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Two-dimensional Flow around a Circular Barrier in a Rotating Spherical Shell

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Abstract

An experiment has been recently set up in the hydrodynamics laboratory to study the flow of fluid around a circular barrier. In a rotating spherical shell of liquid, a small cylindrical obstacle is moved zonally in mid-latitudes and observations are made of the resulting perturbed flow. If the flow relative to the obstacle is westerly, a strong anticyclonic circulation is set up faround the barrier and wave motions exist around the globe. When the basic flow is easterly with the same relative speed, no circulation is found, but the wake is very long and broad and the fluid in it has little motion relative to the obstacle. When viewed with respect to the shell, the obstacle moving eastward creates a strong westerly jet at its latitude.

1. Introduction

The following note contains some very preliminary results of a new experiment in the hydrodynamics laboratory of the University of Chicago. This experiment was designed to simulate the flow of air around a circular mountain on the earth.

Some previous work in this laboratory concerned the dynamics of a fluid polar vortex moving on the earth or in a rotating spherical shell. The first results of this work have been described by FULTZ (1950). One of the factors of importance in this problem involved the pressure force exerted by the surrounding fluid on the boundary of the vortex. Since a knowledge of this force requires, in general, a prior knowledge of the fluid motion in the environment, a search was made to discover the nature of the flow around circular obstacles immersed in a uniform zonal current. A study by STEWART (1948), involving this type of motion in a steady state, introduced assumptions which were too severe in our case due to the great size of the vortex compared to the earth model. A paper by ERTEL (1943), however, suggested that a solution could be obtained on a sphere by using certain of the harmonic components of the wave motion in a zonal current.[1] Due to various complications the knowledge of the pressure forces computed from this theoretical motion was not particularly useful in an interpretation of the polar vortex experiments. It was decided, however, that the environmental motions were of independent interest and that a new experiment should be set up to study these motions in the laboratory.

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[1] This solution was found prior to our work (Long) by C. C. LIN of the Massachusetts Institute of Technology. Dr LIN kindly provided us with a copy of a manuscript of this work which was unpublished at that time. These solutions of the vorticity equation are related to those given by S. M. NEAMTAN, 1946: The motion of harmonic waves in the atmosphere, *J. Meteor.*, **3**, 53—56.

Fig. 1. Spherical apparatus adapted to the obstacle experiment.

2. Description of the apparatus

The experimental apparatus is the same as that used in the polar vortex work and in certain experiments on the general circulation (FULTZ, 1949). It consists of two concentric hemispheres of mean radius 10 cm with water filling the space between them. This shell has a depth of 1.6 cm. The two bowls are rigidly connected and are rotated as a unit by an outer shaft. After a sufficiently long period of rotation, the water moves as a solid at the same angular velocity as the bowls. Fig. 1 is a picture of this apparatus. The obstacle in this photograph is a cylindrical piece of wood of angular radius 10 degrees of latitude, shaped to the curvature of the outer and inner spheres. Up to the present time its center has been kept at 45 deg. latitude. It fits rather snugly between the two bowls but is loose enough to move relative to them. The obstacle is connected by a thin arm to a shaft within the one used to rotate the bowls. The inner and outer shaft are driven by two different motors so that the sphere and the obstacle rotate independently. Thus the obstacle can be given any desired zonal motion relative to the basic rotation of the bowls.

Several different interpretations of the experiment are kinematically possible. Thus the motion may be viewed relative to the obstacle and the problem interpreted as that of the flow around a stationary circular mountain extending through the entire atmosphere. In this case if the cylinder is moved faster than the bowls it will appear as though an east wind is flowing past the obstacle. If it is moved slower than the bowls a relative west wind is effectively created. On the other hand the motion of the fluid may be viewed relative to the spheres and interpreted as the escape motion around a moving circular cylinder. It is not strictly correct to go from one interpretation of the experiment to the other by addition or subtraction of a constant angular velocity. Among other points the "no-slip" boundary condition at the bowl surfaces is always present. Mechanically speaking, however, it is much easier to arrange to move the cylinder relative to the shell than to generate a simple zonal current relative to the spheres. We have actually checked this interpretation to some extent by rotating both obstacle and bowls together. We then slow them down or speed them up suddenly in unison. When this is done the water maintains its angular momentum for quite some time and moves past the barrier. We have found that the features of the flow described below are still present.

3. Observing techniques

The motion of the water must be made visible by putting some sort of foreign element in the fluid to serve as a tracer. This presented many difficulties which have not yet been overcome in our particular case, but the most satisfactory tracer was found to be an aluminum tristearate powder from F. Stresen-Reuter Co. Although the particles are ordinarily too small for photography, some sodium hydroxide mixed in the water causes them to hydrolyse into a partial precipitate of aluminum hydroxide. The particles swell and aggregate into sizeable lumps which are then easily visible. We have not been successful yet with the usual aluminum powder types of internal tracers. The motion of the water seems to be too slow to hold the samples we have tried in good suspension.

The motion relative to the rotating sphere is observed both visually and photographically by using a rotoscope modelled after that designed by D. Thoma (FULTZ, 1950). This consists of a Dove reversing prism fixed in a rotating barrel. The line of sight is placed along the axis of the rotating object. Since the image in the prism rotates twice during each revolution of the prism itself, either the bowls or the revolving cylinder will appear stationary if the barrel is turned one-half their rate of rotation. Fig. 2 is a picture of the spherical apparatus as seen through this instrument.

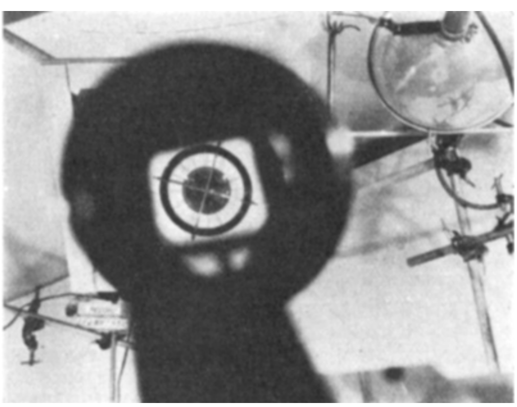

Fig. 2. Spherical apparatus as seen through the rotoscope.

4. Description of observations

The actual observations are quite meager at this time and, aside from a few photographs, are qualitative in nature. Moreover, unless a somewhat different technique is used, rotoscope observations are not possible at latitudes lower than about 30 degrees. The following description then, is subject to change in detail although the general pattern is well established.

Fig. 3, 4 and 5 are a sketch and two photographs of the flow relative to the obstacle for a basic west wind current. In these and in the following figures the north pole is at the center of the picture. Westerly winds flow counterclockwise and easterly winds, clockwise. Although the sketch is largely schematic the spacing between the streamlines is a

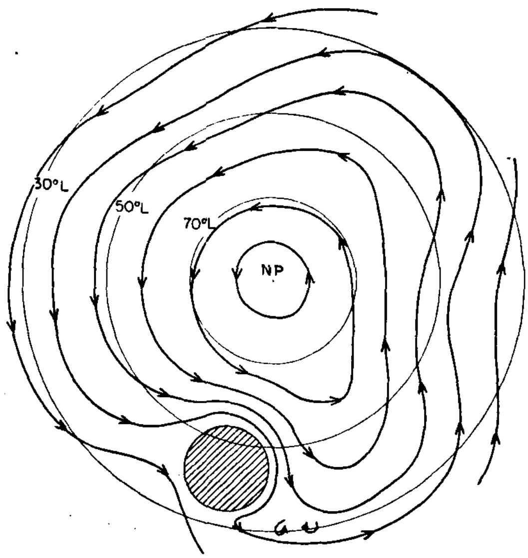

Fig. 3. Schematic sketch of the flow relative to the obstacle in the west wind case. The north pole is at the center of the drawing and the region south of 30 deg. latitude is omitted.

rough measure of the speed of the flow. In this case and in the case of a basic easterly flow, the ratio of the angular velocities of the obstacle and the bowls is about 0.07 to 0.20. The general features, however, are largely unaffected by wide variations of bowl speed and relative obstacle speed. Up to the present time the bowl has been rotated in a range of 15 to 115 r. p. m.

The most significant point, perhaps, is the strong anticyclonic circulation around the barrier.[1] As seen in the sketch, a broad northerly current exists east of the obstacle and curls to the south so that a stagnation point is located on the southern side. The region southeast of the obstacle is a narrow stagnation zone with small eddies but no significant motion. On either side the fluid motions are quite strong. The waves that are set up in the rear are most pronounced when the relative zonal flow is high. When the cylinder is moving slowly, they are damped out by viscosity even faster than shown in the drawing. If the relative speed of the obstacle is in-

[1] "Circulation" is used here in the hydrodynamic sense of the line integral of the tangential velocity around a contour enclosing the cylinder.

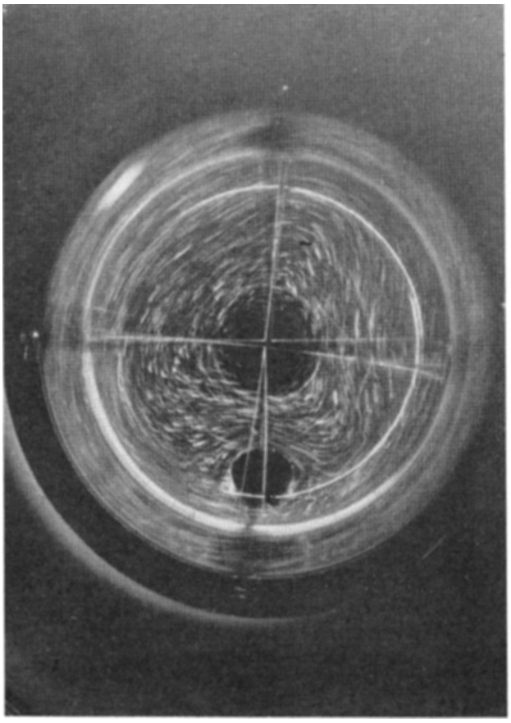

Fig. 4. Photograph of a relative westerly flow past the obstacle. The general direction of flow is counterclockwise.

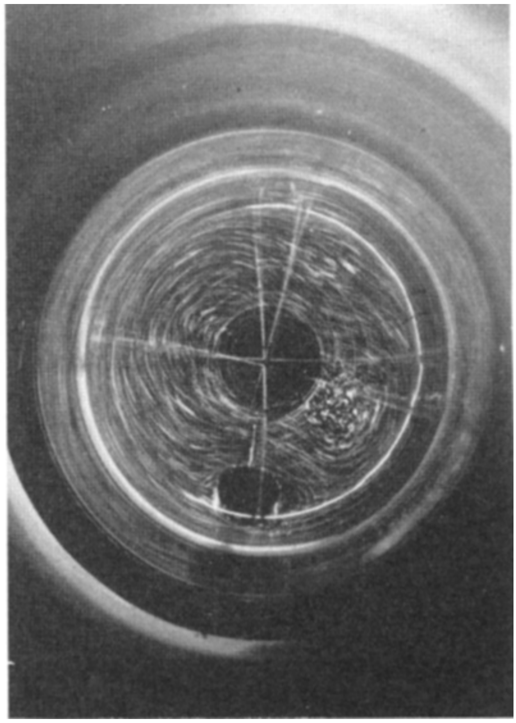

Fig. 5. Photograph of a relative westerly flow past the obstacle. The general direction of flow is counterclockwise and is about twice as strong as in figure 4.

creased sufficiently, the wave number changes from three to two and finally to one.

The relative anticyclonic circulation around the cylinder is very strong. A rough estimate would be one-quarter to one-half of the absolute circulation about the cylinder corresponding to the basic bowl rotation. When the cylinder is started from relative rest, the escape motion resembles potential flow for a very brief period of time. In less than one second, however, the anticylonic circulation is well established.

The photograph in figure 4 brings out most of the features in the sketch. This was taken with a one-fifth second time exposure with the motion of the obstacle subtracted out by the rotoscope. The speed of the bowl was close to 75 r.p.m. and the obstacle was moving about 8 r.p.m. relatively. The wave number in this case is three. The anticyclonic circulation around the obstacle is not as impressive as it is visually. The strong flow around the northern half can be seen but the southeast quadrant appears almost completely stagnant. Actually the fluid is still moving toward the southwest in this region but too slowly to register streaks on the film. Fig. 5 shows the effect of increasing the relative obstacle speed. It is moving at 63 r.p.m. in this picture, while the bowl is rotating at 76 r.p.m. The wave number is now two but the wave west of the obstacle is weakening and would not be present for somewhat higher relative west winds.[1]

Fig. 6, 7 and 8 show the case of an east wind. The situation is somewhat more confused and because of the lack of quantitative data it is difficult to draw an accurate picture of the flow relative to the obstacle for the entire hemisphere. Instead, figure 6 shows

[1] Very recent experiments have revealed that the wave number is a function of the ratio of the relative obstacle speed to the bowl speed. This relationship is very close to that given by a frequency equation for planetary waves in HAURWITZ, B., 1940: The motion of atmospherical disturbances on the spherical earth. *Journ. of Marine Research*, 3, pp. 254—267.

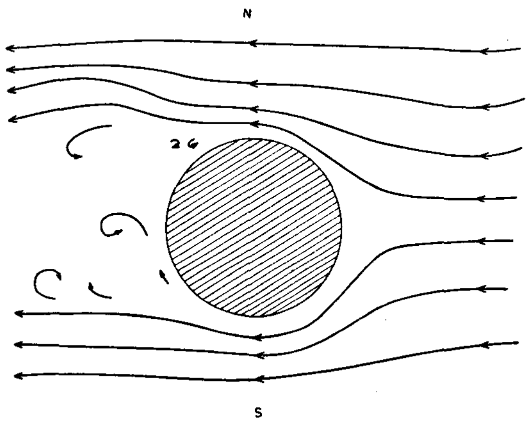

Fig. 6. Schematic sketch of the flow relative to the obstacle in the east wind case.[1]

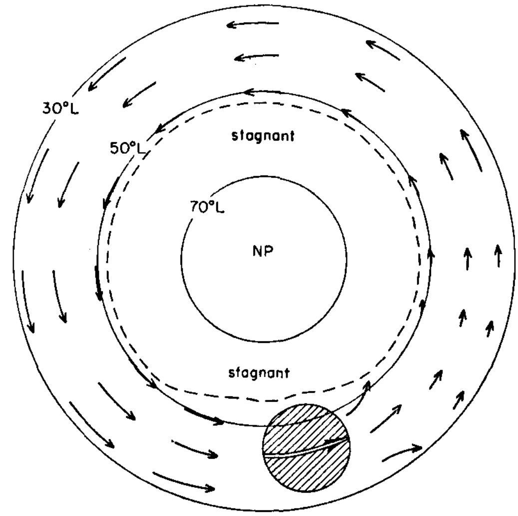

Fig. 7. Schematic sketch of the flow relative to the spheres when the obstacle is moved eastward. The north pole is at the center of the drawing and the region south of 30 deg. latitude is omitted.

this flow in the vicinity of the obstacle and fig. 7 is the flow relative to the bowl. It is apparent that there is no circulation around the barrier in this case. There is, moreover, a broad region of fluid that has almost no motion at all relative to the obstacle. This zone is as wide as the obstacle and extends almost around the globe. This feature is shown in fig. 7 where the zone appears as a marked westerly jet when viewed with respect to the bowl.

Fig. 8 is a photograph of the east wind case. Here the motion of the bowl has been subtracted out rather than the motion of the obstacle. This was also taken with a one-fifth second time exposure. The speed of the bowl was 76 r.p.m. while the obstacle moved at 83 r.p.m.

The two cases are thus quite dissimilar. The circulation in the westerly case is absent in the easterly case. In the latter the obstacle is very efficient in dragging the fluid in its path along with it. In the westerly case there is little tendency for this phenomenon to occur except at very high relative obstacle speeds. That these features are due to the basic rotation of the fluid is quite certain. Preliminary experiments show that they are absent when the bowl is held stationary and the obstacle is moved slowly in either direction.

In the easterly case early experiments showed that in addition to the westerly jet relative to the bowls in the latitude band of the obstacle, a secondary westerly jet appeared at about 72 deg. latitude. Fig. 9 is a photograph showing both jets. Further investigation revealed that this was caused by a widening of the space between the two bowls in the vicinity of the pole. Thus the inner bowl is spherical in shape over the entire hemisphere. The outer bowl loses its spherical shape, however, from about 70 to 90 deg. latitude where it is joined to the outer shaft. The distance between the two bowls is close to 1.6 cm, then, except for the 70 to 90 deg. latitude belt where this distance is about 2.5 cm. When the cavity in the outer bowl was filled in with aquarium cement, the widening of the space between the bowls was eliminated and the secondary jet no longer appeared.

Apparently the secondary jet is caused by

[1] It will be noted that Fig. 6 is not quite consistent with Fig. 7 in front of the obstacle. However, the streamlines are quite approximate and it is very difficult to estimate their form in the coordinate system moving with the cylinder. The frictional slowing of the west wind band (relative to the spheres) far from the cylinder produces some streaming around the front depending both on the basic and relative angular velocities but this is hard to observe precisely. Possibly the streamlines directly in front of the obstacle should be more widely spaced and should refract where they meet the north and south boundaries of stagnation described for Fig. 7.

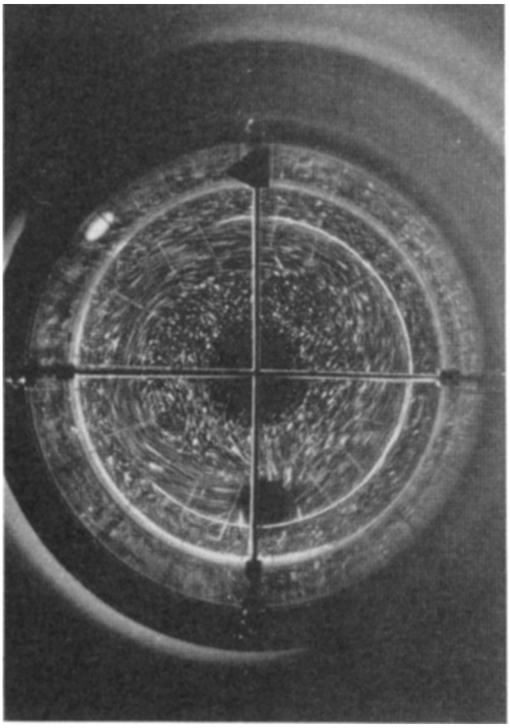

Fig. 8. Photograph of flow relative to the hemispheres when the obstacle is moved eastward. In the latitude band of the obstacle the flow is generally counterclockwise.

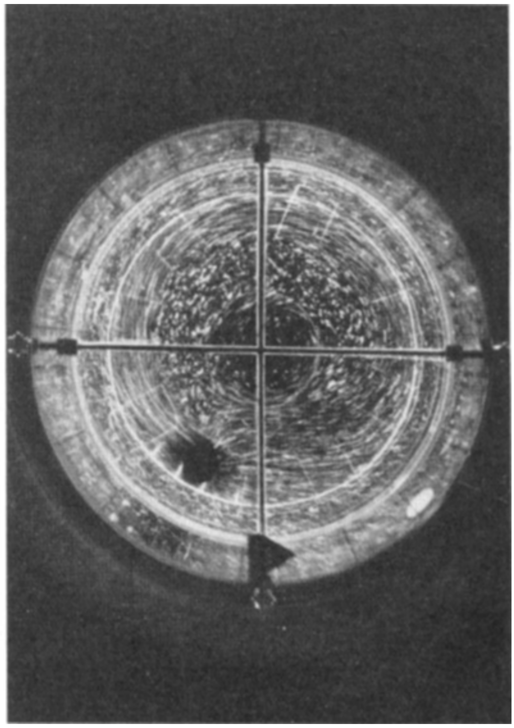

Fig. 9. Photograph of flow relative to the hemispheres when the obstacle is moved eastward. The secondary west wind jet near the pole is due to a local widening in the thickness of the sphericel shell.

an accumulation of cyclonic relative vorticity around the pole. Fluid columns entering this region from lower latitudes are stretched vertically and horizontal convergence results. This causes a local increase in vorticity in this region in excess of that due to solid rotation. The circulation theorem then requires that there be a westerly circulation around latitude circles near the pole.

5. Theoretical remarks

Very little progress has been made yet to explain the above observations. Some of the details seem to be viscous phenomena and these always present great difficulties. A description may be given, however, of the inviscid solution mentioned in the introduction. As is well known in two-dimensional, incompressible, steady motion the streamlines are lines of constant vorticity. When applied to the earth this can be written:

$$\zeta = H(\Psi) \tag{1}$$

and

$$\zeta = 2\Omega \sin\Theta + \frac{1}{a^2} \triangle \Psi \tag{2}$$

where Θ is latitude, Ψ is the streamfunction of the relative motion, a is the radius of the earth and ζ is the absolute vorticity. The operator $\triangle$ is given by

$$\triangle = \frac{1}{\cos^2\Theta} \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\cos\Theta} \frac{\partial}{\partial\Theta} \left(\cos\Theta \frac{\partial}{\partial\Theta}\right). \tag{3}$$

Although the arbitrary function of Ψ is not known we will assume that it is linear. Then

$$\triangle \Psi + 2\Omega a^2 \sin\Theta = k\Psi. \tag{4}$$

If the basic current is one of solid rotation about the polar axis with an angular velocity

ω_z relative to the circular barrier, the general solution is

$$\Psi = - a^2 \omega_z \sin \Theta + \sum_{s=0}^{\infty} (A_s \cos s \varphi_1 + B_s \sin s \varphi_1) S_n^s \tag{5}$$

where S_n^s satisfies

$$\frac{d}{d \sin \Theta_1} \left\{ \cos^2 \Theta_1 \frac{d S_n^s}{d \sin \Theta_1} \right\} + \left\{ n (n + 1) - \frac{s^2}{\cos^2 \Theta_1} \right\} S_n^s = 0. \tag{6}$$

In these equations Θ_1 and φ_1 are latitude and longitude measured with respect to a north pole at the center of the barrier. n is given by

$$n (n + 1) = 2 \frac{(\Omega + \omega_z)}{\omega_z} = - k. \tag{7}$$

Evidently n need not be an integer and may be complex if the basic zonal wind is easterly ($\omega_z < 0$). In the case of integral n, S_n^s is an associated Legendre polynomial.

According to the kinematic boundary condition to be applied shortly, s can only be 0 and 1. In addition it is easily shown that in the new coordinates,

$$\sin \Theta = \sin \Theta_1 \sin \alpha - \cos \Theta_1 \cos \alpha \cos \varphi_1 \tag{8}$$

where α is the latitude of the center of the obstacle. The solution becomes

$$\Psi = - a^2 \omega_z \{ \sin \Theta_1 \sin \alpha - \cos \Theta_1 \cos \alpha \cos \varphi_1 \} + A_0 S_n^0 + A_1 S_n^1 \cos \varphi_1 + B_1 S_n^1 \sin \varphi_1. \tag{9}$$

Since Ψ is constant on the boundary we obtain finally

$$\Psi = - a^2 \omega_z \left\{ \sin \Theta_1 \sin \alpha - \cos \Theta_1 \cos \alpha \cos \varphi_1 + \frac{\cos \Theta_0 \cos \alpha \cos \varphi_1}{S_n^1 (\Theta_0)} \cdot S_n^1 (\Theta_1) + A S_n^0 (\Theta_1) \right\}. \tag{10}$$

A is arbitrary and Θ_0 is the value of Θ_1 at the boundary. The determination of A requires a knowledge of the circulation of the fluid at the solid boundary. If Γ is this circulation

$$A = \frac{\Gamma - 2 \pi a^2 \omega_z \cos^2 \Theta_0 \sin \alpha}{2 \pi a^2 \omega_z \cos \Theta_0 S_n^1 (\Theta_0)}. \tag{11}$$

The form of the motions given in equation (10) cannot be described in detail in this brief paper. For a west wind and integral n they can easily be deduced and for non-integral n a reasonable interpolation can be made. The flow consists of wave motions over the entire sphere with a closed streamline at the boundary of the cylinder and other centers at various parts of the globe. In the case of an east wind, the solution of equation (6) involves conal harmonics[1] which are not tabulated. The solution in the vicinity of the boundary, however, is closely given by Bessel functions of an imaginary argument. These functions decrease rapidly with distance from the boundary so that the perturbed motion is not wave-like but instead drops off rapidly and steadily to zero with distance.

It would appear that the experimental results are not in disagreement with the theory, at least to the extent that wave motions exist in the westerly case and are not visibly present in the easterly case. Some of the experimental features that are absent in the theory may well be a result of viscosity effects. It seems possible that there may be analogies between the appearance of the anticyclonic circulation and the problem of the lift exerted on an air foil. If so, the basic rotation may play a role comparable to the form and orientation of the air foil.

6. Future plans and suggestions

As is evident from the meager nature of the results above, there is much theoretical and experimental work left to do. In addition to obtaining quantitative data on all the aspects of the motion the effects of changes in size, height, shape and position of the obstacle must be studied. Detailed information should be

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[1] See, for example, PRASAD, G., *A Treatise on Spherical Harmonics and the Functions of Bessel and Lamé*, Part II, 1932, p. 22.

acquired regarding the influence of changes in the angular velocities of the bowl and the obstacle, and of changes in viscosity. Finally, we will put a rigid cover at the equator in order to eliminate the free surface that is now present. This step is a necessary one in determining the nature of the waves that are observed in the fluid.

There are several general implications connected with this experiment. According to private advice from Prof. H. RIEHL, the flow of a westerly current past the obstacle shows a pattern which strongly resembles the normal high altitude pattern around and east of the Himalayan massif in Asia. In addition, it is conceivable that in the atmosphere an air mass could serve as a barrier of the magnitude of the one used in this experiment. One type that comes to mind is the warm, blocking high pressure system of high and middle latitudes.

In the realm of hydrodynamic instability is the problem of the formation of the wakes behind the obstacle. As seen above, there are velocity discontinuities or vortex sheets on either side of the wake when the cylinder is moved eastward. There is only one such surface when the obstacle moves westward. The vortices that form on these surfaces at various velocity conditions resemble those in diagrams by ROSENHEAD (1931). The long wake and associated discontinuity surfaces in the eastward case resemble the "free streamlines" of Helmholtz (LAMB, 1932). The fact that they can develop in this experiment is an indication of the stability conferred by the rotation. In this connection the work on the stability of curved flow begun by RAYLEIGH (1917) and reviewed by WATTENDORF (1935) should be useful.

Finally it might be of interest to extend these experiments with a view to completing, for rotations, the picture of the wake motion behind cylinders and air foils (GOLDSTEIN, 1938). For example, the question of the manner in which vorticity is shed might be illuminated by comparison with the rotational case which seems to intensify some aspects of the occurrences[1]. Somewhat similar work has been done by BOUASSE (1939) who moved cylinders around in a resting annular ring of liquid. He found, for example, that under conditions that might ordinarily give rise to a Kármán street, the vortices on the inner side were suppressed. In this connection we intend to carry out corresponding work with obstacles moved relative to a rotating disk of fluid. Intercomparisons should be fruitful in choosing between various hypotheses as to the detailed mechanics of the motions both in the spheres and the disks.

[1] In our experiment the Reynolds number for the relative motion of the cylinder ranges between 500 and 2,500.

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