrm{d}A = 0.$$

But $\bar{\zeta}^2 + \overline{\zeta'^2} = \overline{\zeta^2}$. In short, the total enstrophy is conserved in the ensemble average, despite the approximations of (5).

To show that an analogous result applies in the case of total kinetic energy, we multiply (3) by $\bar{\psi}$, writing it in the form:

$$\bar{\psi}\nabla\cdot\frac{\partial}{\partial t}\nabla\bar{\psi} + \bar{\psi}\nabla\cdot\bar{\zeta}\bar{V} + \bar{\psi}\nabla\cdot\overline{\zeta' V'} = 0$$

or as

$$\nabla\cdot\bar{\psi}\frac{\partial}{\partial t}\nabla\bar{\psi} + \nabla\cdot\bar{\psi}\bar{\zeta}\bar{V} + \nabla\cdot\bar{\psi}\overline{\zeta' V'} - \nabla\bar{\psi}\cdot\frac{\partial}{\partial t}\nabla\bar{\psi} - \bar{\zeta}\bar{V}\cdot\nabla\bar{\psi} - \overline{\zeta' V'}\cdot\nabla\bar{\psi} = 0.$$

Let us first observe that the 5th term is identically zero. Moreover, the area integrals of the first three terms may be transformed into line integrals around the boundary; the latter vanish. (Note that, if the flow is contained by a closed rigid boundary, ψ is constant on the boundary. Since ψ is defined only to within an arbitrary additive function of time, we may choose that constant to be zero without loss of generality.) Thus, the equation above reduces to:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \nabla\bar{\psi}\cdot\nabla\bar{\psi}\,\mathrm{d}A = -\int_A \overline{\zeta'\boldsymbol{V}'}\cdot\nabla\bar{\psi}\,\mathrm{d}A. \tag{9}$$

Similarly, we multiply (5) by ψ', writing it as:

$$\psi'\nabla\cdot\frac{\partial}{\partial t}\nabla\psi' + \psi'\nabla\cdot\zeta'\bar{\boldsymbol{V}} + \psi'\nabla\cdot\bar{\zeta}\boldsymbol{V}' = 0,$$

or as

$$\nabla\cdot\psi'\frac{\partial}{\partial t}\nabla\psi' + \nabla\cdot\psi'\zeta'\bar{\boldsymbol{V}} + \nabla\cdot\psi'\bar{\zeta}\boldsymbol{V}' - \nabla\psi'\cdot\frac{\partial}{\partial t}\nabla\psi'$$
$$- \zeta'\bar{\boldsymbol{V}}\cdot\nabla\psi' - \bar{\zeta}\boldsymbol{V}'\cdot\nabla\psi' = 0.$$

In this case, the 6th term is identically zero, and the area integrals of the first three terms vanish for the reasons cited earlier. The equation above then becomes:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \nabla\psi'\cdot\nabla\psi'\,\mathrm{d}A = -\int_A \zeta'\bar{\boldsymbol{V}}\cdot\nabla\psi'\,\mathrm{d}A$$
$$= \int_A \zeta'\boldsymbol{V}'\cdot\nabla\bar{\psi}\,\mathrm{d}A.$$

Finally, averaging both sides of this equation, we find that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_A \overline{\nabla\psi'\cdot\nabla\psi'}\,\mathrm{d}A = \int_A \overline{\zeta'\boldsymbol{V}'}\cdot\nabla\bar{\psi}\,\mathrm{d}A. \tag{10}$$

Adding (9) and (10), we conclude that

$$\frac{\partial}{\partial t}\int_A \left(\frac{\bar{\boldsymbol{V}}\cdot\bar{\boldsymbol{V}}}{2} + \frac{\overline{\boldsymbol{V}'\cdot\boldsymbol{V}'}}{2}\right)\mathrm{d}A = 0.$$

Thus, since $\overline{\boldsymbol{V}\cdot\boldsymbol{V}} = \bar{\boldsymbol{V}}\cdot\bar{\boldsymbol{V}} + \overline{\boldsymbol{V}'\cdot\boldsymbol{V}'}$, the total kinetic energy is also conserved in the ensemble average. It appears then, that, although the approximations of (5) do not guarantee conservation of total kinetic energy and enstrophy for each member of the ensemble, they do imply exact conservation in the ensemble average.

Returning to (3), we see that the evolution of the ensemble-averaged flow depends on $\overline{\zeta'\boldsymbol{V}'}$, the mean transport of vorticity by the "small-scale" motions. Initially, when the motions of unresolvable scale are statistically isotropic, $\overline{\zeta'\boldsymbol{V}'} = 0$, simply because there is no preferred direction of mean transport. At later times, however, the small-scale motions become anisotropic, owing to the straining or deformation of the small-scale velocity-fields by the large-scale resolvable motions. It is this process that provides the strong coupling between the resolvable and unresolvable scales of motion. The crux of the problem, then, is to predict $\overline{\zeta'\boldsymbol{V}'}$.

3. Departure from isotropy

Our next concern is to calculate the time-derivative of $\overline{\zeta'\boldsymbol{V}'}$ at *initial time*, when the unresolvable fields are statistically isotropic. We start with its y-component:

$$\frac{\partial}{\partial t}\overline{\zeta' v'} = \left\langle v'\frac{\partial\zeta'}{\partial t} + \zeta'\frac{\partial v'}{\partial t}\right\rangle.$$

Here and in what follows, x and y are cartesian coordinates in the plane of motion, and u and v are the corresponding components of $\boldsymbol{V}$. Since $v' = \partial\psi'/\partial x$ and $\zeta' = \nabla^2\psi'$, we may write the previous equation in the form:

$$\frac{\partial}{\partial t}\overline{\zeta' v'} = \left\langle v'\nabla^2\frac{\partial\psi'}{\partial t} - \frac{\partial\psi'}{\partial t}\nabla^2 v'\right\rangle + \frac{\partial}{\partial x}\left\langle\zeta'\frac{\partial\psi'}{\partial t}\right\rangle. \tag{11}$$

We observe that the first term on the right-hand side of this equation is suggestively antisymmetric in v' and $\partial\psi'/\partial t$.

We now digress briefly to discuss a simple inversion of the Laplace operator. By making use of the properties of the logarithmic potential, it is readily shown that

$$\phi(x,y) = \tilde{\phi}(x,y) + \iint_O G(r)\nabla^2\phi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{12}$$

where ϕ is any continuous and twice differentiable function, ξ and η are variables of integration corresponding to x and y,

$$G = \frac{1}{2\pi}\ln\frac{r}{R}, \quad r = [(\xi - x)^2 + (\eta - y)^2]^{\frac{1}{2}},$$

and the integration is taken over the area O bounded by a circle of radius R with center at (x, y); $\tilde{\phi}$ is the average value of ϕ on the circle. In particular, if ϕ fluctuates randomly around zero, $\tilde{\phi}$ is negligibly small for sufficiently large R. Since we suppose that the unresolvable motions are of this character, (12) reduces to

$$\phi(x,y) = \iint_O G(r)\nabla^2\phi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{13}$$

G is singular at $\xi = x$, $\eta = y$ and behaves as a Green's function.

Let us now apply (13) to calculate v' and $\partial\psi'/\partial t$ in (11). Since $\nabla^2 \partial\psi'/\partial t = \partial\zeta'/\partial t$, we see that

$$\frac{\partial}{\partial t}\overline{\zeta' v'} = \iint_O G(r)\left\langle \nabla^2 v'(\xi,\eta)\frac{\partial\zeta'}{\partial t}(x,y) - \nabla^2 v'(x,y)\frac{\partial\zeta'}{\partial t}(\xi,\eta)\right\rangle \mathrm{d}\xi\,\mathrm{d}\eta + \frac{\partial}{\partial x}\iint_O G(r)\left\langle \zeta'(x,y)\frac{\partial\zeta'}{\partial t}(\xi,\eta)\right\rangle \mathrm{d}\xi\,\mathrm{d}\eta. \tag{14}$$

In SDP it was shown that, if the ensemble of deviations from the ensemble average is statistically isotropic, the second term on the right-hand side of (14) is negligible in comparison with the first. The remaining step is to substitute the expression given by (5) for $\partial\zeta'/\partial t$, where it appears in the integrand of (14). Let us first note that (5) may be written as:

$$\frac{\partial\zeta'}{\partial t} = -\left(\bar{u}\nabla^2 v' - \bar{v}\nabla^2 u' + u'\frac{\partial\bar{\zeta}}{\partial x} + v'\frac{\partial\bar{\zeta}}{\partial y}\right),$$

so that (14) becomes:

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\zeta' v'} = -\iint_O G(r)\Big[&\bar{u}(x,y)\langle\nabla^2 v'(x,y)\nabla^2 v'(\xi,\eta)\rangle \\ &- \bar{v}(x,y)\langle\nabla^2 u'(x,y)\nabla^2 v'(\xi,\eta)\rangle \\ &+ \frac{\partial\bar{\zeta}}{\partial x}(x,y)\langle u'(x,y)\nabla^2 v'(\xi,\eta)\rangle \\ &+ \frac{\partial\bar{\zeta}}{\partial y}(x,y)\langle v'(x,y)\nabla^2 v'(\xi,\eta)\rangle \\ &- \bar{u}(\xi,\eta)\langle\nabla^2 v'(\xi,\eta)\nabla^2 v'(x,y)\rangle \\ &+ \bar{v}(\xi,\eta)\langle\nabla^2 u'(\xi,\eta)\nabla^2 v'(x,y)\rangle \\ &- \frac{\partial\bar{\zeta}}{\partial \xi}(\xi,\eta)\langle u'(\xi,\eta)\nabla^2 v'(x,y)\rangle \\ &- \frac{\partial\bar{\zeta}}{\partial \eta}(\xi,\eta)\langle v'(\xi,\eta)\nabla^2 v'(x,y)\rangle\Big]\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{15}$$

The right-hand side of (15) may be considerably simplified by making use of the statistical isotropy of the unresolvable initial fields. For example, examining the second term of the integrand, we see that

$$\begin{aligned}\langle\nabla^2 u'(x,y)\nabla^2 v'(\xi,\eta)\rangle &= -\left\langle\frac{\partial\zeta'}{\partial y}(x,y)\frac{\partial\zeta'}{\partial\xi}(\xi,\eta)\right\rangle \\ &= -\frac{\partial^2}{\partial y\,\partial\xi}\langle\zeta'(x,y)\zeta'(\xi,\eta)\rangle.\end{aligned}$$

Owing to the isotropy of the ensemble of ζ'-fields, the correlation function $\langle\zeta'(x,y)\zeta'(\xi,\eta)\rangle$ depends only on the distance from (x,y) to (ξ,η), i.e., it depends only on r. Thus, according to the equation above, $\langle\nabla^2 u'(x,y)\nabla^2 v'(\xi,\eta)\rangle$ is antisymmetric around both $\xi = x$ and $\eta = y$. $G(r)$, however, is symmetrical around $\xi = x$ and $\eta = y$. Since $\bar{v}(x,y)$ may be taken outside the integral, the integral of the second term of the integrand vanishes exactly.

Similarly, considering the third term of the integrand in (15), we observe that

$$\begin{aligned}\langle u'(x,y)\nabla^2 v'(\xi,\eta)\rangle &= -\frac{\partial^2}{\partial y\,\partial\xi}\langle\psi'(x,y)\zeta'(\xi,\eta)\rangle \\ &= -\frac{\partial^2}{\partial y\,\partial\xi}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\langle\psi'(x,y)\psi'(\xi,\eta)\rangle.\end{aligned}$$

Since the correlation function $\langle\psi'(x,y)\psi'(\xi,\eta)\rangle$ depends only on r, so also does its Laplacian. Hence, according to the preceding equation $\langle u'(x,y)\nabla^2 v'(\xi,\eta)\rangle$ is antisymmetric around $\xi = x$ and $\eta = y$, with the result that the integral of the third term of the integrand also vanishes exactly. Moreover, although $\bar{v}(\xi,\eta)$ and $\partial\bar{\zeta}/\partial\xi$ are not constant over the area O, they vary little over the small region around the singularity of G: hence the integrals of the 6th and 7th terms of the integrand of (15) are small for the same reasons that the integrals of the 2nd and 3rd terms vanish exactly.

In contrast, the correlation functions that enter into the 1st, 4th, 5th, and 8th terms of the integrand of (15) are symmetrical around $\xi = x$ and $\eta = y$, rather than antisymmetric. Thus those terms do not consist of compensating positive and negative contributions and are the dominant terms of the integrand. Then, observing that $\langle v'(x,y)\nabla^2 v'(\xi,\eta)\rangle = \langle v'(\xi,\eta)\nabla^2 v'(x,y)\rangle$, we may write (15) in the form:

$$\frac{\partial}{\partial t}\overline{\zeta' v'} = \iint\limits_{O} G[\bar{u}(\xi,\eta) - \bar{u}(x,y)]$$
$$\times \langle \nabla^2 v'(x,y) \nabla^2 v'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta$$
$$+ \iint\limits_{O} G\left[\frac{\partial \bar{\zeta}}{\partial \eta}(\xi,\eta) - \frac{\partial \bar{\zeta}}{\partial y}(x,y)\right]$$
$$\times \langle v'(x,y) \nabla^2 v'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta. \quad (16)$$

We may put (16) in a more compact form by expressing $\bar{u}(\xi,\eta)$ as a truncated Taylor expansion around (x,y), i.e.,

$$\bar{u}(\xi,\eta) - \bar{u}(x,y) = (\xi - x)\frac{\partial \bar{u}}{\partial x}(x,y)$$
$$+ (\eta - y)\frac{\partial \bar{u}}{\partial y}(x,y) + \frac{1}{2}(\xi - x)^2 \frac{\partial^2 \bar{u}}{\partial x^2}(x,y)$$
$$+ \frac{1}{2}(\eta - y)^2 \frac{\partial^2 \bar{u}}{\partial y^2}(x,y)$$
$$+ (\xi - x)(\eta - y)\frac{\partial^2 \bar{u}}{\partial x \partial y}(x,y)$$
$$+ \text{3rd degree terms.}$$

Now, imagining that this expression is substituted into the first integral on the right-hand side of (16), we first observe that the differential coefficients in the foregoing series do not depend on ξ or η and can thus be taken outside the integral. We also note that both G and $\langle \nabla^2 v'(x,y) \nabla^2 v'(\xi,\eta)\rangle$ are symmetric around $\xi = x$ and $\eta = y$, so that the only terms in the series that contribute to the integral are the 3rd and 4th. An analogous conclusion applies to a Taylor expansion of $\partial\bar{\zeta}/\partial\eta(\xi,\eta)$ around (x,y). Eq. (16) may then be written as:

$$\frac{\partial}{\partial t}\overline{\zeta' v'} = \frac{1}{4}\nabla^2 \bar{u} \iint\limits_{O} r^2 G(r)$$
$$\times \langle \nabla^2 v'(x,y) \nabla^2 v'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta$$
$$+ \frac{1}{4}\nabla^2 \frac{\partial \bar{\zeta}}{\partial y} \iint\limits_{O} r^2 G(r)$$
$$\times \langle v'(x,y) \nabla^2 v'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta. \quad (17)$$

By following an exactly analogous procedure in calculating the time-derivative of the x-component of the small-scale vorticity transport, we find that

$$\frac{\partial}{\partial t}\overline{\zeta' u'} = -\frac{1}{4}\nabla^2 \bar{v} \iint\limits_{O} r^2 G(r)$$
$$\times \langle \nabla^2 u'(x,y) \nabla^2 u'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta$$
$$+ \frac{1}{4}\nabla^2 \frac{\partial \bar{\zeta}}{\partial x} \iint\limits_{O} r^2 G(r)$$
$$\times \langle u'(x,y) \nabla^2 u'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta. \quad (18)$$

Finally, owing to the isotropy of the initial $\boldsymbol{V}'$ fields and because $\nabla^2 \bar{u} = -\partial\bar{\zeta}/\partial y$ and $\nabla^2 \bar{v} = \partial\bar{\zeta}/\partial x$, we may combine (17) and (18) in a single vector equation:

$$\frac{\partial}{\partial t}\overline{\zeta' \boldsymbol{V}'} = -\frac{1}{4}\nabla\bar{\zeta} \iint\limits_{O} r^2 G(r)$$
$$\times \langle \nabla^2 v'(x,y) \nabla^2 v'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta$$
$$+ \frac{1}{4}\nabla^2(\nabla\bar{\zeta}) \iint\limits_{O} r^2 G(r)$$
$$\times \langle v'(x,y) \nabla^2 v'(\xi,\eta)\rangle \, \mathrm{d}\xi \, \mathrm{d}\eta. \quad (19)$$

This equation reflects the systematic organization of the average small-scale vorticity transport, both through the deformation of the ensemble of unresolvable motions by the resolvable motion and, conversely, through the deformation of the resolvable motion by the ensemble of unresolvable motions.

4. A second-moment closure

In interpreting (19) it is convenient to introduce the normalized correlation function $\boldsymbol{\Phi}(r)$, defined by:

$$\langle \nabla^2 v'(x,y) \nabla^2 v'(\xi,\eta)\rangle$$
$$= \langle \nabla^2 v'(x,y) \nabla^2 v'(x,y)\rangle \boldsymbol{\Phi}(r)$$
$$= \left\langle \frac{\partial \zeta'}{\partial x}\frac{\partial \zeta'}{\partial x}\right\rangle \boldsymbol{\Phi}(r)$$
$$= \frac{1}{2}\left\langle \frac{\partial \zeta'}{\partial x}\frac{\partial \zeta'}{\partial x} + \frac{\partial \zeta'}{\partial y}\frac{\partial \zeta'}{\partial y}\right\rangle \boldsymbol{\Phi}(r)$$
$$= \frac{1}{2}\overline{\nabla\zeta' \cdot \nabla\zeta'}\, \boldsymbol{\Phi}(r).$$

We also define the normalized correlation function $\chi(r)$ by:

$$\begin{aligned}&\langle v'(x,y)\nabla^2 v'(\xi,\eta)\rangle\\ &= \langle v'(x,y)\nabla^2 v'(x,y)\rangle\,\chi(r)\\ &= [\langle \nabla\cdot v'\nabla v'\rangle - \langle \nabla v'\cdot\nabla v'\rangle]\,\chi(r)\\ &= \left[\frac{1}{2}\nabla^2\langle v'^2\rangle - \frac{1}{2}\langle \nabla v'\cdot\nabla v' + \nabla u'\cdot\nabla u'\rangle\right]\chi(r)\\ &= -\frac{1}{2}\overline{\zeta'^2}\,\chi(r).\end{aligned}$$

In deriving the expressions above, we have freely exploited the statistical isotropy and homogeneity of the small-scale initial velocity fields. Substituting the foregoing results into (19), we arrive at:

$$\frac{\partial}{\partial t}\overline{\zeta'\boldsymbol{V}'} = -\frac{\pi}{4}\left[\overline{\nabla\zeta'\cdot\nabla\zeta'}\,\nabla\bar{\zeta}\int_0^R r^3 G(r)\,\boldsymbol{\Phi}(r)\,\mathrm{d}r + \overline{\zeta'^2}\,\nabla^2(\nabla\bar{\zeta})\int_0^R r^3 G(r)\,\chi(r)\,\mathrm{d}r\right]. \quad (20)$$

This result is, of course, most nearly valid at initial time, when the ensemble of small-scale velocity fields is statistically isotropic. One might expect, however, that the small-scale fields would remain quasi-isotropic at later times.

Inspecting (3), (6), and (20) and recalling that $\bar{\boldsymbol{V}} = \boldsymbol{k}\times\nabla\bar{\psi}$ and $\bar{\zeta} = \nabla^2\bar{\psi}$, we see that those three equations involve the three variables $\bar{\psi}$, $\overline{\zeta'^2}$ and $\overline{\zeta'\boldsymbol{V}'}$, but note that (20) also contains the moment $\overline{\nabla\zeta'\cdot\nabla\zeta'}$. The latter may be expressed as:

$$\overline{\nabla\zeta'\cdot\nabla\zeta'} = k_s^2\,\overline{\zeta'^2}, \quad (21)$$

where k_S is a characteristic wavenumber of the unresolvable vorticity fields. Owing to the statistical homogeneity of the initial small-scale fields, $\overline{\nabla\zeta'\cdot\nabla\zeta'}$ and $\overline{\zeta'^2}$ (and, therefore, k_S) are spatially uniform initially. Under the usual assumption that the local kinetic energy spectrum of the unresolvable motions follows a similarity law, k_S is also independent of *time*.

To show this, we express k_s^2 in terms of the spectral energy density $E(k)$. That is, from (21),

$$k_s^2 = \int_{k_0}^{\infty} k^4 E(k)\,\mathrm{d}k\left[\int_{k_0}^{\infty} k^2 E(k)\,\mathrm{d}k\right]^{-1}, \quad (22)$$

where k is the two-dimensional scalar wavenumber.

Differentiating (22) with respect to time, we see that:

$$\frac{\partial}{\partial t}k_s^2 = \left[\int_{k_0}^{\infty} k^2 E(k)\,\mathrm{d}k\right]^{-2} \times\left[\int_{k_0}^{\infty} k^2 E(k)\,\mathrm{d}k\int_{k_0}^{\infty} k^4\frac{\partial E}{\partial t}\,\mathrm{d}k - \int_{k_0}^{\infty} k^4 E(k)\,\mathrm{d}k\int_{k_0}^{\infty} k^2\frac{\partial E}{\partial t}\,\mathrm{d}k\right]. \quad (23)$$

Now, in order to maintain a similarity spectrum, the time-variations of $E(k)$ must be proportional to $E(k)$ by a factor that is constant for all $k_0 < k < \infty$. Thus, under that condition, the rhs of (23) vanishes, and $\partial k_s^2/\partial t = 0$. In short, according to (21), k_s is a constant determined by the initial kinetic energy spectrum of the unresolvable motions and is thus presumed known. The moments $\overline{\nabla\zeta'\cdot\nabla\zeta'}$ and $\overline{\zeta'^2}$, however, are free to vary in accordance with (21).

Substituting from (21) into (20), we may rewrite the latter as:

$$\frac{\partial}{\partial t}\overline{\zeta'\boldsymbol{V}'} = -\frac{\pi}{4}\overline{\zeta'^2}\left[k_s^2\nabla\bar{\zeta}\int_0^R r^3 G(r)\,\boldsymbol{\Phi}(r)\,\mathrm{d}r + \nabla^2(\nabla\bar{\zeta})\int_0^R r^3 G(r)\,\chi(r)\,\mathrm{d}r\right]. \quad (24)$$

For convenience (3) and (6), the equations that complete the system, are repeated here:

$$\frac{\partial\bar{\zeta}}{\partial t} = -\bar{\boldsymbol{V}}\cdot\nabla\bar{\zeta} - \nabla\cdot\overline{\zeta'\boldsymbol{V}'}, \quad (25)$$

$$\frac{\partial}{\partial t}\overline{\zeta'^2} = -\bar{\boldsymbol{V}}\cdot\nabla\overline{\zeta'^2} - 2\overline{\zeta'\boldsymbol{V}'}\cdot\nabla\bar{\zeta}. \quad (26)$$

Again recalling that $\bar{\boldsymbol{V}} = \boldsymbol{k}\times\nabla\bar{\psi}$ and $\bar{\zeta} = \nabla^2\bar{\psi}$, we now observe that (24)–(26) is a closed system of stochastic differential equations in which the variables are $\overline{\zeta'\boldsymbol{V}'}$, $\bar{\psi}$ and $\overline{\zeta'^2}$. Those equations take into account the mutual nonlinear interactions between the resolvable large-scale motions and the unresolvable small-scale motions.

5. Some qualitative interpretations

Reasoning qualitatively from (24)–(26), one can anticipate the main effects of interaction between the resolvable and unresolvable motions during the early stages of evolution. At initial time, when the ensemble of unresolvable fields is presumed to be statistically isotropic and homogeneous, $\overline{\zeta'^2}$ is spatially uniform, and $\overline{\zeta' \boldsymbol{V}'} = 0$. Thus from (26) we infer that $\overline{\zeta'^2}$ is initially unchanging. Moreover, from (25), we see that $\bar{\zeta}$ is initially conserved following the mean motion. Indeed, if $\overline{\zeta' \boldsymbol{V}'}$ were zero at all times, then $\overline{\zeta'^2}$ would remain constant, and $\bar{\zeta}$ would be advected with $\bar{\boldsymbol{V}}$ at all times. In short, the coupling between the resolvable and unresolvable scales is brought about by the small-scale transport of vorticity induced by deformation of the unresolvable fields by the resolvable motion.

The effect of the latter process is described in (24). Let us recall that $G(r)$ is negative, and that the normalized correlation-functions $\boldsymbol{\Phi}(r)$ and $\chi(r)$ are positive for small r, so that the integrals appearing in (24) are both negative. We also observe that, in typical atmospheric flows, $\nabla^2 \bar{\zeta}$ is highly correlated with $\bar{\zeta}$, but in the negative sense: thus the two terms in the square brackets of (24) tend to compensate each other. The sign of the quantity in square brackets is initially determined by the "local" scale of the mean vorticity field $\bar{\zeta}$ relative to the characteristic scale of the initial ensemble of ζ'-fields. Where the gradient of $\bar{\zeta}$ is rather featureless, $\partial \overline{\zeta' \boldsymbol{V}'}/\partial t$ is proportional to $\nabla \bar{\zeta}$ by a *positive* factor; however, in concentrated regions of strong gradients of $\bar{\zeta}$, $\partial \overline{\zeta' \boldsymbol{V}'}/\partial t$ tends to be proportional to $\nabla \bar{\zeta}$ by a *negative* factor. One would thus expect that the effects of scale interactions in relatively featureless regions of the mean flow would be quite different from those in regions where there is considerable fine structure.

To be more specific, let us consider the most common situation, in which the local scale of the $\bar{\zeta}$-field is much greater than that of the ζ'-fields. Then $\partial \overline{\zeta' \boldsymbol{V}'}/\partial t$ is proportional to $\nabla \bar{\zeta}$ by a positive factor. Thus, since $\overline{\zeta' \boldsymbol{V}'} = 0$ initially, $\overline{\zeta' \boldsymbol{V}'}$ at slightly later times is also proportional to $\nabla \bar{\zeta}$ by a positive factor. Then, according to (26), $\partial \overline{\zeta'^2}/\partial t$ is proportional to $\nabla \bar{\zeta} \cdot \nabla \bar{\zeta}$ by a negative factor, and $\overline{\zeta'^2}$ decreases with time. Moreover, according to (25), the average effect of small-scale vorticity transport is to diffuse mean vorticity, but with a *negative* coefficient of eddy diffusivity.

At the other extreme, if the "local" scale of the $\bar{\zeta}$-field is only slightly greater than that of the ζ'-fields, the second term in the square brackets of (24) may dominate. This arises from the fact that the normalized correlation-function $\chi(r)$ decreases more slowly than $\boldsymbol{\Phi}(r)$ as r increases, simply because it involves derivatives of lower order: hence, the second integral in the square brackets of (24) is appreciably larger than the first. Thus, shortly after initial time, $\overline{\zeta' \boldsymbol{V}'}$ becomes nearly proportional to $\nabla \bar{\zeta}$, but by a *negative* factor. In such cases, $\overline{\zeta'^2}$ increases with time, and the average effect of the small-scale vorticity transport is to diffuse mean vorticity, with a *positive* coefficient of eddy diffusivity.

We conclude that the statistical effect of unresolvable motions is to transport mean vorticity, but either down-gradient or counter-gradient, depending on the local scale of the $\bar{\zeta}$-field relative to the characteristic scale of the unresolvable ζ'-fields. Whichever is actually the case, the form of (25) and (26) guarantees that the total enstrophy is conserved in the ensemble average.

6. Summary and conclusions

In the foregoing sections, the main purpose has been to analyze the interactions between motions on resolvable scales and those whose scale is too small to be resolved by values at a finite number of discrete points. In particular, our aim has been to derive a second-moment closure for the evolution of the average of a large ensemble of predicted states, originating in an ensemble of initial states. Each member of the ensemble of initial states has the same resolvable part, but its small-scale unresolvable part is chosen at random from an unbiased population, whose frequency distribution matches a given power spectrum of the unresolvable motions. The initial ensemble of unresolvable states has zero mean and is statistically isotropic and homogeneous.

Starting with the equation for conservation of vorticity in two-dimensional inviscid flow, we first derive evolution equations for the ensemble-averaged vorticity, which initially is just the resolvable vorticity, and for the deviation from that average. An approximate form of the latter, corresponding to second-moment closure, is shown to guarantee conservation of total kinetic energy and enstrophy in the ensemble average.

The ensemble-averaged vorticity and the quadratic statistics of deviations from that average are coupled through the mean transport of vorticity by the unresolvable small-scale motions. That transport, which depends on the anisotropy of the small-scale fields, is induced by the deformation or straining of the small-scale vorticity fields by the larger-scale resolvable motions, and conversely. The rate of change of the mean vorticity transport is found to depend primarily on the local gradient of mean vorticity, the local variance of the small-scale vorticity fields and on their scale. The evolution equations for those quantities, taken together with the evolution equation for the mean transport of vorticity by small-scale motions, comprise a closed system of stochastic differential equations that reflect the statistical effect of interactions between the ensemble average (resolvable) and deviations (unresolvable) from that average.

The effects of those interactions in the early stages of evolution depend crucially on the local scale of the average vorticity field, relative to the characteristic scale of the unresolvable vorticity fields. If the local scale of the average vorticity field is large, the effect of the mean small-scale transport of vorticity is to diffuse mean vorticity, but *counter*-gradient. If, on the other hand, the scale of the average vorticity field is only slightly larger than the characteristic scale of the unresolvable vorticity fields, it tends to diffuse mean vorticity *down*-gradient. In either case, the rate of diffusion is proportional to the local variance of vorticity, for at least short times. The effects at long times are not readily interpretable. It is, however, clear that the statistical effect of interactions between the resolvable and unresolvable scales of two-dimensional motion is not universally that of down-gradient diffusive transport of mean vorticity.

One of the reviewers has pointed out that our general conclusions are supported by the results of Lorenz (1953), for a special case of mean flow subjected to random perturbations.

Since (26)–(29) comprise a closed system, they provide the basis of a method of "subgrid-scale parameterization" of the statistical effects of unresolvable motions. In order to test the hypothesis underlying this method, we have planned a series of experiments in which numerical solutions of the system of stochastic eqs. (24)–(26) can be compared directly with the average of an ensemble of high resolution numerical solutions of (1). The ensemble of small-scale "unresolvable" fields, generated by the procedure outlined in the Introduction, will be the same in either calculation, but will be treated as "resolvable" in solving (1). An existing code for solving the barotropic vorticity equation has been modified for the latter purpose, and a spectral version of (24)–(26) is being developed. The results will be reported separately.

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REFERENCES

Daly, B. J. and Harlow, E. H. 1970. Transport equations in turbulence. *Phys. Fluids 13*, 2634–2649.

Deardorff, J. 1973. The use of subgrid transport equations in a three-dimensional model of atmospheric turbulence. *J. Fluids Eng. 95*, 429–438.

Deardorff, J. 1974. Three-dimensional numerical study of the height and mean structure of a heated planetary boundary layer. *Boundary Layer Met. 7*, 81–106.

Launder, B. E., Reece, G. J. and Rodi, W. 1975. Progress in the development of a Reynolds-stress turbulence closure. *J. Fluid Mech. 68*, 537–566.

Lilly, D. K. 1962. On the numerical simulation of buoyant convection. *Tellus 14*, 148–172.

Lorenz, E. N. 1953. The interaction between a mean flow and random disturbances. *Tellus 5*, 238–250.

Smagorinsky, J. 1963. General circulation experiments with the primitive equations. I. The basic experiment. *Mon. Wea. Rev. 91*, 99–164.

Starr, V. P. 1968. *Physics of negative viscosity phenomena*. New York: McGraw-Hill Book Company. 256 pp.

Thompson, P. D. 1986. A simple approximate method of stochastic-dynamic prediction for small initial errors and short range. *Mon. Wea. Rev. 114*, 1709–1715.