On the Effective Radiation to Various Parts of a Cloudless Sky

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Abstract

From the author's formula for the relative radiation as a function of temperature and humidity conditions in the atmosphere, a formula is deduced for the zenith distance variation of the effective radiation. This formula is compared with observational data and empirical formulae. Finally, the constancy of the representative zenith distance, for which the effective radiation equals the effective radiation integrated over the whole hemisphere, is investigated.

Introduction

The long-wave radiation from the sky may be characterized in quite different, but closely related, ways. We could either give the figure for *the downward radiation D* itself, or the figure for the difference between the black-body radiation at the surface temperature and the downward radiation, i.e. *the effective radiation F*. To eliminate, at least to a very large extent, the temperature dependence of the radiation, we could thirdly give the percentage figure for the ratio between the effective radiation and the earth's black-body radiation. This ratio has been called *the relative radiation r* by the present author (1950).[1]

Observational data and empirical formulae

Measurements of the effective radiation towards various parts of a cloudless sky by means of small-aperture instruments have been carried out by several investigators. Apart from the rather primitive measurements by HOMÉN (1897), the first measurements of this kind were made by ÅNGSTRÖM (1915). Since then investigations have been made by DINES & DINES (1927), DUBOIS (1929), SÜSSENBERGER (1935) and RAMDAS, SREENIVASIAH & RAMAN (1937). From all such investigations it is obvious that the effective radiation, and so the relative one, is largest in the vertical direction and that it decreases with increasing zenith distance.

LINKE (1931) found an empirical formula (1), i.e. a trigonometric power law, which fitted very closely the measurements by DUBOIS, at least for zenith distances up to about 70°, viz.:

$$\frac{F(z)}{F(0)} = \frac{r(z)}{r(0)} = \cos^L z, \tag{1}$$

where $F(z)$ is the effective radiation and $r(z)$ the relative radiation at the zenith distance z. The exponent L was supposed by LINKE to depend upon the humidity at surface. It has a value generally between 0.2 and 0.5. In fig. 1 are shown curves for these two values of L as well as for $L = 1$, i.e. the simple cosine function. The solid dots represent observed values by RAMDAS and collaborators (see Table II), for which L approximately equals 0.4.

SÜSSENBERGER (1935) tried to find a more

[1] Also the figure for $(1-r)$ could be used and defined as e.g. *the relative downward radiation.*

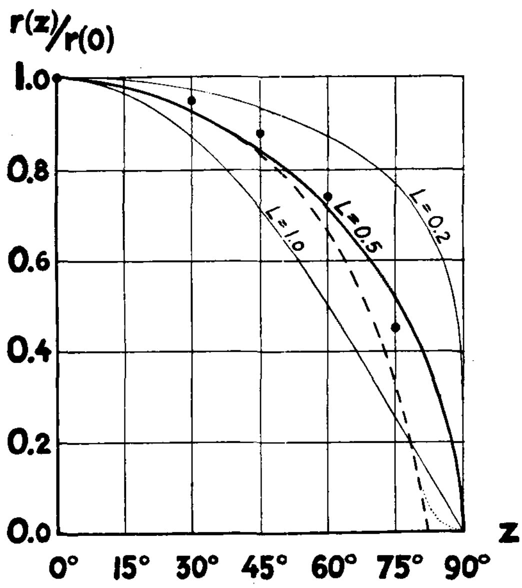

Fig. 1. The ratio $r(z)/r(0)$ given as a function of the zenith distance z. Full lines represent LINKE's formula (1) for various values for the exponent L. The broken line represents our formula (7) for $L = 0.5$, while the dotted line gives a correction to this curve, arising from certain limitations of our basic formula (3). Solid dots show values observed by RAMDAS, SREENIVASIAH & RAMAN.

accurate formula, and he found one, which followed the empirical data up to about 85°, viz.:

$$\frac{r(z)}{r(0)} = 1.1 - 0.1 \left\{ \frac{90}{100 - z} \right\}^S, \tag{2}$$

where z is given in degrees(!) and where S has values generally between 1.0 and 1.6. The formula was pointed out by SÜSSENBERGER not to be valid for $z < 10°$, where instead $r(z)/r(0) = 1.00$.

Formulae previously given by the present author

In a recent paper (1950) the author presented a general method, based on the water-vapour spectrum, for calculating effective radiation. The method was applied to a set of cloudless atmospheres, where the temperatures and humidities were varied systematically. It was then possible to deduce a formula, which for all these atmospheres gave values in good agreement with the results obtained by the general method. As the computations were made under the assumption of purely vertical radiation, the formula was properly valid only for $z = 0$:

$$r(0) = A + 0.06\,t_0 + (0.18 + 0.12\,w)\,\gamma_m - 10 \ln w\varrho_0\,\%, \tag{3}$$

where A is a constant, somewhat uncertain as to its numerical value, and t_0 is the temperature (in °C). ϱ_0 is the absolute humidity (in gram/m³) at the surface, γ_m is the mean value (up to about 2 km) of the temperature lapse-rate (in °C/km), and w is the "humidity factor", defined as

$$w = \frac{1}{\varrho_0} \int_0^\infty \varrho \sqrt{\frac{p}{1000}}\, dh,$$

where p is the pressure in mb and h is the height in km. It is readily seen that $w\varrho_0 \approx 0.9\,W$, where W is the amount of precipitable water, given in mm.

To obtain now a formula for $r(z)$ we must introduce the apparent lapse-rate $\gamma_m \cos z$ in place of γ_m in (3), and $w/\cos z$ for w, as shown by the author (1951). We then find

$$r(z) = r(0) - 0.18\,\gamma_m + 0.18\,\gamma_m \cos z + 10 \ln \cos z \tag{4}$$

Denoting the relative radiation integrated over the whole hemisphere by r, we have

$$r = 2 \int_0^{\frac{\pi}{2}} r(z) \sin z \cos z \, dz \tag{5}$$

From (4) and (5) we obtain

$$r = r(0) - 0.18\,\gamma_m + \frac{2}{3}\,(0.18\,\gamma_m) + \frac{1}{2}\,(10),$$

which can be written

$$r(0) - r = 5.0 + 0.06\,\gamma_m \tag{6}$$

Derived formulae for the ratio r(z)/r(0)

Dividing equation (4) with $r(0)$, we obtain

$$\frac{r(z)}{r(0)} = 1 - \frac{0.18\,\gamma_m}{r(0)}\,(1 - \cos z) - \frac{10}{r(0)} \ln \frac{1}{\cos z} \tag{7}$$

But

$$-L \ln \frac{1}{\cos z} = \ln \cos^L z = \cos^L z - 1 -$$
$$- 1/2\,(\cos^L z - 1)^2 + 1/3\,(\cos^L z - 1)^3 - \ldots$$

Substituting in (7)

$$L = \frac{10}{r(0)} \qquad (8)$$

and ignoring terms of third and higher order, we obtain

$$\frac{r(z)}{r(0)} = \cos^L z - 1/2\ (\cos^L z - 1)^2 -$$
$$- 0.018\ L\gamma_m (1 - \cos z), \qquad (9)$$

where L takes values generally between 0.2 and 0.5, corresponding to $r(0) = 50\,\%$ and $20\,\%$, respectively. At least for small zenith distances we may neglect the last two terms in (9), thus obtaining LINKE's empirical formula (1). We have also shown by (8) and (9) that his exponent L is a function of $r(0)$ and thus chiefly depends on ϱ_0, as supposed by LINKE. We have further obtained in (9) two correction terms to LINKE's formula; the maximum magnitudes of these terms are as follows (computed for $L = 0.5$ and $\gamma_m = 10$):

Table I. Maximum magnitudes of correction terms to Linke's formula

z	$1/2\,(\cos^L z - 1)^2$	$0.018\,L\gamma_m(1-\cos z)$
10	0.00	0.00
20	0.00	0.01
30	0.00	0.01
40	0.01	0.02
50	0.02	0.03
60	0.04[1]	0.05
70	0.09[1]	0.06
80	0.17[1]	0.07

[1] These figures are somewhat increased (to 0.05, 0.13 and 0.29, respectively) by terms ignored in (9), viz. $- 1/3\,(\cos^L z - 1)^3$, etc.

We see from the table that also under extreme conditions, the LINKE formula is valid with good approximation up to about 60° or even 70°, which was the value found empirically by LINKE himself, when determining L from the observed ratio $r(z)/r(0)$ at $z = 60°$.

In fig. 1 we have also drawn a broken curve representing equation (7) for $L = 10/r(0) = 0.5$ and $\gamma_m = 0$. We see that according to this curve the ratio $r(z)/r(0)$ equals zero for a zenith distance less than 90°. The author has given in his last paper (1951) an explanation of this fact, viz. the basic formula (3) is not valid for the very large values of the apparent humidity factor $w/\cos z$, which occur when the zenith distance approaches 90°. The author has also given a formula, viz.

$$r(z) = K_1 \cdot e^{-0.0124 \frac{w\varrho_0}{\cos z}},$$

which holds for $z > z_Q$, where z_Q is called the critical zenith distance, defined by

$$\cos z_Q = \frac{w\,\varrho_0}{K_2}$$

The above quantities K_1 and K_2 vary somewhat with the temperature of the atmosphere but take values approximately equal to 25 and 80, respectively.

For the case studied in fig. 1, the critical zenith distance approximates 69°. The dotted curve in the figure gives that correction to the broken curve, which results from the above discussion. It can be seen from the figure that this correction to formula (7) is of very small significance.

Comparisons with empirical data

As mentioned above, LINKE recommends that L be determined with $z = 60°$. In fig. 2 we have plotted values of L, obtained in this way, as a function of the surface humidity, using measurements of DUBOIS (according to LINKE, 1931), SÜSSENBERGER (1935) and RAMDAS and collaborators (1937). Also shown are curves for extreme values of L, obtained from (8) by aid of (3) when using those extreme values of w, γ_m and t_0, which were found by the author (1950) from Swedish data. We find from the figure that most of the empirical data fall within the theoretical spread; the SÜSSENBERGER data, however, are too low to lie within the spread area.

To supplement the empirical formulae (1) and (2) we have obtained a third formula for the zenith distance dependence of the relative radiation, viz. equation (7). The three formulae

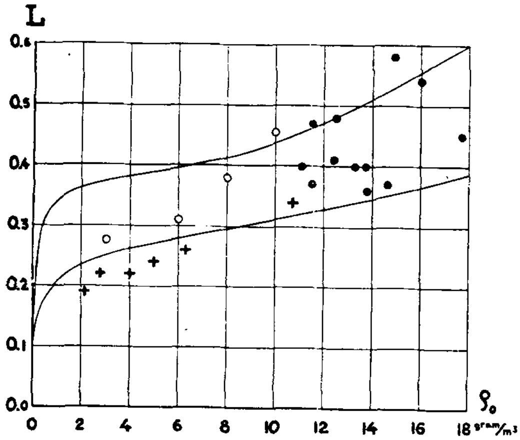

Fig. 2. The exponent L in LINKE's formula given as a function of the absolute humidity at the surface. Circles represent values observed by DUBOIS; crosses refer to SÜSSENBERGER and solid dots to RAMDAS, SREENIVASIAH & RAMAN. The curves bound the area of extreme spread, according to our theory.

will now be compared between themselves and with empirical data. The first series of such data is the one by RAMDAS and collaborators which was mentioned above. The measurements were all made during one single night (May 2, 1936) under "ideal conditions" as the authors say. In Table II we find in the first column average values computed from these twelve observations. The values for L and S, used in the columns to the right, have been obtained with $z = 60°$. It is seen from the table that up to this zenith distance there are no significant differences between empirical and computed values. For greater zenith distances, however, our values seem to be more accurate than values obtained from (1) and (2).

Table II. Comparison between computed values and Ramdas' data for the ratio r (z)/r (o)

z	Observed by RAMDAS and coll.	Calculated from (1) $L = 0.43$	(2) $S = 1.58$	(7) $L = 0.38$[1]
0	1.00	1.00	(1.00)	1.00
30	0.95	0.94	0.95	0.95
45	0.88	0.86	0.88	0.87
60	0.74	0.74	0.74	0.74
75	0.45	0.56	0.34	0.49

[1] $\gamma_m = 0$. This assumption influences the results but slightly.

Another comparison can be made with measurements by DINES & DINES (1927), viz. their annual means of the radiation to various parts of the sky. Here again we have obtained L and S with $z = 52{,}5°$. We see here, too, that there are no significant differences up to this value. For $z = 67.5°$ however, our formula gives the closest agreement with empirical data, while SÜSSENBERGER's formula seems to be the best one for $z = 82.5°$.

Table III. Comparison between computed values and the data by Dines & Dines for the ratio r (z)/r (o)

z	Observed by DINES & DINES	Calculated from (1) $L = 0.30$	(2) $S = 1.37$	(7) $L = 0.28$[1]
7.5	1.00	1.00	(1.00)	1.00
22.5	0.99	0.97	0.98	0.98
37.5	0.94	0.93	0.94	0.94
52.5	0.86	0.86	0.86	0.86
67.5	0.73	0.75	0.70	0.73
82.5	0.20	0.54	0.15	0.43

[1] $\gamma_m = 0$. This assumption influences the results but slightly.

The representative zenith distance

DINES & DINES (1927) have pointed out that the downward radiation from the cloudless atmosphere as a whole is very closely the same as the radiation at the zenith distance 52.5°[2]. They also refer to RICHARDSON (1922), who found by some kind of deduction that what we call the representative zenith distance has a value about 55°. What is here said as to downward radiation, obviously holds for effective and relative radiation as well.

It is clear that the relative radiation $r(z)$ of necessity equals r for a certain z, as it is, for $z = 0$, larger than the hemispherical value r, and approaches zero for $z = 90°$. We denote this zenith distance z_r and will attempt to deduce it from the equations given above. Then the question is if this representative zenith distance may be treated as a constant.

[2] In fact it is seen from their data that the representative zenith distance has a variation between 52° and 57°. ROBINSON (1947) has obtained the same result.

Starting from the formula by LINKE (1), we have according to (5):

$$r = \frac{2\ r(0)}{L + 2},$$

which was indeed found by LINKE himself.

By definition $r(z_r) = r$ we find from (1)

$$r = r(0) \cos^L z_r,$$

and eliminating $r(0)$ in these two equations, we obtain

$$\cos z_r = \sqrt[L]{\frac{2}{L + 2}}. \qquad (10)$$

For $L = 0.2$ equation (10) gives $z_r = 51.6°$, and for $L = 0.5$ it gives $z_r = 50.2°$; values not too far from the empirical result, given above.

As SÜSSENBERGER (1935) has himself pointed out, his formula (2) is not as suitable as LINKE's for deriving a formula for the hemispherical radiation r, nor could we obtain one like (10) for the representative zenith distance. By graphical integration for certain values of S, however, we find that z_r equals 54°, apart from a small variation with S.

We can also start from (4) and insert there the value of $r(0)$ to be found in (6). For $z = z_r$ we then obtain

$$r(z_r) = r + 5.0 + 0.18\ \gamma_m \cos z_r - 0.12\,\gamma_m + \\ + 10 \ln \cos z_r,$$

and again setting $r = r(z_r)$, we find

$$10 \ln \cos z_r + 0.18\ \gamma_m \cos z_r = \\ 0.12\ \gamma_m - 5.0$$

Inserting $\gamma_m = + 8°/\text{km}$, we obtain $z_r = 52.2°$, while a lapse-rate of $- 8°/\text{km}$ gives $z_r = 53.0°$. In the simple case with isothermal conditions, we find $z_r = 52.7°$.[1]

Obviously both the constancy of the representative zenith distance and the empirical value itself seem to be well confirmed. We may then echo these words of DINES & DINES, viz.: "This result is a very fortunate one, for it renders it possible to get a good idea of the sky radiation as a whole — an important meteorological quantity — from one single observation, and will greatly faciliate the possibility of using a self-recording radiometer."

Conclusion

Concerning the empirical formula given by LINKE for the zenith distance variation of the effective radiation, we have shown that his formula may be derived from the author's formula for the effective radiation, but that it is strictly valid for very small zenith distances only. For larger zenith distances we have given the necessary correction terms. As to the exponent L in LINKE's formula, it has been shown that it varies with the relative radiation and thus essentially with the humidity at surface. The formula empirically given by SÜSSENBERGER, on the other hand, has no theoretical background.

From comparisons made between observed data and computed values from our formula and from the empirical ones, we find a good agreement, at least for $z \leq 60°$. Our formula and that of SÜSSENBERGER seem to be the most accurate ones.

Finally we have shown that there exists an almost constant zenith distance which is representative in the sense that parallel radiation in that direction equals the integrated hemispherical radiation.

[1] ROBINSON (1947) has called attention to the connection between what we call the representative zenith distance and the factor 1.66, found by ELSASSER (1942); a factor which makes computations based on the assumption of vertical radiation applicable to diffuse radiation. The present author (1951) found for the same factor the mean value 1.65, which for isothermal atmospheres could also be found from the equation

$$x = \frac{1}{\cos z_r}.$$

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