The Boundary Conditions at the Tropopause

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Abstract

In this paper the general properties of surfaces are investigated at which the derivatives of pressure, temperature, density and velocity may suffer discontinuities. The assumed continuity of the meteorological variables implies severe conditions on the properties of the surface, so that only two types are possible: either the surface moves at the speed of sound relative to the fluid or it is a material surface and moves with the fluid. The tropopause, like a front, is of the latter type. It is shown that the pressure gradient, all particle time derivatives and the velocity divergence are continuous. A knowledge of the discontinuity in temperature lapse rate determines the discontinuities in all the first derivatives of density and potential temperature. In general, the components of the discontinuities in the vertical wind shear parallel to the tropopause can be prescribed arbitrarily (the normal component is zero), but when the wind is geostrophic on both sides of the tropopause, these components too are determined by the jump in the temperature lapse rate.

The importance of these considerations for the internal consistency of mathematical models in dynamical meteorology is discussed.

The fundamental properties of frontal discontinuities have been fairly well known ever since the introduction in meteorology of the air mass concept. Even so, the derivation of these properties given in most textbooks on dynamical meteorology conceals the mathematical simplicity of the problem in a welter of unnecessary assumptions. With regard to the tropopause the situation is even worse: the "simplifying" assumptions are in some cases contradictory, though in no treatments known to the author does the analysis go deep enough to derive conclusions about the tropopause which are obviously inconsistent. This state of affairs is the more remarkable since all types of fluid dynamical discontinuity surfaces were catalogued and discussed by the French mathematician HADAMARD (1903). A more detailed analysis was carried out by the German mathematician LICHTENSTEIN (1929) and the essential results of this paper are implicit in his work, although Lichtenstein was apparently unaware of the possible application of his results to dynamical meteorology.

The adiabatic, non-viscous motion of an air parcel in a baroclinic atmosphere on a rotating earth is described by the equations:

$$(1) \quad \dot{\varrho} + \varrho \operatorname{div} \boldsymbol{q} = 0$$

$$(2) \quad \dot{E} + p \operatorname{div} \boldsymbol{q} = 0$$

$$(3) \quad \varrho\dot{\boldsymbol{q}} + 2\,\varrho\boldsymbol{\Omega} \times \boldsymbol{q} + \operatorname{grad} p + \varrho \operatorname{grad} \varphi = 0$$

where ϱ is the density, T is the temperature, $p = p\,(\varrho, T)$ is the pressure, $E = E(\varrho, T)$ is the internal energy, $\boldsymbol{q} \equiv (u, v, w)$ is the velocity, $\boldsymbol{\Omega}$ is the angular velocity of the Earth and φ is the geopotential. All dependent variables are assumed to depend on x, y, z and t. A dot over any of the dependent variables indicates the total derivative following the motion.

We assume a surface $F(x, y, z, t) = 0$ along which a first (and possibly higher) derivative of one or more of the quantities ϱ, T, p, $\boldsymbol{q}$, etc. may be discontinuous, but along which these quantities themselves are continuous. The surface $F = 0$ (stationary in the four-dimensional x, y, z, t-space) divides the space into two regions $F < 0$ and $F > 0$. In the physical x, y, z-space we can think of $F = 0$

as a one-parameter family of surfaces depending on the parameter t or as a moving surface whose position depends on the time t. With the latter interpretation we denote the unit normal to $F = 0$ in the x, y, z-space by $\boldsymbol{n}$ and the speed of the surface measured normal to itself by G. Then

$$F_t + G\,\boldsymbol{n} \cdot \operatorname{grad} F = 0$$

or

$$G = -r^{-1} F_t$$

where $r^2 = F_x^2 + F_y^2 + F_z^2$. The direction cosines of the normal are given by

$$\boldsymbol{n} \equiv (\alpha,\ \beta,\ \gamma) = (r^{-1} F_x,\ r^{-1} F_y,\ r^{-1} F_z)$$

We now investigate the conditions on the derivative jumps imposed by the continuity of the functions themselves. Consider, for example, the component u of the velocity $\boldsymbol{q}$. In the hypersurface $F = 0$, $u^{(1)} = u^{(2)}$, where the superscripts denote the value of u at a point P approached from the two sides of F. If we differentiate u on both sides while remaining in the surface, we have

$$u_x^{(1)} F_x + \ldots + u_t^{(1)} F_t = u_x^{(2)} F_x + \ldots + u_t^{(2)} F_t$$

which implies

(4) $[u_x] : [u_y] : [u_z] : [u_t] = F_x : F_y : F_z : F_t$

where $[u_x]$ denotes $u_x^{(1)} - u_x^{(2)}$, etc. Similar relations hold for all other quantities continuous in the surface F.

We assume the tropopause is not vertical, i.e. $\gamma \neq 0$, and write all velocity gradient jumps in terms of the vertical derivative jumps:

(5) $[\operatorname{grad} u] = \gamma^{-1}[u_z],\ [u_t] = -\gamma^{-1} G\, [u_z]$

(6) $[\operatorname{grad} v] = \gamma^{-1}[v_z],\ [v_t] = -\gamma^{-1} G\, [v_z]$

(7) $[\operatorname{grad} w] = \gamma^{-1}[w_z],\ [w_t] = -\gamma^{-1} G\, [w_z]$

With the notation $G_0 = G - \boldsymbol{q} \cdot \boldsymbol{n}$ for the speed of the discontinuity relative to the fluid, we easily derive the relations

(8) $[\dot{u}] = -\gamma^{-1} G_0 [u_z], \ldots\ldots [\dot{w}] = -\gamma^{-1} G_0 [w_z]$

(9) $[\operatorname{div} \boldsymbol{q}] = \gamma^{-1}(\alpha[u_z] + \beta[v_z] + \gamma[w_z]) = \gamma^{-1} \boldsymbol{n} \cdot [\boldsymbol{q}_z]$

(10) $[\dot{\varrho}] = -\gamma^{-1} G_0 [\varrho_z]$

(11) $[\dot{E}] = -\gamma^{-1} G_0 [E_z] = -\gamma^{-1} G_0 (E_\varrho [\varrho_z] + E_T [T_z])$

(12) $[\operatorname{grad} p] = \gamma^{-1} \boldsymbol{n}\, (p_\varrho [\varrho_z] + p_T [T_z])$

After these preliminaries, which were independent of the dynamical equations and depended solely on the continuity of the meteorological variables at the discontinuity surface, we return to the equations of motion (1)—(3). Since these equations hold on both sides of $F = 0$, we may apply them at a point P in the discontinuity surface and, recalling the continuity of all undifferentiated quantities, obtain the following equations in the jump quantities:

(1A) $[\dot{\varrho}] + \varrho\, [\operatorname{div} \boldsymbol{q}] = 0$

(2A) $[\dot{E}] + p\, [\operatorname{div} \boldsymbol{q}] = 0$

(3A1) $\varrho\, [\dot{u}] + [p_x] = 0$

(3A2) $\varrho\, [\dot{v}] + [p_y] = 0$

(3A3) $\varrho\, [\dot{w}] + [p_z] = 0$

Substitution from relations (8)—(12) leads to

(13) $M X = 0$

where M is the 5×5 matrix

$$M = \begin{pmatrix} \alpha\varrho & \beta\varrho & \gamma\varrho & 0 & -G_0 \\ \alpha p & \beta p & \gamma p & -G_0 E_\varrho & -G_0 E_T \\ -G_0\varrho & 0 & 0 & \alpha p_\varrho & \alpha p_T \\ 0 & -G_0\varrho & 0 & \beta p_\varrho & \beta p_T \\ 0 & 0 & -G_0\varrho & \gamma p_\varrho & \gamma p_T \end{pmatrix}$$

and X is the column matrix whose transpose is

$$X' = ([u_z],\ [v_z],\ [w_z],\ [\varrho_z],\ [T_z])$$

Equations (13) are linear, homogeneous and algebraic. Hence a necessary condition for the system to have a non-trivial solution is that the determinant of M must vanish. But it is easily verified that

(14) $\det M = G_0^3 (G_0 + c)(G_0 - c)$

where

(15) $c^2 = p_\varrho + p_T E_T^{-1} (p\varrho^{-2} - E_\varrho)$

By thermodynamic arguments it may be shown that the expression on the right in (15) is $(p_\varrho)_S = \partial p/\partial\varrho$ at constant entropy $S = S(\varrho, T)$. In other words, c is the local Newtonian sound speed.

The requirement that det M vanish is satisfied by either of two mutually exclusive possibilities:

1. $G_0 = 0$, i.e., the surface $F = 0$ is a material surface consisting always of the same fluid particles; or

2. $G_0 = \pm c$, i.e., the surface $F = 0$ is not a material surface and its speed relative to the moving fluid is equal to the local sound speed.

The latter alternative leads to the conditions at a gas dynamical "shock of second order" in an arbitrary fluid and, in fact, as has been shown by J. B. Keller and the author (in a report which has not yet been published), corresponds in the limit to an actual shock of weak strength. No discontinuity of this type has ever been observed in the atmosphere, nor is it likely to be of meteorological significance if ever observed.

Accordingly we accept the first alternative and identify the discontinuity surface $F = 0$ for which $G = \boldsymbol{q} \cdot \boldsymbol{n}$ with the ideal tropopause. The further mathematical properties of the tropopause are obtained very easily by inserting the value $G_0 = 0$ in equations (13) and relations (8)—(12). We find

(16) $[\text{grad } p] = 0$

(17) $[\dot{\boldsymbol{q}}] = [\dot{\varrho}] = [\dot{E}] = 0$

(18) $[\text{div } \boldsymbol{q}] = \gamma^{-1}\boldsymbol{n} \cdot [\boldsymbol{q}_z] = 0$

(19) $[\varrho_z] = -(p_T)_\varrho (p_\varrho)_T^{-1}[T_z] = (\varrho_T)_p[T_z]$

(20) $[\Theta_z] = (p_0/p)^\varkappa [T_z]$ where $\Theta = T(p_0/p)^\varkappa$

(21) $[\text{div}(\varrho\boldsymbol{q})] = \gamma^{-1}\boldsymbol{q} \cdot \boldsymbol{n}(\varrho_T)_p[T_z]$

(22) $[\text{curl } \boldsymbol{q}] = \gamma^{-1}\boldsymbol{n} \times [\boldsymbol{q}_z]$

We may state these conclusions with respect to the material tropopause as follows:

1. The pressure gradient is continuous.
2. There is no difference in the acceleration of particles compared on either side.
3. There is no velocity convergence or divergence.
4. The vertical wind shear is normal to the tropopause.
5. The change in the mass divergence is proportional to the change in temperature lapse rate, to the solenoidal term $\partial\varrho/\partial T$ in an isobaric surface at the same level, to the wind component perpendicular to the tropopause, and is inversely proportional to the vertical component of the unit normal to the tropopause.
6. The change of density lapse rate is proportional to the change of temperature lapse rate and to the solenoidal term.
7. The change of vorticity is proportional to the parallel component of the change in wind shear.

Since the tropopause is quasi-horizontal, one may also conclude that the change in mass divergence is small, that the change in the vertical shear of the vertical wind component is nearly zero and that the change in vorticity is horizontal and proportional to the change in the vertical shear of the horizontal wind component.

In dynamical meteorology, and especially in the study of the general circulation, one customarily assumes a model for the unperturbed atmosphere. Usually, frontal surfaces are omitted from this model, but almost always a tropopause, or what is equivalent, a discontinuity in temperature lapse rate, is assumed at some level. It frequently happens that too many dynamic and thermodynamic conditions are imposed at this interior boundary surface. Internal inconsistencies of this sort are hard to detect and may, because the equations governing atmospheric motions are not totally hyperbolic, spread their influence far from their origin. It is clear from the preceding analysis that the jumps in the gradients of all thermodynamic quantities (temperature, density or potential temperature) are fixed when the jump in any one component of these gradients (the temperature lapse rate, for example) is prescribed. In a general atmosphere one may also prescribe the change in wind shear at the tropopause (or any derivative jump equivalent to it), but if geostrophic equilibrium is prescribed on both sides of the tropopause, the fixing of the change in temperature lapse rate is sufficient to fix the values of all other discontinuities, since the geostrophic wind shear is proportional to the temperature gradient.

It is worth emphasizing that the results obtained do not give any information about the structure of the atmosphere, or even about the motion of the tropopause or its genesis. A purely kinematic description, similar to the treatment of Petterssen and others on frontogenesis and frontolysis, of the processes influencing the genesis, decay and persistence of tropopause-like surfaces (including such phenomena as subsidence inversions) could almost certainly be given, but it is unlikely that such a description would lead to a very deep understanding of the most important processes involved. For one thing, it would involve a knowledge of the space derivatives of the temperature and velocity gradients and these would be impossible to measure or even estimate at the present stage of meteorology. A more important endeavor would be the consideration of viscosity, turbulence, radiation and heat conduction and their effects on the tropopause. Strictly speaking, discontinuities such as we have been discussing would not even exist mathematically if the presence of these factors were taken account of in the equations of atmospheric motion. Experience from gas dynamics suggests, however, that these processes would exert a modifying influence only, for example by producing a zone of finite thickness through which the gradients of wind, temperature and density would change rapidly.

REFERENCES

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