On the Vertical and Horizontal Concentration of Momentum in Air and Ocean Currents

I. Introductory Comments and Basic Principles, with Particular Reference to the Vertical Concentration of Momentum in Ocean Currents

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Abstract

In this and a following paper an investigation is made into the factors which are responsible for the characteristic "streakiness" of certain atmospheric and oceanic current systems. As a first step a search is made for the factors which might give rise to the striking vertical concentration of momentum observed in many well-defined atmospheric and oceanic jet streams. A straight, parallel, heteorogeneous and incompressible current with a free surface, hydrostatic pressure distribution and a continuous density field is analyzed theoretically to determine the particular distribution of velocity with depth which, for a prescribed volume transport within each isopycnic layer, leads to a minimum value for the transfer of momentum. This requirement leads to a certain "critical" relationship between the velocity and density distribution which in the case of a two-layer system reduces to the well-known critical relative speed at an internal fluid boundary. In the general case of a continuous density distribution it is found that currents which are subjected to momentum losses and hence tend to approach the "critical" velocity distribution in many cases must shrink vertically and develop a sharp velocity maximum at or below the free surface. The nature of the structural changes to be anticipated depends ultimately upon the numerical value of a certain "internal" Froude Number which is a function of the initial velocity distribution as well as of the total density range.

A. Introductory comments

The immediate purpose of the theoretical analyses presented below is to suggest a mechanism through which the momentum of certain stratified air and ocean currents may be concentrated vertically into a shallow layer or jet, and to establish a criterion from which the limiting vertical distribution of velocity in such a jet may be determined. Recent aerological investigations into the structure of the upper atmosphere over North America have shown that the prevailing upper westerlies of our latitudes, which may be said to constitute a "statistical" or climatological jet, frequently consist of two or three narrow zones of extremely high winds, stretched out along the lines of flow and separated by regions of weaker winds. It is with these individual jets, or streaks of high wind, that we wish to concern ourselves. One major aim of the investigation into the general circulation of the atmosphere of which the present analyses form a part, is to provide a satisfactory answer for this "streakiness" which is becoming increasingly apparent as detailed synoptic data for the upper atmosphere accumulate.

Through the courtesy of the Woods Hole Oceanographic Institution the writer has had the opportunity to examine some of the data collected during a recent, broadly planned synoptic survey of the Gulf Stream system east of Cape Hatteras. Some of the results of this survey are described by FUGLISTER and WORTHINGTON (1951) in this issue of Tellus. These observations show that the current, after leaving the continental shelf of North America, and in spite of considerable entrainment of "slope" water along its left edge, remains narrow and well-defined, with a surface maximum speed of about five knots and a total width of approximately fifty nautical miles. In general, the behavior of the current during the survey period was strikingly similar to that of an atmospheric jet stream. Waves and meander-patterns were observed, including a complete cut-off of a cyclonic vortex on the Bermuda side of the stream. It is true that the length scale of these formations was considerably smaller than that of the corresponding phenomena in the atmosphere. However, if it may be assumed that the horizontal meander-patterns are controlled by the planetary vorticity effect such a reduction in scale must be expected and the ratio should have the value $\left(\frac{u_g}{u_{\text{jet}}}\right)^{1/2}$ where u_g is the maximum speed of the Gulf Stream and u_{jet} the maximum speed of a fully developed atmospheric jet. The observed ratio of length scales seems to be in fair accord with this theoretical conclusion. Hence, in view of the similarity between these apparently unrelated current systems it is not unreasonable to assume, as a working hypothesis, that the factors controlling the shape and behavior of jets must be fairly independent of their driving mechanism, and derivable from quite general dynamic principles.

Recent theoretical investigations by STOMMEL (1948), HIDAKA (1949) and MUNK (1950 a, 1950 b) have shown that the broad-scale *statistical* features of the horizontal current distribution in the North Atlantic Ocean may be interpreted as the result of the permanent North Atlantic wind system. These highly successful analyses are based on the assumption that the vorticity of vertical water columns may be expressed as a balance between the changes in vorticity brought about through wind stresses, through latitude displacements, and through lateral mixing. The theory accounts in a remarkable manner for the large-scale features of the ocean circulation, in particular for the westward displacement of the North Atlantic eddy, but is incapable of accounting for such features as the presence of an extremely narrow meandering current in the western part of the ocean.

This point of view is in accord with the results of an analysis undertaken by Mr H. Stommel of the Woods Hole Oceanographic Institution and presented as a separate note in the present issue of Tellus (STOMMEL, 1951). Mr Stommel utilizes a composite map which contains the individual synoptic Gulf Stream patterns observed on various occasions during the survey referred to above, as well as individual positions of the current determined during earlier cruises (fig. 2 in FUGLISTER and WORTHINGTON, 1951). He then computes a mean value of the horizontal eddy viscosity (K) that would correspond to this statistical picture of the current distribution. The calculation indicates that if the meander patterns are interpreted as a manifestation of lateral turbulence the associated eddy viscosity must have a value of about 2.10^8 cm^2 sec^{-1}; thus it is about five times as large as the value used by Munk in his theoretical analyses. This may seem to be a large discrepancy. However, the irregular and transient horizontal currents of the North Atlantic are certainly more intense in the Gulf Stream region than elsewhere. Furthermore, the horizontal eddy viscosity enters into Munk's calculations in the form of a "wave" number which contains the third root of K. Hence the computed current patterns are relatively insensitive to inaccuracies in the assumed value of K. It may, therefore, be asserted that Mr Stommel's calculations lend support to the statistical interpretation proposed above for the phenomena described by the Munk-Hidaka-Stommel theory.

Similarly, the initial studies of the atmospheric jet stream (ROSSBY 1947, UNIVERSITY OF CHICAGO 1947) did not distinguish clearly between the permanent, climatological features of the middle latitude west wind belt on the one hand, and the details of transient local jets of extreme sharpness on the other hand. These first analyses were based on the assumption that the basic features of the permanent jet could be accounted for by quasi-horizontal exchange processes.[1] The meteorological theories which may be outlined on the basis of this hypothesis account in a fair manner for the average strength of the middle latitude westerlies, and for the average distribution of horizontal wind shear at the tropopause level, from polar latitudes southward to the vicinity of the center of the west wind belt. However, the rapid decrease of wind speed observed on the south side of the west wind maximum in synoptic cross sections through individual jets can hardly be accounted for by quasi-horizontal motions, and such features as the occurrence of highly concentrated double or multiple jets are not likely to be explained in terms of quasi-barotropic models.

Such evidence as is now available indicates that strong individual atmospheric jets are limited both in length and in life span. Likewise there are reasons to believe that the concentrated current found in the Woods Hole survey of the Gulf Stream may be limited in extent, and breaks up into several branches as the current reaches the Grand Banks. One is thus led to search for mechanisms capable of bringing about *local* concentrations of momentum within the broad "climatological" circulation patterns of the oceans and the atmosphere. These local jets always appear to be associated with a strong solenoid concentration, as required by the thermal wind equation (circulation theorem) and with intense vertical shear in the fluid immediately below the jet, suggesting that the vertical stability distribution may plan an important role in the concentration of momentum.

It should be stressed that the small width and the high speed of the upper layers of the Gulf Stream rather definitely exclude the possibility of interpreting the current in this part of the Atlantic as the result of momentum added locally by the prevailing winds. In this connection it is significant that the California current, which is considered as driven by the local wind systems off the Pacific Coast of North America is broader, more diffuse and far slower than the Gulf Stream. This comparison strengthens the impression *that individual sharp jets result from a concentrating mechanism which is a basic characteristic of stratified air and ocean currents.*

A clue to the possible nature of the vertical concentrating mechanism may be contained in the results of certain aerological case studies which are being undertaken at the present time in Chicago.

On several occasions during the summer of 1950 it was observed that westerly currents impinging on the Pacific Coast of North America and characterized by a fairly flat wind speed maximum near the 200-mb level, in crossing the mountains experienced a marked change in vertical structure. East of the mountains these westerly currents showed very slight air motion below the 300-mb level. Above 300 mb, however, a sharp jet appeared, concentrated both vertically and horizontally into a maximum of much greater intensity than the one observed on the West Coast. It is hard to avoid the impression that the concentration must have been associated with the momentum losses in the lower portion of the air stream which resulted from the passage over the mountainous parts of the continent.

On the basis of such clues it appears reasonable to attempt to ascertain what the vertical profile of velocity would be in an immiscible, stratified current which is subjected to momentum losses through contact with the

[1] To the extent that these analyses have made use of vorticity considerations they differ from the corresponding oceanographic theory in one basic aspect. Munk's calculations consider the effect of lateral exchange processes on the distribution of the vertical component of the *relative* vorticity while the meteorological analyses assume that it is the vertical component of the *absolute* vorticity which is being redistributed by lateral mixing. Apparently because of the fact that the oceanographic problems considered until now have dealt with closed basins in low or middle latitudes the omission of the earth's own vorticity component from the diffusion terms does not seem to invalidate the results of the calculations.

The kinetic energy and the vorticity of the large-scale quasi-horizontal eddy motions in the atmosphere must be derived from barocline processes; the distribution and intensity of the zonal motions established through such lateral exchange processes should be fairly independent of the specific nature of the mechanisms through which the lateral "eddy" motion is created. In the ocean, the lateral exchange motion may derive some of its energy from the effect of stresses exerted by the non-permanent wind systems, but may also to a considerable extent be fed by the manifest meandering instability of such narrow currents as the Gulf Stream.

underlying (rigid or fluid) boundary, or at its lateral boundaries, but required to transport, within each material fluid sheet, a prescribed amount of fluid per unit time. It would be particularly valuable to know the nature of the particular velocity profile which corresponds to a *minimum* value of the transfer of momentum per time across a vertical plane normal to the current axis. It is reasonable to assume that this profile represents a limiting state which would be approached gradually by any stratified current subjected to momentum losses but prevented from escaping to the sides.

The withdrawal of momentum from the interior of a stratified current demands frictional (turbulent) interaction between layers. If we nevertheless are permitted to apply the principle of mass continuity to the individual fluid layers, this must imply that the individual fluid sheets remain immiscible, or that the eddy diffusion of mass properties is small in comparison with the eddy diffusion of momentum. This is in good agreement with oceanographic experience, which indicates that in stable water masses the ratio of eddy conductivity or diffusivity to eddy viscosity may sink to 0.1 or less. This point has been discussed by GOLDSTEIN (1938, pp. 229—231).

The velocity profiles obtained from the principle outlined above are characterized by a rapid increase of current speed with increasing height above the underlying boundary, and also by a marked increase of vertical stability upward. When applied to well developed atmospheric jet streams the theory appears to lead to a reasonable value for the wind speed at the upper inversion. It may likewise be applied to well-defined ocean currents in which case it leads to surface current speeds in good agreement with the observational data collected during the recent Gulf Stream survey.

The pattern for a theoretical determination of the profile of minimum momentum transfer is well known from fluid mechanics. In the case of flow of water in an open flume the transfer of momentum is given by

$$MT = \int_0^D (\varrho u^2 + p)\, dz = \varrho \left(u^2 D + g \frac{D^2}{2} \right), \tag{1}$$

D being the depth and u the velocity, assumed to be independent of depth. The water pressure p is computed from the hydrostatic equation and counted from the free surface. The volume transport V is given by

$$V = uD = \text{constant}, \tag{2}$$

and hence

$$MT = \varrho \left(\frac{V^2}{D} + \frac{gD^2}{2} \right) \tag{3}$$

Variation shows that

$$\delta\,(MT) = \varrho \left(gD - \frac{V^2}{D^2} \right) \delta D, \tag{4}$$

and thus the extreme (minimum) value of MT occurs when

$$gD = \frac{V^2}{D^2} = u^2 \tag{5}$$

If the flow takes place in a shallow layer D, flowing on top of a deep non-moving layer of density ϱ_b ($> \varrho$), the corresponding value of the critical current velocity is given by

$$u^2 = g\,\frac{\varrho_b - \varrho}{\varrho_b}\, D = \gamma D \quad \left(\gamma = g\,\frac{\varrho_b - \varrho}{\varrho_b} \right). \tag{6}$$

We shall now attempt to use the principle outlined above in an analysis of the vertical concentration of momentum in straight water currents in which density varies with depth. In part II, to be published later, we shall consider the vertical distribution of momentum in air currents and finally the mechanism which appears to be responsible for the horizontal concentration of momentum in air and ocean.

B. Vertical concentration of momentum in water currents with a free surface

We consider a straight, parallel stream incapable of escape motion sideways. The water is considered incompressible and the density varies with depth. Under these assumptions, the transfer of momentum across a vertical strip normal to the current axis is given by

$$MT = \int_0^D (\varrho u^2 + p)\, dz, \tag{7}$$

where the vertical coordinate z is counted upward from the bottom and where u is current speed, p the water pressure (assumed to be hydrostatic). We shall now introduce a function of the density as the independent vertical coordinate, to help us incorporate and utilize the fact that the density is constant individually,

$$\frac{d\varrho}{dt} = 0. \tag{8}$$

We write

$$\varrho = \varrho_b (1 - 2\varkappa\sigma) \tag{9}$$

where ϱ_b is the density of the bottom water, and

$$\varrho_s = \varrho_b (1 - 2\varkappa) \tag{10}$$

is the surface density. Hence σ, our new independent variable, ranges from 0 in the bottom layer to 1 at the sea surface, and

$$2\varkappa = \frac{\varrho_b - \varrho_s}{\varrho_b} \tag{11}$$

is the percentage range of density over the entire water column.

In terms of the new variable,

$$MT = \int_0^1 (\varrho u^2 + p)\, \dot{z}\, d\sigma, \tag{12}$$

where

$$\dot{z} = \frac{dz}{d\sigma}. \tag{13}$$

We shall next assume that the mass transport in every infinitesimal isopycnic layer remains constant during the variation process, though it may vary from layer to layer. Hence

$$\varrho u \dot{z} d\sigma = \varrho u_0 \dot{z}_0\, d\sigma = v(\sigma) d\sigma, \tag{14}$$

where the subscript o refers to the initial conditions. This expression v for the mass transport may be used to eliminate u from MT and one finds

$$MT = \int_0^1 \left(\frac{v^2}{\varrho \dot{z}} + p\dot{z}\right) d\sigma. \tag{15}$$

Because of the hydrostatic balance

$$\dot{p} = -g\varrho\dot{z} \tag{16}$$

and thus one finally obtains

$$MT = -\int_0^1 \left(\frac{gv^2}{\dot{p}} + \frac{p\dot{p}x}{\varrho g}\right) d\sigma. \tag{17}$$

Our variation problem consists in determining the particular function p of σ which reduces MT to a minimum for the prescribed distribution of v with σ. The variation of p vanishes at the sea surface and we shall assume that it vanishes also at great depths.[2] Under these circumstances the minimum value of MT is characterized by the fact that

$$\delta(MT) = \\ = \int_0^1 \left[\left(\frac{gv^2}{{}^2p} - \frac{p}{g\varrho}\right) \delta\dot{p} - \frac{\dot{p}}{g\varrho}\, \delta p\right] d\sigma = 0. \tag{18}$$

This is true for arbitrary values of δp provided the function p satisfies Euler's equation

$$\frac{\dot{p}}{g\varrho} + \frac{d}{d\sigma}\left[\frac{gv^2}{\dot{p}^2} - \frac{p}{g\varrho}\right] = 0, \tag{19}$$

which, after substitutions, reduces to

$$\frac{du^2}{d\sigma} = p\, \frac{d}{d\sigma}\left(\frac{1}{\varrho}\right). \tag{20}$$

Introducing the specific volume α instead of the density, one finds

$$\boxed{du^2 = p d\alpha} \tag{21}$$

[2] If the latter assumption is dropped a critical velocity corresponding to the speed of external gravity waves is obtained. This type of flow is clearly inconsistent with the prescribed external conditions of motion and it may be assumed that such variations in the bottom pressure quickly are dispersed through gravity waves.

To determine the final velocity distribution from the initial mass transport distribution it is necessary to combine (21) with (14), or

$$\varrho u dz = v(\sigma)\, d\sigma \qquad (22)$$

We may test the character of our basic result (21) by applying it to a simple, double current system in which a lighter fluid of depth D_2 and specific volume α_2 flows over a deep, immobile fluid of specific volume α_1 ($< \alpha_2$). In this case du^2 and $d\alpha$ vanish everywhere except at the internal boundary where

$$p_2 = \varrho_2 g D_2 \quad \left(\varrho_2 = \frac{1}{\alpha_2}\right) \qquad (23)$$

Hence, (21) reduces to

$$u_2^2 = \varrho_2 g D_2 (\alpha_2 - \alpha_1) \qquad (24)$$

or

$$u_2^2 = g \frac{\varrho_1 - \varrho_2}{\varrho_1} D_2 = \gamma D_2 \qquad (25)$$

in agreement with the result stated in (6).

The basic equation (21) is particularly simple to evaluate for ocean currents, in view of the exceedingly small percentage variation of ϱ, which permits us to replace the variable ϱ with a suitable mean value $\bar{\varrho}$ in the computation of p. One then finds

$$p = \bar{\varrho} g z,$$

where the vertical coordinate now is counted downward.

Before attempting to determine theoretically examples of current profiles of minimum momentum transfer which correspond to prescribed initial volume transports we shall rewrite our basic equation so as to permit us to calculate the limiting velocity profiles which correspond to the observed density distribution in the central part of a fully developed ocean current. We have, with good approximation,

$$\frac{du^2}{dz} = -g z\, 10^{-3} \frac{d\sigma_t}{dz}, \qquad (26)$$

where z increases downward and σ_t represents the quantity

$$\sigma_t = 1{,}000\,(\varrho - \varrho_{35,0,0}). \qquad (27)$$

In this expression $\varrho_{35,0,0}$ is the standard density corresponding to a salinity of 35 ‰, atmospheric pressure and a temperature of 0° C. With sufficient accuracy (26) may be written in the form

$$\frac{du^2}{dz} = -z \frac{d\sigma_t}{dz}, \qquad (28)$$

where z and u are in cm and cm sec^{-1} respectively.

We shall at first attempt to represent the σ_t-distribution curve by the following analytical expression:

$$\sigma_t = \sigma_{tb} - \triangle e^{-\mu z^2}. \qquad (29)$$

It is evident that $\triangle$ measures the total range in σ_t. In the Gulf Stream region an adequate value for $\triangle$ appears to be 4.5. To determine μ, note that (29) upon differentiation gives

$$\frac{d\sigma_t}{dz} = 2\mu \triangle \cdot z \cdot e^{-\mu z^2}, \qquad (30)$$

indicating that the σ_t-gradient disappears at the sea surface (within the homogeneous layer) and at great depths. The maximum gradient occurs at

$$z = \frac{1}{\sqrt{2\mu}} \equiv D \qquad (31)$$

which may be considered as the depth of the center of the thermocline.

One then finds, from (28)

$$u^2 - u_s^2 = -2\mu \triangle \int_0^z z^2 e^{-\mu z^2} dz, \qquad (32)$$

u_s being the current speed at sea level. Since u must vanish at great depths,

$$u_s^2 = 2\mu \triangle \int_0^\infty z^2 e^{-\mu z^2} dz =$$

$$= \triangle D \sqrt{\frac{\pi}{2}} \approx 1.25 \triangle D. \qquad (33)$$

For $\triangle = 4.5$ this gives

$$u_s^2 = 5.625\, D. \qquad (34)$$

Expressing u_s in m.p.s. and D in meters this is equivalent to

$$u_s = 0.237\sqrt{D}, \tag{35}$$

or, for a thermocline depth D of 100 m,

$$u_s = 2.37 \text{ mps},$$

in good agreement with the observed maximum surface speed of the Gulf Stream. For a D-value of 200 m the value of u_s is raised to about 3.3 mps, somewhat in excess of the observed maximum surface speed.

The vertical distribution of u obtained from (32) is such that the maximum vertical shear of the current occurs well below the center of the thermocline.

The distribution of σ_t assumed above gives a more gradual density variation through the thermocline than normally observed in the Gulf Stream. We shall attempt to investigate the effect of a steep density gradient by assuming the same total range $\triangle$ in σ_t as before, but limited to a layer of thickness D. The thickness of the superimposed homogeneous layer is H. Then

$$\frac{d\sigma_t}{dz} = 0 \tag{36}$$

for $z < H$ and for $z > H + D$, and

$$\frac{d\sigma_t}{dz} = \frac{\triangle}{D} \tag{37}$$

for $H < z < H + D$. It follows that

$$u_s^2 = \triangle\left(H + \frac{1}{2}D\right). \tag{38}$$

Using 100 m for the depth of the homogeneous layer and 200 m as the total thickness of the total thermocline one finds, for the same $\triangle$ as before,

$$u_s = 3.0 \text{ mps}.$$

We shall next consider an initial value problem. A uniformly stratified current of speed u_0 and depth D_0 flows on top of a homogeneous bottom layer of density ϱ_b in which the volume transport vanishes, and is permitted to readjust itself to a minimum momentum transfer current profile. We shall attempt to determine the distributions of density and velocity in the final state. Introducing the new variable

$$\varrho = \varrho_s(1 + 2\varkappa\sigma), \tag{39}$$

where ϱ_s is the surface density and

$$\varrho_b = \varrho_s(1 + 2\varkappa) \tag{40}$$

the deep water density, it follows from the assumption of a uniform initial stratification that initially (subscript 0)

$$\sigma = \frac{z}{D_0}, \quad \left(\frac{dz}{d\sigma}\right)_0 = D_0. \tag{41}$$

Hence the continuity requirement takes the form

$$u\frac{dz}{d\sigma} = u_0 D_0. \tag{42}$$

Our basic equation gives, with good approximation,

$$\frac{du^2}{dz} = -\gamma z\frac{d\sigma}{dz}, \quad (\gamma = 2\varkappa g) \tag{43}$$

or, through elimination of $\frac{d\sigma}{dz}$ and integration,

$$u = \frac{\gamma}{4u_0 D_0}(D^2 - z^2). \tag{44}$$

Substitution in (42) gives

$$\sigma = \frac{\gamma}{4u_0^2 D_0^2}\left(D^2 z - \frac{z^3}{3}\right). \tag{45}$$

It follows from our basic equation that σ must be continuous wherever u is continuous. Hence, at the depth D, where u vanishes, σ must reach its maximum value of unity, corresponding to the bottom water density. Thus

$$\sigma_{z=D} = 1 = \frac{\gamma D^3}{6u_0^2 D_0^2}, \tag{46}$$

and D may then be determined from

$$\frac{D}{D_0} = \left(\frac{6u_0^2}{\gamma D_0}\right)^{1/3}. \tag{47}$$

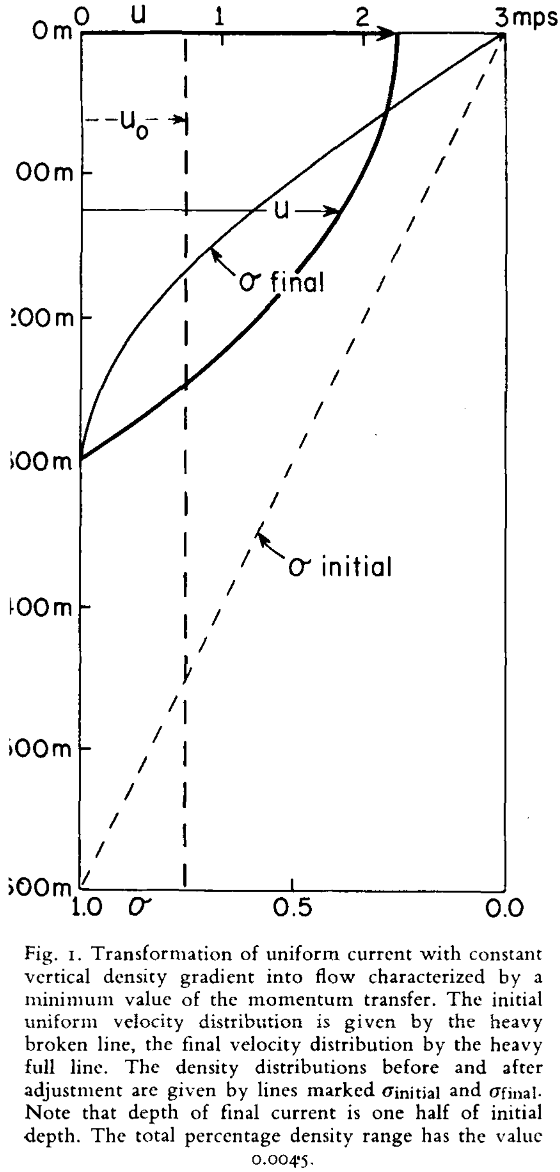

Fig. 1. Transformation of uniform current with constant vertical density gradient into flow characterized by a minimum value of the momentum transfer. The initial uniform velocity distribution is given by the heavy broken line, the final velocity distribution by the heavy full line. The density distributions before and after adjustment are given by lines marked σ_{initial} and σ_{final}. Note that depth of final current is one half of initial depth. The total percentage density range has the value 0.0045.

It follows that the current must become shallower, and the bottom layer increase in thickness, whenever u_0 falls below the critical value $u_{0\,\text{critical}}$ defined by

$$u_0 < u_{0\,\text{critical}} = \sqrt{\frac{\gamma D_0}{6}}, \tag{48}$$

The final surface current speed u may be computed from u_0 by means of the expression

$$\frac{u_s}{u_0} = \frac{3}{2}\left(\frac{D_0}{D}\right). \tag{49}$$

The process of adjustment is illustrated graphically in fig. 1 for $\triangle = 4.5$, $D_0 = 600$ m, $u_0 = 0.75$ mps. In this case $D = 300$ m and $u_s = 2.25$ mps.

It is clear that the non-dimensional quantity F, defined by

$$F = \frac{u_0^2}{\gamma D_0}, \tag{50}$$

has the structure of a Froude number, in which the acceleration of gravity has been reduced in proportion to the total percentage density range of the fluid. The results obtained show that this new number determines the *nature of the internal, or baroclinе, motions* of the current, whereas the standard Froude number controls the external, or barotropic motions, such as long gravity waves at the free surface.

The analysis presented above indicates that a uniformly stratified current, subjected to momentum losses through frictional action, will be concentrated to the lighter surface layers whenever the internal Froude number is less than a certain critical value (in this case 1/6). For Froude numbers in excess of this critical value momentum losses must result in an increase in depth. Also in this case some shearing motion develops, but it will be less than in the sub-critical case.

It should be noted that whenever the final surface velocity u_s exceeds the initial speed u_0, the free surface must slope downward in the direction of the current axis. The minimum slope required is easily determined from the Bernoulli equation.

Though it is difficult to make exact determinations of the internal Froude number for real ocean currents with their arbitrary velocity profiles it would nevertheless appear, from rough estimates, that these currents generally are characterized by sub-critical values of F. Hence, when subjected to decelerations (momentum losses), they would have a tendency to develop strong shearing motion with increasing velocities and increasing stability near the surface, decreasing velocities and stabilities further down. As far as the Gulf Stream is concerned, this would not necessarily be the case in the Straits of Florida or over the Continental Shelf, where the range in σ_t is smaller than further downstream and where no homogeneous deep water is available to facilitate separation of the current from the bottom. After the current leaves Cape Hatteras, however, the momentum it gains through direct wind action on the narrow strip exposed to the atmosphere is presumably incapable of balancing the losses which result from interaction with the deeper water masses and through lateral mixing. Hence the current tends to become more and more superficial; through this process the high surface velocities are maintained.

The stagnant water masses which surround the swift moving upper layers of the current possess, as a consequence of the earth's rotation, a high degree of inertia stability. They will therefore tend to resist changes in width of the current, in proportion to its superficial concentration. It follows that the vertical concentration of momentum must be accompanied by a corresponding sharpening of the horizontal current profile. This effect will be discussed in Part II. There is, however, another possible consequence of the earth's rotation which deserves mention at the present time, namely the effect on the current direction. As the upper layers shrink vertically and stretch out in the direction of motion they will necessarily acquire increased anticyclonic vorticity. They will therefore bend more and more sharply clockwise, away from shore, while the deeper strata, which are subjected to vertical stretching, tend to bend cyclonically toward the coast. It is reasonable to assume that this process must facilitate upwelling of colder water along the left edge of the current. The anticyclonic deflection may ultimately turn the current around a full right angle. This turning, in combination with the tendency toward upwelling along the left edge, may at least in part contribute to the formation of the peculiar, long-necked north-south meander pattern observed during the recent Gulf Stream survey. Similar patterns have been observed in the atmosphere, at the tropopause level, east of the Rocky Mountains.

Our basic result up to this point, equation (21), is derived under the assumption that the mass transport does not vanish anywhere within the medium; the equation shows that the highest current speed then necessarily must be associated with the largest specific volumes. Hence the swiftest current must flow right at the surface. This would be true even in a current system which before adjustment had been characterized by a velocity maximum in the interior of the fluid. If the homogeneous top-layer before adjustment is characterized by a small but non-vanishing volume transport its thickness would decrease until the appropriate maximum speed is reached.

This result may be in accord with normally prevailing conditions, even though certain cross sections calculated in earlier years from the more sparse data then available show current maxima well below the free surface. This is the case, for instance, with the current values published by Iselin (1936) for a section across the Gulf Stream off Chesapeake Bay, in which a maximum of 120 cmps is located at a depth of about 200 m. New and detailed measurements are required to settle this point.

The problem of current maxima below the free surface may conceivably be of some interest in connection with the analysis of river flow into harbors or tidal basins, where water masses of different density are brought into contact with each other. It is certainly of considerable meteorological importance. We shall therfore attempt to extend our theoretical analysis to include certain types of flow in which a stable maximum may develop also in the interior of the medium.

It will be assumed that the moving, stratified fluid is overlain by a homogeneous layer, in which the volume transport vanishes. Underneath the current there is another homogeneous layer of vanishing volume transport.

Furthermore we shall assume that the specific volume α in the stratified current varies continuously from α_b, the value in the homogeneous bottom layer, to α_s, the value in the surface homogeneous layer. We introduce, as before,

$$\alpha = \alpha_b\,(1 + 2\,\varkappa\sigma),\ \alpha_s = \alpha_b\ (1 + 2\,\varkappa),\quad (51)$$

and hence σ runs from 0 at the bottom of the stratified layer to 1 at its top.

In the analysis of the momentum transport of this stratified current, it can no longer be assumed that the pressure variations along the upper and lower boundaries (a and b) disappear. Because of the absence of motion in the surroundings (apart from transient motions during adjustment), it follows that

$$\delta p_a = -g\varrho_s \delta a, \; \delta p_b = -g\varrho_b \delta b. \quad (52)$$

It should be noted that under the assumptions made above, the level of the free surface must remain constant. This is readily seen from the fact that in the absence of all other motions but transient gravity waves no dissipation of energy can occur within the homogeneous top layer. Hence the Bernoulli equation may be applied to this layer; since the kinetic energy vanishes and the sea level pressure remains constant it follows that the height of the free surface can not change. For similar reasons it follows that the pressure at any fixed level within the lower homogeneous layer cannot change.

To calculate the momentum transfer we extend our integral from some fixed level H_b in the lower layer through the stratified layer up to another fixed, but otherwise arbitrary level H_a in the upper homogeneous layer. Hence the variable portion of the momentum transport is contained in the expression

$$MT = \int_{H_b}^{H_a} (\varrho u^2 + p)\, dz = \int_{H_b}^{b} p\, dz + \int_{a}^{H_a} p dz + \int_0^1 (\varrho u^2 + p)\, \dot{z} d\sigma, \left(\dot{z} = \frac{dz}{d\sigma}\right) \quad (53)$$

or

$$MT = p_{H_b}(b - H_b) - \varrho_b g \frac{(b - H_b)^2}{2} + p_{H_a}(H_a - a) + \varrho_s g \frac{(H_a - a)^2}{2} + \int_0^1 (\varrho u^2 + p)\, \dot{z} d\sigma. \quad (54)$$

It will next be shown that in computing the variation of MT from the expression given above one must take into account a restrictive condition associated with the lack of motion in the two homogeneous layers.

There can be no energy losses in these two media since they remain motionless. Since they, furthermore, are incompressible and homogeneous it follows that the energy per unit mass (E) is constant from point to point and with time in each of these two strata. One finds

$$E_s = p\alpha_s + gz = gh \quad (55)$$

and

$$E_b = p\alpha_b + gz = \alpha_b p_{00}, \quad (56)$$

where h is the constant height of the free surface above the bottom ($z = 0$) and p_{00} the constant bottom pressure. In the intermediate stratified current E varies with height and has the value

$$E = \frac{u^2}{2} + p\alpha + gz. \quad (57)$$

The change from layer to layer is given by

$$\frac{dE}{d\sigma} = \frac{1}{2}\frac{du^2}{d\sigma} + p\frac{d\alpha}{d\sigma} + \alpha\frac{dp}{d\sigma} + g\frac{dz}{d\sigma}. \quad (58)$$

The two last terms on the right side of this equation vanish in consequence of the hydrostatic requirement. Hence

$$\frac{dE}{d\sigma} = \frac{1}{2}\frac{du^2}{d\sigma} + p\frac{d\alpha}{d\sigma} \quad (59)$$

and, if this equation be integrated across the current,

$$E_s - E_b = \int_0^1 p \frac{d\alpha}{d\sigma} d\sigma = 2\varkappa\alpha_b \int_0^1 p\, d\sigma = gh - \alpha_b p_{00} = \text{constant}. \quad (60)$$

Following standard procedures we multiply the constant integral for $E_s - E_b$ with an arbitrary parameter λ and add to the expression for MT. The variation of the sum is easily computed. Keeping in mind that

$$p_b = p_{H_b} - \varrho_b g\,(b - H_b), \; p_a = p_{H_a} + \varrho_s g\,(H_a - a), \quad (61)$$

one finds

$$\delta\,[MT + \lambda\,(E_s - E_b)] = p_a \frac{\delta p_a}{g\varrho_s} - p_b \frac{\delta p_b}{g\varrho_b} + \delta \int_0^1 [\varrho u^2 \dot{z} + p\dot{z} + \lambda \cdot 2\varkappa\alpha_b p]\, d\sigma. \quad (62)$$

For extreme values of momentum transfer this gives, following the same procedure as before,

$$0 = \frac{p_a \delta p_a}{g \varrho_s} - \frac{p_b \delta p_b}{g \varrho b} +$$

$$+ \int_0^1 \left[\left(2 \varkappa \alpha_b \cdot \lambda - \frac{\dot{p}}{g \varrho} \right) \delta p + \right.$$

$$\left. + \left(\frac{g v^2}{\dot{p}^2} - \frac{p}{g \varrho} \right) \delta \dot{p} \right] d\sigma, \qquad (63)$$

where v represents the mass transport per unit interval of σ. The integral may be evaluated by parts and one then finds

$$\frac{p_a \delta p_a}{g \varrho_s} - \frac{p_b \delta p_b}{g \varrho_b} + \left[\left(\frac{g v^2}{\dot{p}^2} - \frac{p}{g \varrho} \right) \delta p \right]_b^a +$$

$$+ \int_0^1 \left[2 \varkappa \alpha_b \lambda - \frac{\dot{p}}{g \varrho} - \right.$$

$$\left. - \frac{d}{d\sigma} \left(\frac{g v^2}{\dot{p}^2} - \frac{p}{g \varrho} \right) \right] \delta p \, d\sigma = 0. \qquad (64)$$

Replacing v by the current speed one finds, after a few reductions, as condition for an extreme value of the momentum transfer

$$\frac{1}{g} [u^2 \delta p]_b^a +$$

$$+ \int_0^1 \left[2 \varkappa \alpha_b \left(\lambda + \frac{p}{g} \right) - \frac{1}{g} \frac{d u^2}{d \sigma} \right] \delta p \, d\sigma = 0. \qquad (65)$$

Hence,

$$u_a = 0, \; u_b = 0 \qquad (66)$$

and

$$\frac{d u^2}{d \sigma} = (p + g\lambda) \, 2 \varkappa \alpha_b \qquad (67)$$

or

$$d u^2 = (p + g \lambda) \, d\alpha \qquad (68)$$

Since u_a and u_b vanish, one may determine λ by integration of this equation across the current. One finds

$$\lambda = -\frac{1}{g} \int_{\alpha_b}^{\alpha_s} p \, d\alpha = -\frac{\bar{p}}{g}, \qquad (69)$$

where $\bar{p}$ is the arithmetic mean value of the pressure with respect to specific volume. Hence

$$\boxed{d u^2 = (p - \bar{p}) \, d\alpha.} \qquad (70)$$

The modification so derived of our previous criterion is perhaps of greater potential usefulness when applied to the atmosphere, for instance in the analysis of cold air outbreaks in the lower troposphere. Nevertheless, a simple application to incompressible media will be given. It will be assumed that initially, within the stratified layer,

$$u = u_0 = \text{constant}, \; \frac{dz}{d\sigma} = D_0, \qquad (71)$$

where D_0 is the initial thickness of the stratified layer. Hence, in the final state

$$\varrho u \frac{dz}{d\sigma} = \varrho u_0 D_0 \qquad (72)$$

and substitution of this expression in (70) gives

$$\frac{du}{dz} = \frac{p - \bar{p}}{2 u_0 D_0} \cdot 2 \varkappa \alpha_b. \qquad (73)$$

To integrate this equation we make use of the fact that to a very good degree of approximation

$$\alpha_b (p - \bar{p}) = g (\bar{z} - z), \qquad (74)$$

where $\bar{z}$ is the level of $\bar{p}$ and the vertical coordinate z is counted upward. Hence

$$\frac{du}{dz} = \frac{\gamma (\bar{z} - z)}{2 u_0 D_0}, \qquad (75)$$

and after integration,

$$u = \frac{\gamma}{2 u_0 D_0} \left[\bar{z} (z - b) - \frac{z^2 - b^2}{2} \right], \qquad (76)$$

b as before being the lower boundary. u must vanish at the upper boundary a and hence

$$\bar{z} (a - b) = \frac{(a^2 - b^2)}{2} \qquad (77)$$

or

$$\bar{z} = \frac{a+b}{2}. \quad (78)$$

A critical Froude number may now be obtained from the requirement that σ must vary from *0* to *1* as z goes from b to a. This may be seen from the calculation presented below.

If one introduces a vertical coordinate y, measured upward from b, the expression for u given in (76) may be changed into

$$u = \frac{\gamma}{4\, u_0 D_0}\, y\,(D-y), \quad (79)$$

where D is the final depth of the stratified current,

$$D = a - b, \; \bar{z} = b + \frac{D}{2}. \quad (80)$$

The expression for $\frac{d\sigma}{dy}$ follows from a combination of (72) and (79). One finds

$$\frac{d\sigma}{dy} = \frac{\gamma}{4\, u_0^2 D_0^2}\, y\,(D-y), \quad (81)$$

and hence, after integration

$$\sigma = \frac{\gamma}{4\, u_0^2 D_0^2}\left[\frac{y^2 D}{2} - \frac{y^3}{3}\right]. \quad (82)$$

At the top of the stratified layer ($y = D$) this must have the value 1. Hence

$$1 = \frac{\gamma}{4\, u_0^2 D_0^2}\,\frac{D^3}{6} \quad (83)$$

and

$$\left(\frac{D}{D_0}\right)^3 = 24\,\frac{u_0^2}{\gamma D_0}. \quad (84)$$

The critical Froude number now has the value 1/24. When

$$F = \frac{u_0^2}{\gamma D_0} < \frac{1}{24} \quad (85)$$

the stratified current will contract and speed up at its center. For $F > 1/24$ the current will expand its cross section and slow down, even though there still will be a concentration of motion at its center. One may introduce an average value of the stability for the stratified layer in its initial state,

$$g\bar{s} = \frac{g\,(\alpha_s - \alpha_b)}{\alpha_b D_0} = \frac{2\,\varkappa g}{D_0} = \frac{\gamma}{D_0}. \quad (86)$$

Then

$$F = \frac{u_0^2}{g\bar{s}\, D_0^2} \quad (87)$$

and the critical relationship becomes

$$u_0 = D_0 \sqrt{\frac{g\bar{s}}{24}}. \quad (88)$$

Further discussions of the velocity profiles derived above must be postponed until observational data are available.

Professor A. Craya, of Grenoble, with whom the author has had the opportunity to discuss the basic principles of this paper, has suggested that the restrictive condition (60) derived above has a much broader validity than here indicated. Indeed, from the hydrostatic condition it follows that

$$g h - \alpha_b p_{00} = \int_{\alpha b}^{\alpha_s} p\, d\alpha, \quad (89)$$

where h is the height of the free surface above the bottom, p_{00} the constant bottom pressure and α_b and α_s the specific volumes at the bottom and the free surface respectively. Since the value of h is very nearly, even though not completely, determined by the depth of the surrounding fluid, it follows that the integral on the right side of (89) must be kept constant during the variation of *MT*. Professor Craya's analysis is presented in a separate contribution to this issue of Tellus (CRAYA, 1951).

We shall finally rewrite our basic equation (21) through the introduction of the vertical stability s. Counting z positive upward one finds

$$s = -\frac{1}{\varrho}\frac{d\varrho}{dz} = \frac{1}{\alpha}\frac{d\alpha}{dz} \quad (90)$$

and thus, through substitution in (21),

$$\frac{du^2}{dz} = \frac{p}{\varrho} s. \tag{91}$$

It will be shown in part II of this study that in spite of the compressibility of air the same equation (91) is valid also in the atmosphere.

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