Numerical Tendency Computations from the Barotropic Vorticity Equation

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Abstract

As a preliminary test of the usefulness of the barotropic vorticity equation for numerical forecasting a series of tendency computations have been made for a period of ten days in February 1951. Despite the difficulty of comparing instantaneous tendencies with observed changes over a finite time interval (12 hours) some positive information was obtained concerning computational errors as well as errors in the atmospheric model. The computations support the conclusion reached by CHARNEY and ELIASSEN (1949) that the Rocky Mountains are of importance in their influence on the large scale upper level flow pattern over the United States.

1. Introduction

In a recent article by CHARNEY, FJØRTOFT and VON NEUMANN (1950)[2] a series of numerical forecasts for the 500 mb surface was made using the barotropic vorticity equation. Unfortunately, the number of forecasts was too small to permit general conclusions about the accuracy of the computations, nor was it possible to separate truncation errors from errors due to defects in the model. Final conclusions of this kind cannot be drawn until a large number of complete 24-hour forecasts have been made. Such computations are planned for the high-speed electronic computer being built at the Institute for Advanced Study.

In the four complete forecasts that were described in (I), it was found that the initially computed tendency gives a fairly good idea of the 24-hour change obtained by a complete integration. This is obviously still more true for a 12-hour forecast. Thus one might expect to obtain some useful results from a comparison between initial tendencies and the subsequent observed 12-hour change provided some proper way of comparison can be devised. With this idea in mind a series of tendency computations was carried through for a period in February 1951. It was possible to obtain some information from these computations regarding computational errors as well as defects in the model. It was also instructive to study the computed tendencies in some detail in that they furnish direct illustrations of the quasi-barotropic behaviour of the atmosphere.

It should be stressed that a series of complete forecasts has to be made as a final test of the computational scheme as well as of the model. We can obviously not obtain any information from these computations about errors due to the finite extrapolation in time and the extent to which computational errors put a limit on the time over which a forecast can be made.

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[2] In the following we shall refer to this article by (I).

This study should rather be looked upon as a series of experiments intended to serve as a guide for further work.

2. The solution of the vorticity equation

The reader is referred to (I) for a detailed discussion of the problems that arise when trying to solve the barotropic vorticity equation for a certain initial set of data. Here only a general outline is given of the procedure followed.

Both from empirical and theoretical investigations we know that there exists a level in the mid-troposphere which is approximately non-divergent. As in (I) we shall assume that the 500 mb level coincides with this level. (It will be seen that some of the major errors in the tendency computations are introduced by this assumption.) Thus the vorticity equation may be written

$$\frac{\partial \zeta}{\partial t} = - \mathbf{v} \cdot \nabla (\zeta + f). \qquad (1)$$

ζ is the relative vorticity, $\mathbf{v}$ is the horizontal wind velocity and f the Coriolis parameter. It was shown in (I) how this equation may be transformed into

$$\nabla^2 \frac{\partial z}{\partial t} = J\left(\frac{g m^2}{f} \nabla^2 z + f, z\right) = J(\eta, z), \qquad (2)$$

where the geostrophic wind equation has been used for the evaluation of the vorticity. Here z denotes the height of the 500 mb surface, m is the scale factor in the transformation from spherical coordinates to the particular map used in the computations, η is the absolute vorticity and $J(\alpha_1, \alpha_2)$ is defined as $J(\alpha_1, \alpha_2) = \partial\alpha_1/\partial x \cdot \partial\alpha_2/\partial y - \partial\alpha_2/\partial x \cdot \partial\alpha_1/\partial y$. It is our intention to solve this Poisson equation in order to obtain the initial value of $\partial z/\partial t$ and compare that tendency with the observed 12-hour change of z. For hand computation equation (2) is most rapidly solved by the relaxation method described by SOUTHWELL (1946). However, it is necessary to specify $\partial z/\partial t$ on the boundary. Since these quantities are not known, we are forced to introduce artificial boundary conditions. When carrying out a complete 24-hour forecast, this question deserves special attention, but as we only are going to compute the initial tendency we may assume that $\partial z/\partial t = 0$ on the boundary. This will undoubtedly introduce errors close to the boundary, but the magnitude of these errors decreases rapidly in going away from the boundary. The reason is that the effect of the correct boundary conditions at a fairly small distance from the boundary becomes equivalent to $\partial z/\partial t = 0$ on the boundary because of the interference of positive and negative boundary values. In the verification, no points have been used at a distance less than $2\,\Delta S$ from the boundary, ΔS being the grid interval used in solving the finite difference equation corresponding to (2).

For the solution of equation (2) we choose a rectangular area on the map with sides $p \cdot \Delta S$ and $q \cdot \Delta S$, where ΔS is the distance between the gridpoints in a square grid and p and q are integers. The coordinates of the grid points are given by $i \cdot \Delta S$ and $j \cdot \Delta S$ ($i = 0, 1, 2 \ldots p$; $j = 0, 1, 2 \ldots q$). $\nabla^2 z$ is computed for every other point ($i + j = 2n$) according to the formula

$$\begin{gathered}(\nabla^2 z)_{ij} = \frac{1}{(\Delta S)^2} [z_{i-1,j} + z_{i,j-1} + \\ + z_{i,j+1} + z_{i+1,j} - 4 z_{ij}] \\ i + j = 2n, \\ 1 \leq i \leq p - 1, \qquad 1 \leq j \leq q - 1\end{gathered} \qquad (3)$$

J_{ij} is then obtained for points where $i + j = 2n + 1$ according to

$$\begin{gathered}J_{ij}(\eta, z) = \frac{1}{4(\Delta S)^2} [(\eta_{i+1,j} - \eta_{i-1,j}) \times \\ \times (z_{i,j+1} - z_{i,j-1}) - (\eta_{i,j+1} - \eta_{i,j-1}) \times \\ \times (z_{i+1,j} - z_{i-1,j})] \qquad (4) \\ i + j = 2n + 1, \\ 2 \leq i \leq p - 2, \qquad 2 \leq j \leq q - 2\end{gathered}$$

and where $\eta_{ij} = \frac{g m^2}{f_{ij}} (\nabla^2 z)_{ij} + f_{ij}$. Thus the non-homogeneous term of equation (2) is obtained for points on the map forming a square grid with the grid interval $\Delta S \sqrt{2}$. The equation is solved by relaxation methods.

This procedure differs from (I) in that the vorticity is computed using a gridsize of ΔS, while equation (2) is solved using a grid interval of $\Delta S\sqrt{2}$. The scheme outlined here was chosen for the following reasons: In (I) it was found that the value $\Delta S \approx 736$ km was too large, in particular for the evaluation of the vorticity. Here ΔS was chosen considerably smaller for the computation of the vorticity ($\Delta S = 4°$ long. at lat. $45° = 315$ km). However, the time required for solving equation (2) by relaxation methods increases very rapidly with increasing number of grid points. In order to keep this time within reasonable limits, the relaxation was carried out for the larger grid size ($\Delta S\sqrt{2}$). The method cannot be used for a complete forecast, because tendencies are only obtained for every other point and new vorticities cannot be computed over the smaller grid after the first time extrapolation has been carried out. Finally it might be of some interest to mention that a complete calculation of $\partial z/\partial t$ takes four to five hours for an experienced computer for a grid with $p = 18$ and $q = 15$. This includes the evaluation of the initial field of z from the map.

3. Period under investigation and methods of verification

In all about forty computations were made in which tendencies were computed for every twelve hours for the period February 3, 03 GMT—February 13, 03 GMT over an area covering the United States, the major part of Canada and the western part of the Atlantic Ocean. For these computations the conventional analysis from the Weather Bureau —Air Force—Navy Analysis Center (WBAN) was used. It was pointed out in (I) that an objective analysis would have been preferable and the same conclusion may be drawn from the present computations. It was also realized that the data coverage over the oceans is too poor to permit an analysis that is accurate enough for this type of computation. For this reason it was decided only to verify the computations over the United States and southern Canada. In order to see to what extent the analysis may have influenced the results over these areas, the maps February 4, 15 GMT—February 12, 15 GMT were re-analysed without reference to the WBAN analyses using smoothness in the vorticity field as a basic guiding principle. The same computations then were carried out for these re-analysed maps.

Some simple statistical methods have been used for the estimation of the errors in the computations. From the two independent analyses the probable error, due to uncertainties in the analysis and the replacement of the differential equation (2) by a finite difference equation was computed. This involved the assumption of a Gaussian distribution of errors. For the comparison of the computed tendencies with actual observed changes over twelve hours several methods can be used. For the present study the computed value of $1/2\,[(\partial z/\partial t)_{t=t_0} + (\partial z/\partial t)_{t=t_0+\Delta t}] \cdot \Delta t = x_1$, was compared with the observed value of $(z_{t=t_0+\Delta t} - z_{t=t_0}) = x_2$ (Δt being 12 hours). This method has the advantage that no assumptions concerning the movement of the fall and rise areas have to be made, such estimates being always more or less subjective. On the other hand, it has the disadvantage that two computations are involved in each verification. Furthermore, this verification scheme cannot be used if the changes are quasi-periodic with a period less than 36 to 48 hours. In such cases all changes will be underestimated. During the period under consideration, this has not affected the computations to any considerable extent.

The comparison between x_1 and x_2 was made in a few different ways. The following quantities were computed:

1) The correlation coefficient r
2) The regression equation $x_1 = a\,x_2 + b$
3) A quantity K defined by

$$K = \frac{\sqrt{\Sigma(x_1 - x_2)^2}}{\sqrt{\Sigma x_1^2} + \sqrt{\Sigma x_2^2}}$$

K has the advantage of being a function both of r and a. $K = 0$ if $x_{1i} \equiv x_{2i}$ for all i, and is approximately 0.7 if $r = 0$, the exact value depending upon σ_1 and σ_2. (σ_1 and σ_2 being the root mean square of x_1 and x_2 respectively.) Quite generally $K \leq 1$.

It should be stressed that this study is not intended to be a complete statistical treatment of the computed data. It contains merely some order of magnitude estimates of errors in order to permit the discussion of physical aspects of the problem.

4. Computational errors

Apart from errors due to defects in the barotropic model the following errors arise in the computations:

1) Errors due to uncertainties in the analysis of the initial data.

2) Truncation errors, e.g. errors due to the replacement of the differential equation by a difference equation.

3) Errors depending upon the particular method of verification in replacing $\int_{t_0}^{t_0+\Delta t} (\partial z/\partial t)\, dt$ by x_1. Furthermore, there are inaccuracies in the determination of the observed change, x_2.

By comparing the tendencies that were obtained from the two analyses of the same synoptic situations we can obtain an estimate of the combined effect of 1) and 2). It was found that the probable error of the tendency computation introduced by these two factors was 100 feet/12 hours compared with a root-mean-square of the tendencies themselves of 370 feet/12 hours during this period. These errors may be considered to be distributed at random, thus giving rise to a probable error of 70 feet in the determination of x_1. From the same two sets of maps we find that the probable error in x_2 is 30 feet. We cannot obtain a direct estimate of the errors that are introduced by replacing $\int_{t_0}^{t_0+\Delta t} (\partial z/\partial t)\, dt$ by x_1. In spite of this it seems safe to conclude that the observed probable difference of 160 feet between x_1 and x_2 cannot be explained by computational errors. This is supported by other evidence as shown below.

It is now of some interest to examine the agreement between x_1 and x_2 somewhat more closely. The quantities r and K were computed for an area covering the United States and southern Canada for every tendency calculation that was made. In fig. 1 these values are given as functions of time. The solid curve represents the results that were obtained by using WBAN-analysis, the dashed curve is based on computations from the re-analysed maps. The two curves follow each other fairly closely except for the last few points of the diagram. This again indicates that factors other than analysis and the method of finite differences influence the results, which after all is very likely as we know that the atmosphere is not barotropic. The differences that are observed at the end of the period are mainly due to changes in the analysis on Feb. 11, 15 GMT and Feb. 12, 03 GMT. These were con-

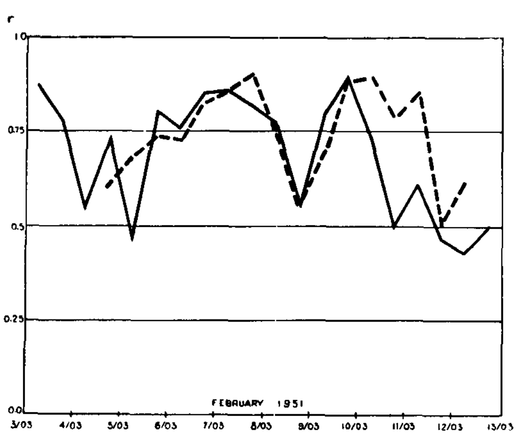

Fig. 1 a.

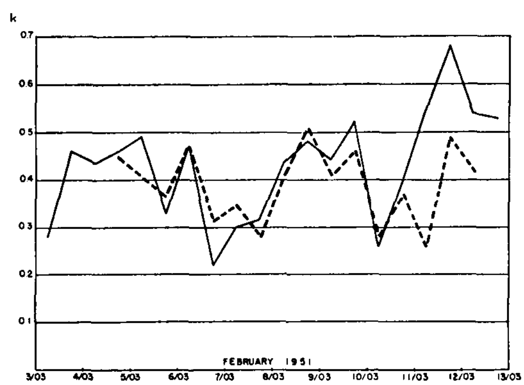

Fig. 1 b.

Fig. 1: Graphical representation of the variation of the correlation coefficient, r (a), and the verification coefficient, K (b). The solid lines represent the results obtained from the WBAN-analyses, the dashed lines those obtained from the reanalysed maps.

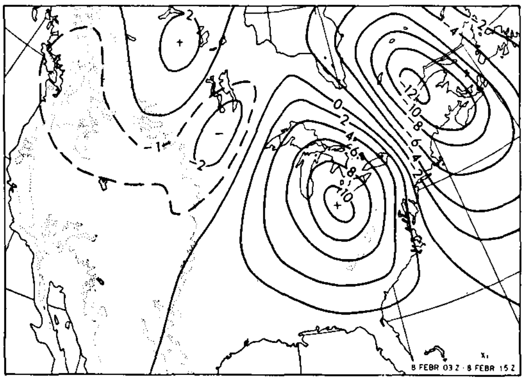

Fig. 2a.

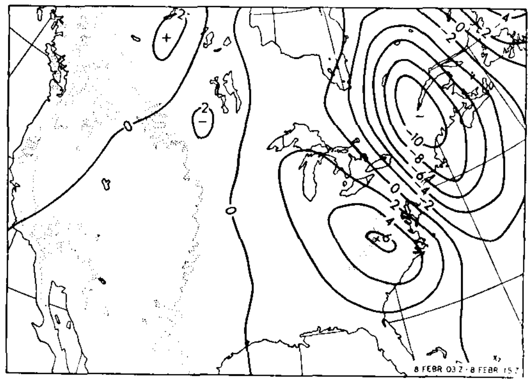

Fig. 2b.

Fig. 2: Computed mean tendency, x_1, in tens of feet/12 hours, for the period 8 Feb. 03 GMT—15 GMT, 1951 (a), and the corresponding observed change, x_2 (b). The situation was chosen to picture an average case ($r = 0.73$, $K = 0.41$).

siderable, in particular on the later occasion. It should be pointed out that when the number of observations is as large as it is over the United States, changes of the analysis to such an extent can be made only if some observations are neglected in one analysis but not in the other.

The mean values of r and K for the whole period are 0.74 and 0.43 for the WBAN-maps and 0.77 and 0.41 for the re-analysed maps, a scarcely significant difference. It is not immediately clear what these values mean, and we can not say what r and K would be for complete 24-hour forecasts. The iterative method used for that computation certainly introduces errors. Tentatively we may say that the values of r and K obtained here are fairly close to the values that would be obtained in a 12-hour forecast but somewhat larger than one might expect for a 24-hour integration. In order to give an idea of what the values of r and K mean the result of a computation is shown in fig. 2. The situation was chosen to picture an average case ($r = 0.73$, $K = 0.41$).

Returing to the curves in fig. 1 we next ask ourselves for the reason for the systematic variation in time of r and K that we observe. A closer analysis quite clearly shows that the relative errors of the computed tendencies decrease when the tendencies themselves become larger, in other words the results are better when well-defined rise and fall areas exist. Thus the computations were most successful on Feb. 6 and 7 when waves in the westerlies moved along the jet in a regular way. Besides this effect certain physical factors probably have given rise to these variations. The material is, however, insufficient for a closer examination of these aspects.

5. The barotropic model

In this whole approach to the problem of forecasting large-scale atmospheric flow patterns the two most important questions are:

1) To what extent does the atmosphere behave as a two-dimensional barotropic fluid?

2) How is the theory for a barotropic atmosphere best applied to the real atmosphere?

These questions have been discussed intensively during the last ten years since Rossby and collaborators presented the first ideas on long waves in the atmosphere (1939). From the results obtained in (I) and in the computations carried out here we can make some further comments on the importance of barotropic processes in the atmosphere.

It is well-known that the rotation of a fluid has a pronounced influence upon its motion. This was already noticed by TAYLOR (1921) who was able to demonstrate this effect in simple model experiments. The stronger the rate of rotation the more pronounced is the tendency for the motion to become quasi-similar in planes perpendicular to the axis of rotation. It seems reasonable to assume that an analogous effect is present in the atmosphere.

Here the motion is already kinematically restricted to take place in a hemispherical shell and for large-scale flow patterns we may thus expect that it is the vertical component of the rotation of the earth that is of importance. Consequently this tendency for similarity from level to level should be found at high and middle latitudes but not in equatorial regions. By and large this agrees with what is observed in the atmosphere. In middle latitudes the similarity from layer to layer is a striking feature of the flow (above the friction layer), while recent investigations of the tropics show large differences between the lower and upper troposphere (RIEHL, 1950). This gives some justification for treating the motion of the atmosphere in the middle latitudes as two-dimensional.

The atmosphere is a thermally active fluid and at first it may seem strange that worthwhile results can be obtained by treating it as a barotropic fluid. However, we know that it would take approximately one week for friction to reduce the kinetic energy of the atmosphere by 50 per cent if this loss were not replaced by baroclinic processes. The total kinetic energy is approximately constant, thus the baroclinic processes in the atmosphere do not replace more than about 10 per cent of the total kinetic energy in twenty-four hours. There are several reasons to believe that this change of potential energy into kinetic energy does not take place simultaneously all over the hemisphere but mainly in connection with what is known as real deepening or intensification of atmospheric systems. Thus outside such restricted areas the change in the atmospheric flow patterns are primarily the result of a *redistribution of kinetic energy*. Such a redistribution essentially is a barotropic process.

Even if the total kinetic energy of a barotropic fluid is conserved (apart from frictional dissipation) local intensifications of the flow may occur in a barotropic fluid. It has been shown (KUO, 1949; FJØRTOFT, 1950) that waves in parallel flow are unstable if certain criteria are fulfilled and YEH (1949) has pointed out that dispersion of kinetic energy may give rise to intensification of disturbances in the flow. Some examples have also been given indicating that these processes are of importance in the atmosphere. In most cases, however, it is difficult to tell whether baroclinic processes have not also affected the development. Having been able to compute the changes in the fluid that would take place if the atmosphere were barotropic, we may obtain an idea of where and when the baroclinicity plays a significant role. From this point of view it is of some interest to study the series of computations from Feb. 6, 03 GMT, to Feb. 8, 15 GMT. During this period a wave developed from a fairly small disturbance in the westerlies to a very intense trough. The 500 mb maps for Feb. 6, 03 GMT, Feb. 7, 03 GMT and Feb. 8, 15 GMT are shown in figs. 3a to 5a. The heavy lines give the distribution of absolute vorticity. The com-

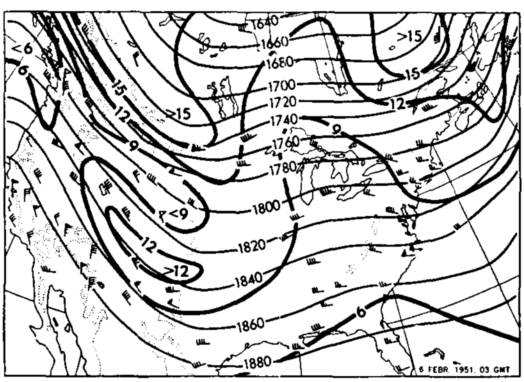

Fig. 3a.

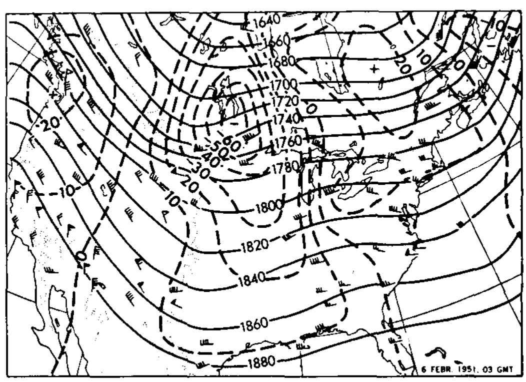

Fig. 3b.

Fig. 3: 500-mb chart at 03 GMT, 6 Feb. 1951. Thin lines are the contours labelled in tens of feet. The heavy lines in fig. a give the distribution of absolute vorticity in units of 10^{-5} sec^{-1} and the dashed lines in fig. b show the computed tendency in tens of feet/12 hours.

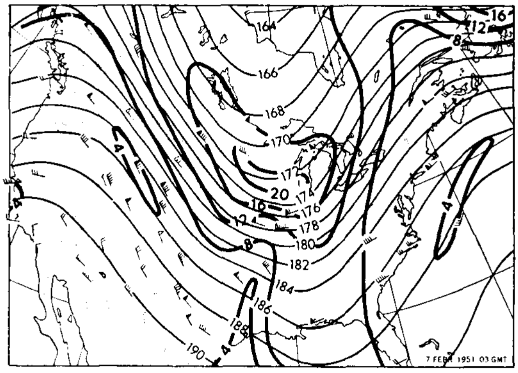

Fig. 4 a.

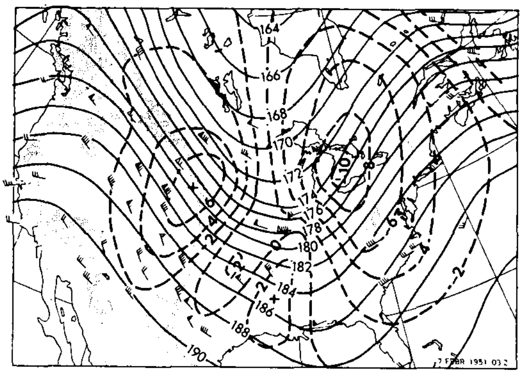

Fig. 4 b.

Fig. 4: 500-mb chart at 03 GMT, 7 Feb. 1951. Legend as in fig. 3.

puted tendencies are given in figs. 3 b—5 b (dashed lines). We observe that the changes during the first 24 hours seem mainly to be the result of a redistribution of the vortex lines, and there is only a slight increase of the intensity of the vorticity maximum in spite of the fact that the wave is amplifying. This indicates that the changes are quasi-barotropic (the diagram in fig. 1 shows that the most successful tendency computations for the whole 10-day period were made during these 24 hours). It is now interesting to see that the development of the wave, at least qualitatively, is indicated by the barotropic tendencies. Both on Feb. 6, 03 GMT and Feb. 7, 03 GMT (in fact also on Feb. 6, 15 GMT, not shown here) the zero-line of the tendency field is located well to the west of the trough line as computed barotropically and the fall area in front of the trough is more intense than the rise area behind (figs. 3 b, 4 b). On Feb. 8, on the other hand, we observe a pronounced increase of the vorticity in the trough. It is also seen from fig. 1 that the tendency computations become worse during this period and a complete 24-hour forecast probably would give still larger discrepancies between the computed and observed changes of the flow pattern. Here baroclinic effects seem to have played an important role in the development. From this qualitative discussion one therefore gets the impression that the initiation of the development of the trough was barotropic while baroclinic effects became more and more important for the further intensification.

In fig. 3 another illustration of the quasi-

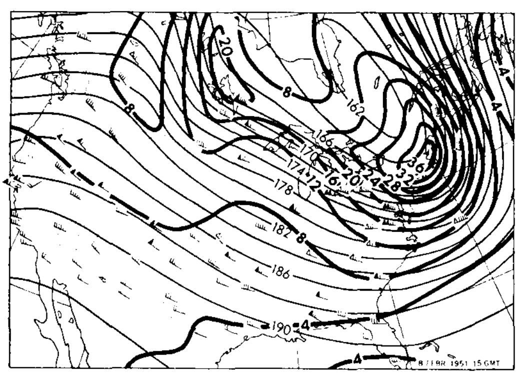

Fig. 5 a.

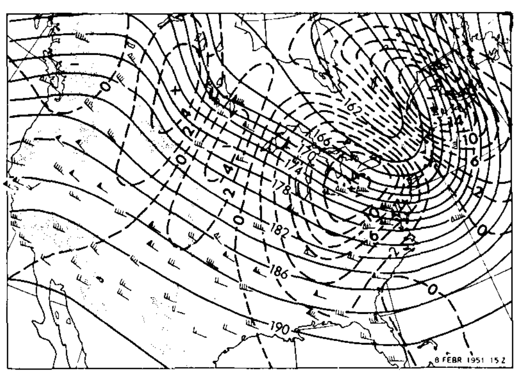

Fig. 5 b.

Fig. 5: 500-mb chart at 15 GMT, 8 Feb. 1951. Legend as in fig. 3.

barotropic behaviour of the atmosphere is shown (northern US around long. 100° W). We see clearly how the contour lines of the 500 mb surface are ajusted when a strong jet pushes into a region where the flow is less intense. The intensification of the gradient is clearly indicated in the tendency computations and agrees by and large with the observed changes.

By inspecting a series of barotropic tendency computations of this kind it is possible to formulate qualitative rules for the changes that may be expected from different flow patterns. This is not the place for a detailed discussion of such rules; it should merely be stated that many of them are in accordance with empirical rules that have proven to be useful for the daily prognosis of the flow pattern at upper levels.

Up to this point very little has been said about the best methods for applying these barotropic tendency computations to the real atmosphere. We have simply assumed that the flow at 500 mb is governed by the barotropic vorticity equation, in other words that the 500 mb surface is the level of non-divergence. This is not in accordance with reality, and considerable errors are introduced by the assumption. Some idea of these errors were obtained in the following way: For each gridpoint in the area of verification the regression coefficient a was computed using all tendency computations that were made. Systematic variations of a were found as shown in fig. 6. During this period a had a maximum value of approx. 1.4 in the Great Lake Region

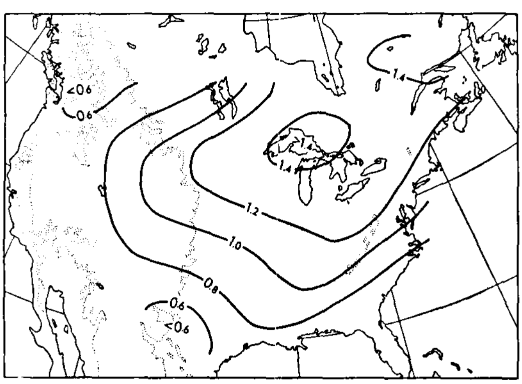

Fig. 6: Geographical distribution of the regression coefficient, a, computed from the whole material.

and was less than 0.6 over parts of the Rocky Mountains. This indicates that the changes as computed for an equivalent barotropic atmosphere were stronger than the observed changes at 500 mb in the Middle West, while the reverse was true over the mountains. To some extent this may have been caused by the use of the geostrophic assumption which results in an overestimate of the vorticity advection in troughs and an underestimate in ridges. During this period a mean trough was located over the middle west where the computed tendencies were too strong and a ridge is found over the Rocky Mountains. However, the differences are too large to be explained merely as an effect of the geostrophic approximation. Instead we may look upon them as indications of a systematic variation in the height of the level of non-divergence, the level lying below the 500 mb surface where $a > 1$ and above where $a < 1$. Thus the lines of constant a as a first approximation may be considered as contour lines of the level of non-divergence. It has been pointed out by CHARNEY (1947) and others that such variations of the height of that level may be expected in the long waves in the westerlies. In this particular area the influence of the Rocky Mountains also seems very important. As the level of non-divergence is located approximately in the middle of the atmosphere (with respect to pressure) one might expect to find it at higher elevation over the mountains. As the computations were fairly restricted both in time and space we were not able to examine more closely the cause of the varation of the height of the level of non-divergence. The values of a as computed here, however, strongly indicate that the assumption of the 500 mb level as the level of non-divergence introduces considerable errors. In (I) it was indicated that integrating through the vertical and thus treating some kind of a mean atmosphere tends to remove this difficulty. However, the interpretation of the flow pattern with respect to weather then becomes more difficult.

Another possible effect of the mountains is indicated in these computations. It was found that the constant factor b in the regression equation has a very characteristic distribution over the area under consideration. (cf. fig. 7). This may be explained as a result of neglecting the divergence and convergence

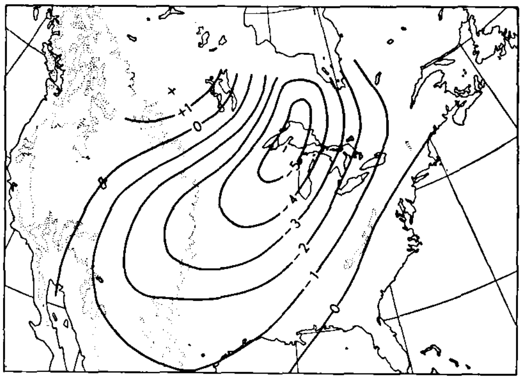

Fig. 7: Geographical distribution of the constant factor, b, in the regression equation, computed from the whole material (tens of feet/12 hours).

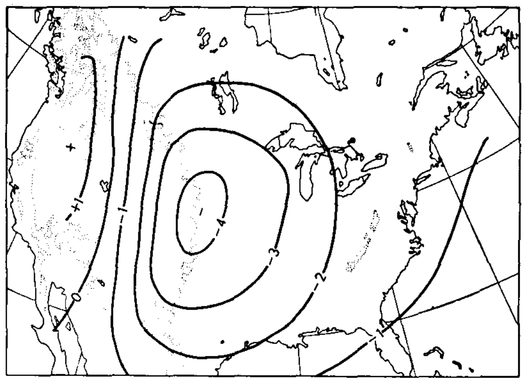

Fig. 8: Computed tendency field (in tens of feet/12 hours) as a result of horizontal divergence and convergence caused by the mountain using $^1/_2\, \bar{\mathbf{v}}$ as representative for the mean surface wind during the period.

that is caused by the mountains when the flow is forced over them. We have made an estimate of the mean 12 hour change of the 500 mb surface caused by this effect during the period by assuming that the divergence term in the vorticity equation may be computed by putting $\mathrm{div}_2\, \mathbf{v} = \mathbf{v}_0 \cdot \nabla h$, where $\mathbf{v}_0$ is the surface wind and h is the height of the mountains. It was assumed that $\mathbf{v}_0 = {}^1/_2\, \mathbf{v}$ and the tendencies thus obtained are shown in fig. 8 expressed in 10 feet/12 hours. The resemblance between fig. 7 and fig. 8 is quite good both with respect to the structure of the pattern and the magnitude of the changes, in particular as the assumption that $\mathbf{v}_0 = {}^1/_2\, \mathbf{v}$ is a fairly crude approximation. By and large the computations support an idea already put forward by CHARNEY and ELIASSEN (1949).

The disagreement between the observed changes of the flow pattern and those derived from the tendency calculations here have been interpreted as a result of the variation of the level of non-divergence and as an effect of the mountains.

A few attempts were made to correct the barotropic computations in a manner indicated in (I) and in CHARNEY (1951) by assuming that the thermal wind rather than the actual wind does not vary in direction with height and that temperature is advected horizontally. The changes in the 500 mb tendency were small and gave no systematic improvement. The surface tendencies, which were also calculated by the method, were for the most part of the right sign. We were, however, led to the conclusion that the advective hypothesis is not satisfactory as the correlation between the computed and observed patterns was poor.

Summarizing the results of these computations the following should be noticed:

1) The data coverage over the United States seems sufficient to give the initial conditions with the accuracy that is required for the present type of computation. Over the oceans a much denser net of upper observations must be established before successful forecasts can be made by the present methods.

2) A severe defect of the model used in these computations seems to be the assumption that the 500 mb level coincides with the level of non-divergence.

3) Computations of this kind are important in that they give us a method of *separating* purely barotropic effects from baroclinic processes. This will be of great help when trying to extend the present model of the atmosphere to incorporate the baroclinicity.

Acknowledgments: The writers wish to acknowledge the benefit they have received from many stimulating discussions about these computations with Dr M. Neiburger, Mr T. V. Davies and Mr J. Smagorinsky.

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