Critical Regimes of Flows with Density Stratification

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Abstract

This study first discusses the analogy between flow in an open channel and either internal or surface currents in large bodies of fluid and shows that the concept of the critical regime is generally applicable to such cases. The critical regime of gravity currents due to density stratification is then examined in terms of both the momentum principle and the somewhat simpler energy principle; contrary to the case of open-channel hydraulics, the two principles result in different flow criteria. In the case of surface currents it is shown that, for constant piezometric or energy level at the surface, the critical regime corresponds to a velocity distribution which reaches a maximum some distance from the surface.

In the foregoing paper, "On the Vertical and Horizontal Concentration of Momentum in Air and Ocean Currents", Dr Rossby makes novel and significant use of the momentum principle, which has long been used in its more elementary form as one of the basic relationships of fluid mechanics. The central theme of his analysis, in fact, is the generalization of the concept of "minimum" conditions of momentum flux which is fundamental to open-channel hydraulics. As a result, not only does it open promising new horizons for hydraulics, but this field in turn may eventually be able to contribute to the extension of Dr Rossby's original analysis — particularly in the parallel application of the equally fundamental energy principle. The writer, as a hydraulician, has hence sought to organize in a manner which is equally useful to each of the fields the pertinent aspects of both principles, and to present some supplementary results concerning surface currents.

It is at once evident, of course, that the transition from the one degree of freedom of schematic open-channel flow to the many degrees of freedom of meteorology and oceanography introduces unavoidable difficulties and obscurities. I will therefore ask the reader's permission to proceed very gradually and to recall briefly certain familiar fundamental notions from the hydraulics of open channels. The analogy between the flow in an open channel and the movement of density currents becomes more clear if we consider first the case of homogeneous gravity currents. We will then analyze the principal object of this study, that of the critical regime of stratified gravity currents.

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I. The critical regime in the hydraulics of open channels

1. Hypothesis

As it is well known, the concept of a critical regime is met essentially in the study of steady flow in open channels. Steadiness naturally implies the constancy of discharge from section to section. Henceforth we shall consider two-dimensional flow (infinitely large channel) where q shall denote the flow per unit width.

An important characteristic of flow in open channels is the quasihydrostatic pressure distribution in a given transverse section, i.e., $p + \varrho g z =$ constant. This assumption is based on the hypothesis that the curvature of the streamlines is negligible.

A further simplification employed in hydraulics is that of a uniform velocity distribution at a section, which thus is equal to the mean velocity of the flow

$$U = \frac{q}{h}$$

2. Use of energy equation

The counterpart of the energy principle for incompressible fluids is Bernoulli's theorem, which states that (for a perfect fluid only) the quantity $p + \varrho g z + \varrho u^2/2$ is constant along the same streamline.

In view of the assumptions made, this quantity is the same for all the streamlines in a given section. If we take the floor of the channel as the plane of reference, we obtain, after dividing by ϱg, the total head

$$\varepsilon = h + \frac{q^2}{2\,gh^2}$$

This quantity varies from section to section of a channel, due to the changes of elevation of the floor and due to energy losses, but for a given discharge it cannot decrease below a minimum value which occurs for the critical depth defined by

$$\frac{gh^3}{q^2} = 1$$

In other words, a given discharge is allowed to flow either at great depth and small velocity or at small depth and great velocity and the critical state corresponds to the intermediate stage when the depth equals twice the velocity head. For a channel with horizontal floor where the total head is constantly decreasing, there is then a tendency towards critical conditions.

3. Use of the momentum principle

Let us consider the integral across a section of the quantity

$$p + \varrho u^2$$

The variation of this quantity from section to section represents, in view of the momentum principle, the horizontal component of the forces which the outer portions exert on the periphery of the flow between the two sections. This at least is true if we neglect the normal forces of viscosity and the turbulent flux of momentum across the transverse sections.

For the flow in a channel we obtain the quantity (which, for the sake of brevity, will be named henceforth the thrust)

$$P = \varrho g \left(\frac{h^2}{2} + \frac{q^2}{hg}\right)$$

For a given discharge there is a minimum value of this thrust, which corresponds to the same critical depth defined previously:

$$\frac{gh^3}{q^2} = 1$$

In a horizontal channel the thrust, according to its mechanical significance, cannot but decrease because of friction forces on the bottom, and we find again and for a slightly different reason a tendency toward critical conditions.

4. The influence of unequal velocity distribution

If we take into account the unequal velocity distribution, we have to express the mean total head as

$$\varepsilon = h + \int_0^h \frac{u^2}{2\,g}\,\frac{u\,dz}{q} = h + \beta\,\frac{U^2}{2\,g}$$

and the thrust as

$$\frac{P}{\varrho g} = \frac{h^2}{2} + \int_0^h \frac{u^2}{g}\,\frac{dz}{h} = \frac{h^2}{2} + \alpha\,\frac{U^2}{g}$$

If, for example, similar velocity curves are considered, α and β will be independent of h and the critical regimes corresponding to the total head and the thrust will be, respectively,

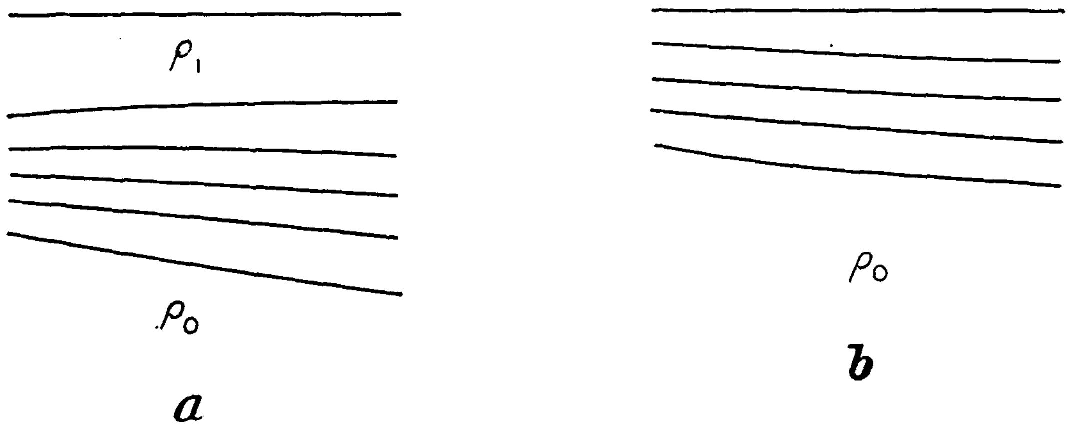

Fig. 1. Gravity currents; (a) submerged, (b) surface.

$$\frac{gh^3}{q^2} = \beta \quad \text{and} \quad \frac{gh^3}{q^2} = \alpha$$

The two lines of attack, then, no longer lead to the same result and we can add that for the particular case under consideration $\beta > \alpha$.

II. Extension to homogenous gravity currents

5. Preliminary remarks

In what follows, we shall denote by "submerged gravity current" a flow confined between two layers of fluid at rest with densities ϱ_0 and ϱ_1, respectively (fig. 1 a). By "surface gravity current" shall be understood a flow with its free surface exposed to the atmosphere with pressure $p = 0$ and taking place on top of a fluid at rest with density ϱ_0 (fig. 1 b).

The homogeneous gravity currents which we intend to examine first are those for which the layer in movement has a uniform density.

For the submerged as well as the surface homogeneous gravity current, it can be seen from fig. 2 that the limits A and B of the current are not independent (the figures represent a hydrostatic pressure distribution, and the slope $-\varrho g$ of AB is fixed).

The interfaces A and B lie on either side of the fixed level Q and coincide with this level only when the current is reduced to an infinitely thin layer. The piezometric head line of each layer is by definition the fictitious free surface, ($p = 0$) which could be obtained by prolonging this layer beyond its actual limits. Thus in fig. 2 a the piezometric head line for the liquid at the bottom of density ϱ_0 is the point N_0, for the intermediate liquid it is N, and for the surface layer it is N_1. We can see from fig. 2 a that the variation N_0N of the piezometric head line around the density discontinuity in B, for example, is equal to $(p_0/g)\,\Delta\,(1/\varrho)$ or $(\Delta\varrho/\varrho)\ \overline{b\,N_0}$. The variations of the piezometric head line are then very small for very small variations of density, but it must be noted that they nevertheless command movements of great amplitude of the current interface.

6. Minimum of energy flux

As in open channels and with reference to the theorem of conservation of energy, we study the flux of the quantity; $p + \varrho g z + \varrho u^2/2 = E$ across a vertical section of the current.

As E is a constant across the gravity current, its total value is given by qE. This flux continuously decreases from section to section by the amount of the internal energy losses.

(a) Let us consider first the case of the submerged flow. Fig. 2 a indicates an immediate geometric relation between N_1P (constant value of $p + \varrho g z$ across the current) and the thickness h of this current. We can express this relation by writing for the upper interface

$$p + \varrho g z = p_1 + \varrho g z_1 = p_1 + \varrho_1 g z_1 + g z_1 (\varrho - \varrho_1)$$

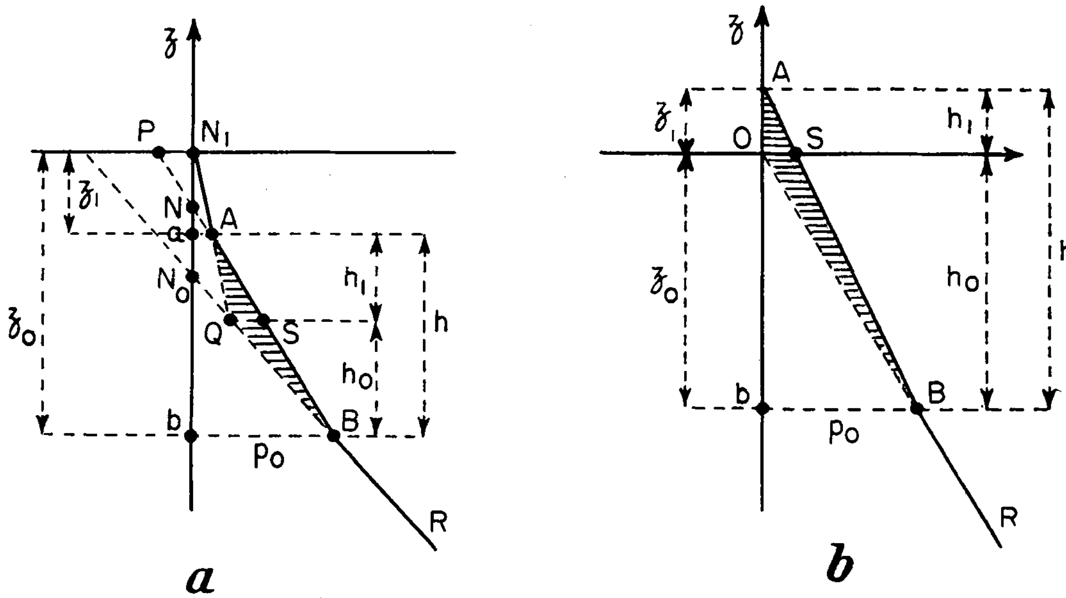

Fig. 2. Pressure distribution in homogeneous currents; (a) submerged, (b) surface.

and for the lower interface

$$p + \varrho g z = p_0 + \varrho g z_0 = p_0 + \varrho_0 g z_0 + \\ + g z_0 (\varrho - \varrho_0)$$

Taking $p + \varrho g z = g Z$, $p_0 + \varrho_0 g z_0 = g Z_0$, $p_1 + \varrho_1 g z_1 = g Z_1$

we obtain $z_1 = \dfrac{Z - Z_1}{\varrho - \varrho_1}$ $\quad z_0 = \dfrac{Z - Z_0}{\varrho - \varrho_0}$

from which we obtain

$$z_1 - z_0 = h = \frac{\varrho_0 - \varrho_1}{(\varrho_0 - \varrho)(\varrho - \varrho_1)} Z + \text{constant}$$

Since E is defined (by the arbitrary datum plane) except for a constant, and since we are interested only in its variation, we can definitely adopt

$$E = \frac{(\varrho_0 - \varrho)(\varrho - \varrho_1)}{\varrho_0 - \varrho_1} g h + \frac{\varrho}{2} \frac{q^2}{h^2}$$

We find consequently for the critical regime which makes E a minimum

$$\frac{(\varrho_0 - \varrho)(\varrho - \varrho_1)}{\varrho(\varrho_0 - \varrho_1)} g \frac{h^3}{q^2} = 1$$

and in the particular case of $\varrho = \dfrac{\varrho_0 + \varrho_1}{2}$

$$\frac{\Delta\varrho}{\varrho} \frac{g h^3}{q^2} = 4$$

(b) The analysis of a surface current proceeds in an analogous manner with the help of fig. 2 b.

We have as previously for the lower interface

$$p + \varrho g z = p_0 + \varrho g z_0 = p_0 + \varrho_0 g z_0 + \\ + g z_0 (\varrho - \varrho_0)$$

which, by setting $p + \varrho g z = g Z$ and $p_0 + \varrho_0 g z_0 = g Z_0$, yields

$$z_0 = \frac{Z - Z_0}{\varrho - \varrho_0}$$

As far as the free surface is concerned, the relation is simply $p + \varrho g z = \varrho g z_1$, which gives $z_1 = Z/\varrho$.

From this we deduce

$$z_1 - z_0 = h = \frac{\varrho_0}{\varrho(\varrho_0 - \varrho)} \cdot Z + \text{constant}$$

and the expression for E is then

$$E = \frac{\varrho\,(\varrho_0 - \varrho)}{\varrho_0}\, g\,h + \frac{\varrho}{2}\,\frac{q^2}{h^2}$$

which for the critical regime becomes, with $\varrho_0 - \varrho = \Delta\varrho$,

$$\frac{\Delta\varrho}{\varrho_0}\,\frac{g h^3}{q^2} = 1$$

It may be remarked that this criterion is the same as the one for the half part of a submerged symmetrical current.

7. Minimum of thrust

With reference to the momentum theorem, we shall analyze the integral of $p + \varrho u^2$ across a vertical plane. This integral continuously decreases from section to section by the amount of the longitudinal shear on the periphery of the current.

If the lower liquid extends to infinity, the integral of the pressure will be infinite, but we can always subtract from it an invariable quantity which comprises the infinite portion.

(a) For a submerged flow (fig. 2 a), it is natural to subtract the constant distribution of pressure N_1AQBR, so that for the variable part only the area AQB remains, which has the value

$$\overline{QS}\cdot\frac{h_1 + h_0}{2}$$

But

$$\overline{QS} = \frac{g\,h_1}{\dfrac{1}{\varrho - \varrho_1}} = \frac{g\,h_0}{\dfrac{1}{\varrho_0 - \varrho}} =$$

$$= g\,(h_1 + h_0)\,\frac{(\varrho_0 - \varrho)\;(\varrho - \varrho_1)}{\varrho_0 - \varrho_1}$$

so that we obtain for the thrust

$$P = \frac{(\varrho_0 - \varrho)\;(\varrho - \varrho_1)}{\varrho_0 - \varrho_1}\; g\,\frac{h^2}{2} + \varrho\,\frac{q^2}{h}$$

and we get for the critical regime the same result as before.

(b) In the same manner, for a surface current we shall subtract the fixed pressure distribution OBR (fig. 2 b), so that the variable part of the pressure integral shall be reduced to the area AOB, i.e.,

$$\overline{OS}\cdot\frac{h_1 + h_0}{2}$$

But

$$\overline{OS} = \frac{g\,h_1}{\dfrac{1}{\varrho}} = \frac{g\,h_0}{\dfrac{1}{(\varrho_0 - \varrho)}} =$$

$$= g\,(h_1 + h_0)\,\frac{\varrho\,(\varrho_0 - \varrho)}{\varrho_0}$$

and the thrust can be written as

$$P = \frac{\varrho\,(\varrho_0 - \varrho)}{\varrho_0}\; g\,\frac{h^2}{2} + \varrho\,\frac{q^2}{h}$$

and the agreement with the critical regime conditions obtained from energy considerations is again maintained.

8. Remarks on surface currents

We have already stressed the fact that in the case of homogeneous currents, A and B necessarily vary together (fig. 2 b). With the very slight differences of density, and reasonable depths encountered in practice, the variation of A is very small; but this reduced degree of freedom of A is the necessary counterpart of the far larger degree of freedom of B.

Thus, if we rigidly fix the free surface A, there is but one regime possible.

The same thing is true if we fix the total head of the topmost layer. In fact, on the basis of the assumption of uniform velocity distribution, we impose the condition of the constancy of the entire flow energy — i.e., according to previous calculations,

$$\frac{(\varrho_0 - \varrho)}{\varrho_0}\,h + \frac{q^2}{2\,g\,h^2} = \varepsilon_1$$

Thus there are at most 2 regimes corresponding to the given conditions.

We shall see that things are different for stratified currents; this is due to the fact that they can vary in a more flexible and less monolithic manner than in the previous schemes.

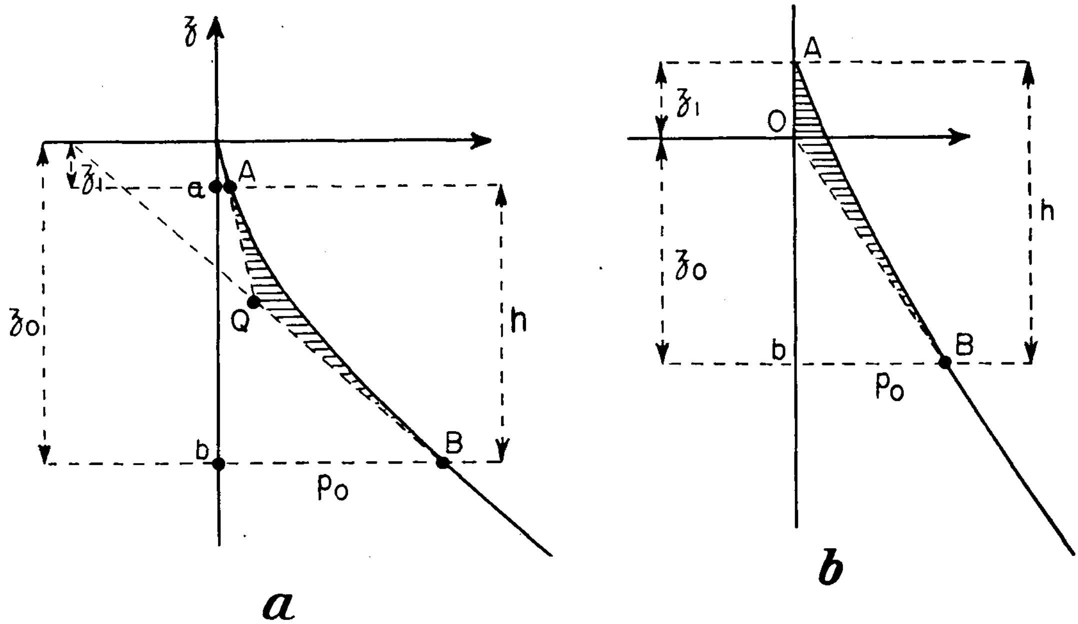

Fig. 3. Pressure distribution in stratified currents; (a) submerged, (b) surface.

III. Stratified gravity currents; critical regime of minimum energy

9. Preliminary mathematical notions

In the course of this analysis we shall have to consider the first variation of an integral of the form

$$I = \int_{x_0}^{x_1} \varphi\ (x, Z, Z', Z'')\ dx$$

if Z is given a variation $\delta Z(x)$ in the fixed interval $x_0 x_1$.

In generalizing the more classical result pertaining to the case where the function φ contains only the first derivative Z', we obtain easily

$$\delta I = \left[\left(\frac{\partial\varphi}{\partial Z'} - \frac{d}{dx}\frac{\partial\varphi}{dZ''}\right)\delta Z\right]_{x_0}^{x_1} + \\ + \left[\frac{\partial\varphi}{\partial Z''}\ \delta Z'\right]_{x_0}^{x_1} + \\ + \int_{x_0}^{x_1}\delta Z\left[\frac{\partial\varphi}{\partial Z} - \frac{d}{dx}\frac{\partial\varphi}{\partial Z'} + \frac{d^2}{dx^2}\frac{\partial\varphi}{\partial Z''}\right] dx$$

If we seek a solution which will make I stationary for the conditions with the given limits, the solution must first of all satisfy the condition

$$\frac{\partial\varphi}{\partial Z} - \frac{d}{dx}\frac{\partial\varphi}{\partial Z'} + \frac{d^2}{dx^2}\frac{\partial\varphi}{\partial Z''} = 0 \qquad (1)$$

The first variation is then reduced to

$$\delta I = \left[\left(\frac{\partial\varphi}{\partial Z'} - \frac{d}{dx}\frac{\partial\varphi}{\partial Z''}\right)\delta Z\right]_{x_0}^{x_1} + \\ + \left[\frac{\partial\varphi}{\partial Z''}\ \delta Z'\right]_{x_0}^{x_1} \qquad (2)$$

Depending upon the nature of the conditions imposed on Z and Z' at the extremities of the interval, we can deduce, by cancelling this residual variation, the four constants of integration of the differential Eq. (1).

10. Condition of minimum of energy flux

Generalizing the previous considerations, we are lead to examine for a stratified current the energy flux across a section:

$$I = \int_0^q \left(p + \varrho g z + \varrho \frac{u^2}{2} \right) dq$$

As the density is constant in each thin layer (fig. 1 a), as well as the elementary discharge dq of the layer, we can set

$$dq = u dz = -k(\varrho) d\varrho \qquad (3)$$

The function $k(\varrho)$, which imposes a fixed distribution of the discharge between different layers, generalizes thus the notion of total given discharge in the preceding calculations.

We shall take as a characteristic distribution function of pressures, densities, and velocities in a section the quantity:

$$gZ = p + \varrho g z \qquad (4)$$

This expression is closely related to the notion of piezometric level and like it, varies relatively little but commands the whole internal structure of the current.

By differentiating Z we obtain

$$g dZ = dp + \varrho g dz + g z d\varrho$$

As $dp + \varrho g\, dz = 0$ from the assumption of hydrostatic distribution, we have

$$z = \frac{dZ}{d\varrho} = Z' \qquad (5)$$

Consequently, by virtue of Eq. (4) the pressure is given by

$$p = g(Z - \varrho Z')$$

On the other hand, because of Eqs. (3) and (5), the velocity is given by

$$u = -k(\varrho) \frac{d\varrho}{dz} = -\frac{k}{Z''}$$

The energy flux is written then in final form with this variable Z

$$I = \int_{\varrho_1}^{\varrho_0} \left(g k Z + \frac{\varrho}{2} \frac{k^3}{Z''^2} \right) d\varrho$$

and in order that this flux becomes stationary (in our case a minimum) Z must satisfy the condition

$$g k - \frac{d^2}{d\varrho^2} \left(\frac{\varrho k^3}{Z''^3} \right) = 0$$

i.e.

$$\frac{d^2}{d\varrho^2} (\varrho u^3) = -g k(\varrho) \qquad (6)$$

II. Profiles of critical velocity

In order to introduce dimensionless variables we shall set

$$r = \frac{\varrho - \varrho_1}{\Delta \varrho} \quad \text{and} \quad k(\varrho) = \frac{q_0}{\Delta \varrho} f(r)$$

where q_0 designates the total discharge and $\Delta \varrho$ the difference of density $\varrho_0 - \varrho_1$ between the extreme layers of the current.

Then, the differential equation (6) of the critical state becomes

$$\frac{d^2}{dr^2} (\varrho u^3) = -g q_0 \Delta \varrho f(r)$$

Introducing also two successive integrals of f

$$F(r) = \int_0^r f(r)\, dr \qquad \Phi(r) = \int_0^r F(r)\, dr$$

and a reference velocity $w = \sqrt[3]{(\Delta \varrho / \varrho) g q_0}$ (which is practically a constant if density varies very little) we have

$$\left(\frac{u}{w} \right)^3 = \alpha r + \beta - \Phi(r) \qquad (7)$$

The function $f(r)$ being positive, $\Phi(r)$ is concave upwards and from fig. 4 we get a general picture of the required velocity distributions for critical conditions. Depending on the values of α and β, the velocity curve continuously increases from bottom to top, or

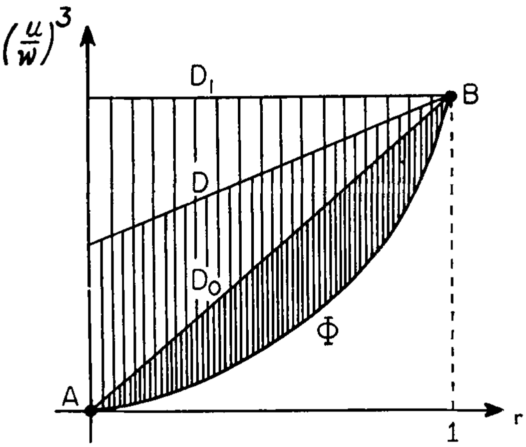

Fig. 4. Critical distributions of velocity.

on the contrary has an intermediate maximum. It remains to eliminate this arbitrariness according to the conditions at the limits.

12. The case of a submerged current

The variation δI for distribution of critical velocity can be written according to Eq. (2) and the value of $\partial\varphi/\partial Z'' = -\varrho k^3/Z''^3 = \varrho u^3$ in the form

$$\delta I = -\left[\frac{d}{d\varrho}(\varrho u^3)\,\delta Z\right]_{\varrho_1}^{\varrho_0} + \left[\varrho u^3\,\delta Z'\right]_{\varrho_1}^{\varrho_0}$$

For a submerged current Z_0 and Z_1 (or the extreme piezometric levels) are imposed; consequently δZ_0 and δZ_1 are zero, whereas values of $\delta Z' = \delta z$ at both extremities are free and independent, (i.e., there is no relation here between the displacement of A and B as in the case for homogeneous current). Then we must have $u_0 = 0$ and $u_1 = 0$, and in fig. 4, $\alpha r + \beta$ is represented by the straight line D_0 which passes through A and B.

The constants α and β are then

$$\beta = 0 \qquad \alpha = \Phi(1).$$

and

$$\left(\frac{u}{w}\right)^3 = r\,\Phi(1) - \Phi(r).$$

The distribution of densities as a function of the height results immediately in

$$\frac{w}{q_0}\,z = -\int \frac{f(r)\,dr}{\sqrt[3]{\alpha r + \beta - \Phi(r)}}$$

The total height h of the current is thus given in a general manner after setting $\Pi = (\Delta\varrho/\varrho)(g\,h^3/q^2)$:

$$\sqrt[3]{\Pi} = \int_0^1 \frac{f(r)\,dr}{\sqrt[3]{\alpha r + \beta - \Phi(r)}}$$

and in particular for a submerged current is given by

$$\sqrt[3]{\Pi} = \int_0^1 \frac{f(r)\,dr}{\sqrt[3]{r\,\Phi(1) - \Phi(r)}}$$

For an initial state of constant velocity and linear density distribution we have $f = 1$, $\Phi(r) = r^2/2$, and

$$\sqrt[3]{\Pi} = \int_0^1 \frac{dr}{\sqrt[3]{\frac{r}{2} - \frac{r^2}{2}}} = 2\int_0^1 \frac{dx}{\sqrt[3]{1 - x^2}} =$$

$$= 2 \cdot \int_0^{\pi/2} \cos^{1/3}\varphi\,d\varphi$$

This last value of the parameter Π is calculated either by means of elliptic functions or by numerical integration. The value of Π thus found is $\Pi \approx 17$.

It is also valuable to derive on hand of this general theory the results for the homogeneous currents obtained more directly at the beginning. If q is the discharge between the layer of density ϱ_1 at the upper interface and a layer of density ϱ, we have

$$\frac{q}{q_0} = F(r)$$

Let us consider the case where this discharge is zero from ϱ_1 to ϱ_d, becomes q_0 for ϱ_d, and remains q_0 from ϱ_d to ϱ_0. This can be considered to be the same as the case of a homogeneous current where the layers comprised between ϱ_1 and ϱ_d and between ϱ_d and ϱ_0 are like two infinitely thin sheets constituting the surfaces of discontinuity of density.

The function $f(r)$ is then zero everywhere except for the value of r_d corresponding to ϱ_d. The function Φ is represented by the broken line AIB of fig. 5 and the value of the parameter becomes

$$\sqrt[3]{\Pi} = \frac{1}{\sqrt[3]{r_d\,\Phi(1) - \Phi(r_d)}} \int_0^1 f(r)\,dr = $$

$$= \frac{1}{\sqrt[3]{r_d(1 - r_d)}}$$

This gives

$$\frac{(\varrho - \varrho_1)(\varrho_0 - \varrho)}{\varrho(\varrho_0 - \varrho_1)} \frac{gh^3}{q^2} = 1$$

which is in complete agreement with the previous result.

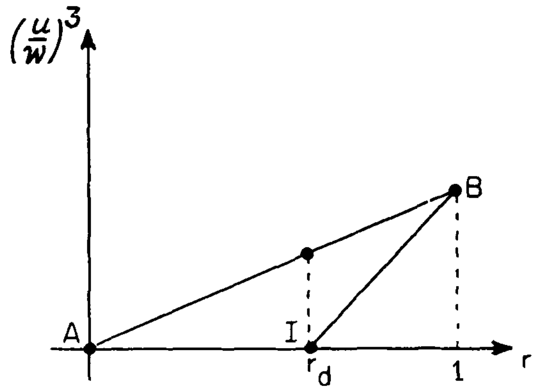

Fig. 5. Velocity diagram for homogenous current.

The conditions that determine α and β become then for the present problem

$$u_0 = 0 \quad \text{and} \quad u_1{}^3 = \left[\frac{d}{d\varrho}(\varrho u^3)\right]_1$$

When expressed in terms of the relative variables previously introduced, this last relation becomes

$$\frac{\Delta\varrho}{\varrho_1}\left(\frac{u}{w}\right)_1^3 = \left[\frac{d}{dr}\left(\frac{u}{w}\right)^3\right]_1$$

i.e.,

$$\frac{\Delta\varrho}{\varrho_1}\beta = \alpha$$

The first condition $u_0 = 0$ requires that $\alpha + \beta = \Phi(1)$, from which we deduce that

$$\beta = \frac{\varrho_1}{\varrho_0}\Phi(1) \qquad \alpha = \frac{\Delta\varrho}{\varrho_0}\Phi(1)$$

It follows, then, that for a surface current the distribution of critical velocity at first increases with elevation above the lower interface, passes through a maximum below the free surface, and then decreases (see fig. 4, where $\alpha r + \beta$ is represented this time by the straight line D_1).

It must be added, however, that this rigorous result (there are no approximations in the preceding calculations) reduces practically for very small differences of density to a maximum velocity at the free surface; the slope α is then in fact very small and we have essentially

$$\beta = \Phi(1) \qquad \alpha = 0$$

13. Case of a surface current

Consider again the variation I for a distribution of critical velocity:

$$\delta I = -\left[\frac{d}{d\varrho}(\varrho u^3)\,\delta Z\right]_{\varrho_1}^{\varrho_0} + \left[\varrho u^3\,\delta Z'\right]_{\varrho_1}^{\varrho_0}$$

For the case of a surface current, the piezometric level of the lower liquid at rest is always imposed. Thus, as previously, Z_0 is given and δZ_0 is zero.

The condition of the surface, however, is different. In fact, the free surface is characterized by $p_1 = 0$;

hence, $\quad gZ_1 = p_1 + \varrho_1 g z_1 = \varrho_1 g z_1$

i.e., $\quad \delta Z_1 = \varrho_1 \delta z_1$

Furthermore, as shown before, $Z' = dZ/d\varrho = z$, so that the expression for the variation δI becomes

$$\delta I = \varrho_0 u_0{}^3 \delta z_0 + \varrho_1\left\{\left[\frac{d}{d\varrho}(\varrho u^3)\right]_1 - u_1{}^3\right\}\delta z_1$$

Under these conditions the critical depth satisfies

$$\sqrt[3]{\Pi} = \int_0^1 \frac{f(r)\,dr}{\sqrt[3]{\Phi(1) - \Phi(r)}}$$

In particular for the case of $f = 1$ and $\Phi = r^2/2$ we obtain

$$\sqrt[3]{\Pi} = \int_0^1 \frac{dr}{\sqrt[3]{\frac{1}{2} - \frac{r^2}{2}}} = \sqrt[3]{2} \int_0^1 \frac{dr}{\sqrt[3]{1 - r^2}} =$$

$$= \sqrt[3]{2} \cdot \int_0^{\pi/2} \cos^{1/3}\varphi \, d\varphi$$

After integrating we get

$$\Pi \approx \frac{17}{4}$$

We could also determine, by exactly the same method, the conditions of critical regime for a homogeneous current established differently at the beginning of this paper.

14. Surface currents — supplementary remarks

For surface currents a question comes up which does not intervene in the study of submerged currents; in the latter case the physical condition of a current limited on either side by liquid at rest is perfectly clear and can be unequivocally translated into mathematical conditions. For a surface current the same thing is true for the lower interface, but the question can be raised whether all required conditions for the free surface were properly taken into account.

In the preceding calculations the free surface was left perfectly free to move along the vertical. Is this always true and will not other physical conditions limit this freedom of movement? One could think, for example of marine currents like the Gulf Stream. The presence of a mass of water at rest on either side of the current of limited width would cause a lateral entrainment, if the level of the current was allowed to vary. On the other hand, even for a two-dimensional flow (laboratory experiment in wide channels), if the drag on the topmost sheet is to be zero, the total energy of this sheet $z_1 + u_1^2/2g$ must remain constant.

The problem then becomes, to paraphrase Kipling, another story, which would require further developments. We want to note here simply that in the case of stratified currents we have the possibility of taking into account such a condition, whereas for a homogeneous current a condition imposed on the free surface would determine completely the flow.

To show this for a relatively simple case let us imagine, for example, that the imposed condition is a free horizontal surface. We have then z_1 given and $\delta z_1 = 0$. The freedom which remains for δz_0 requires for the minimum of the energy flux that $u_0 = 0$; i.e. (*Eq.* 7) $\alpha + \beta = \Phi(1)$ and hence

$$\left(\frac{u}{w}\right)^3 = \alpha(r - 1) + \Phi(1) - \Phi(r)$$

It is then possible to determine the correspondence between α and z_1 (a calculation which seems useless to develop here in detail).

From fig. 4 we can see that now we have a straight line D (representing $\alpha r + \beta$) passing always through B, but susceptible of taking different slopes α depending on the condition of level imposed. The corresponding distribution of velocity has this time a net maximum under the free surface. The study of the case where the energy of the topmost sheet is imposed would led to analogous conclusions.

IV. Stratified gravity currents; critical regime of minimum thrust

15. Variation of the thrust

Changing now the angle of attack, we take up the principle of momentum and consequently the study of the thrust. As for the homogeneous currents, we are led to consider as the variable part of the pressure integral the cross-hatched area of fig. 3, $\int_{\varrho_0}^{\varrho_1} p\,dz - F$ where F designates the area aAQBb for a submerged current (fig. 3 a) and the area OBb for a surface current (fig. 3 b). We shall write the thrust then as

$$P = \int_{\varrho_0}^{\varrho_1} (p + \varrho u^2)\, dz - F$$

Taking again as the principal distribution function the piezometric level, or the quantity related to it,

$$gZ = p + \varrho g z$$

the expression for the thrust becomes

$$P = \int_{\varrho_0}^{\varrho_1} \left[gZZ'' - \varrho g Z' Z'' + \varrho \frac{k^2}{Z''} \right] d\varrho - F$$

The distribution Z which will make P a minimum must first satisfy Eq. (1) of section 9; that is, after simplifying,

$$\frac{d^2}{d\varrho^2} \left(\frac{\varrho k^2}{Z''^2} \right) = g Z'' \qquad (8)$$

For the functions satisfying this differential equation the variation of P is then (Eq. 2, section 9)

$$\delta P = \left[\frac{d}{d\varrho} \left(\varrho \frac{k^2}{Z''^2} \right) \delta Z \right]_{\varrho_0}^{\varrho_1} +$$

$$+ \left[\left(gZ - \varrho g Z' - \varrho \frac{k^2}{Z''^2} \right) \delta z \right]_{\varrho_0}^{\varrho_1} - \delta F$$

or

$$\delta P = \left[\frac{d}{d\varrho} (\varrho u^2)\, \delta Z \right]_{\varrho_0}^{\varrho_1} + \left[(p - \varrho u^2)\, \delta z \right]_{\varrho_0}^{\varrho_1} - \delta F$$

From fig. 3 a, it is clear that for a submerged current we have for δF

$$\delta F = p_1 \delta z_1 - p_0 \delta z_0$$

For a surface current it will be (Fig. 3 b)

$$\delta F = - p_0 \delta z_0$$

On the other hand, for a submerged current Z_0 and Z_1 are imposed, hence $Z_0 = 0$, $Z_1 = 0$, and we have

$$\delta P = \left[\varrho u^2 \delta z \right]_{\varrho_1}^{\varrho_0}$$

For a surface current $p_1 = 0$; $Z_1 = \varrho_1 g z_1$, and Z_0 is given; the remaining variation is then

$$\delta P = \varrho_0 u_0^2 \delta z_0 + \varrho_1 \left[\frac{d}{d\varrho} (\varrho u^2) - u^2 \right]_1 \delta z_1$$

16. Condition of minimum thrust

The first thing that commands attention here is evidently that considerations of the minimum energy flux and of the minimum thrust do not lead to the same result. This will seem less surprising, however, if we recall that in the hydraulics of open channels this agreement is obtained only for the case of uniform velocity distribution. Such is not the case for stratified density currents where the velocity profile varies in accordance with the law of conservation of flow for each sheet.

We can give to Eq. (8) different interesting forms.

(a) If we set $\varrho u^2 = Y$, we have

$$\sqrt{Y} \frac{d^2 Y}{d\varrho^2} = - g\, k(\varrho) \sqrt{\varrho}$$

If ϱ varies very little, it is quite legitimate to use the following approximation.

$$u \frac{d^2 u^2}{d\varrho^2} = - g \frac{k(\varrho)}{\varrho}$$

Introducing relative variables by

$$r = \frac{\varrho - \varrho_1}{\Delta \varrho} \quad k = \frac{q_0}{\Delta \varrho} f(r) \quad w = \sqrt[3]{\frac{\Delta \varrho}{\varrho} g q_0}$$

$$\frac{u}{w} = \zeta$$

we obtain

$$\zeta \frac{d^2 \zeta^2}{dr^2} = - f(r)$$

We recall that the corresponding equation for the energy flux was

$$\frac{d^2 \zeta^3}{dr^2} = - f(r)$$

The present result is thus more difficult to analyse for the general case.

(b) We can also integrate Eq. (8) once and write:

$$\frac{d}{d\varrho}(\varrho u^2) = g(z+\alpha) \qquad (9)$$

Assuming again that ϱ varies very little and using the continuity equation, we get

$$-2\,\varrho\, k(\varrho)\frac{du}{dz} = g(z+\alpha)$$

For the particular case when $k = q_0/\Delta\varrho$ (current with an initial uniform velocity and constant density gradient) we have

$$\frac{du}{dz} = -\frac{1}{2\,q_0}\frac{\Delta\varrho}{\varrho}\,g(z+\alpha) \qquad (10)$$

(c) Lastly, if we integrate Eq. (8) twice, we have the alternative form

$$\varrho u^2 = p + \varrho g z + \varrho g\alpha - p_m \qquad (11)$$

Written in the form $u^2 = p/\varrho + gz + g\alpha - p_m/\varrho$ and differentiated this equation becomes, after taking into account also the hydrostatic equilibrium,

$$du^2 = (p - p_m)\,d\frac{1}{\varrho}$$

which was used by Professor Rossby in this special form.

17. Critical depths

From now on we are going to limit ourselves to the particular case of $k(\varrho) = q_0/\Delta\varrho$, which alone is solvable without excessive calculations.

(a) For a submerged current the conditions at the limits are $u_0 = 0$, $u_1 = 0$. Relation (10) shows that the distribution of velocities is parabolic. By taking the middle of the current as the origin of the z's, this distribution can be written

$$u = -\frac{1}{4\,q_0}\frac{\Delta\varrho}{\varrho}\,g\left(z^2 - \frac{h^2}{4}\right)$$

and the total flow satisfies

$$\int_{-h/2}^{+h/2} u\,dz = q_0$$

From here it follows immediately that

$$\Pi = \frac{\Delta\varrho}{\varrho}\,g\,\frac{h^3}{q^2} = 24$$

This origin of z's corresponds to $\alpha = 0$.

Now Eq. (11) imposes on the two limits of the current where the velocity is zero the conditions

$$gZ_1 + g\alpha\varrho_1 - p_m = 0$$
$$gZ_0 + g\alpha\varrho_0 - p_m = 0$$

from which

$$\alpha = \frac{Z_1 - Z_0}{\varrho_1 - \varrho_0} = 0$$

Consequently, the middle of the current is the juncture of the two straight lines representing the hydrostatic pressure distributions imposed by fluids at rest surrounding the density current, i.e., point Q of fig. 3 a.

(b) Let us consider now a surface current and let us take as origin of ordinates z the piezometric level of the motionless liquid below (fig. 3 b).

From section 15 the actual conditions of minimum are

$$u_0 = 0$$
$$u_1^2 = \left[\frac{d}{d\varrho}\,\varrho u^2\right]_1$$

Equation (10) gives for the free surface ($p = 0$)

$$\varrho_1 u_1^2 = \varrho_1 g z_1 + \varrho_1 g\alpha - p_m$$

and for the lower interface ($p + \varrho g z = gZ_0 = 0$)

$$\varrho_0 u_0^2 = \varrho_0 g\alpha - p_m = 0$$

On the other hand, from Eq. (9)

$$\left[\frac{d}{d\varrho}(\varrho u^2)\right]_1 = g(z_1 + \alpha)$$

We have thus $p_m = 0$ and $\alpha = 0$. The velocity distribution is then, with regard to Eq. (10),

$$u = -\frac{1}{4\,q_0}\frac{\Delta\varrho}{\varrho}\,g(z^2 - z_0^2)$$

We see that the maximum of velocities is not exactly on the free surface ($z = z_1$), but a little below, on the piezometric level $z = 0$ of the motionless fluid (point Q of the figure); this will be a negligible difference if the density varies very little. Consequently we shall take $z = 0$ for the level of the free surface and, taking $\int_{z_0}^{0} u dz = q_0$, we obtain

$$\Pi = \frac{\Delta\varrho}{\varrho}\,\frac{gh^3}{q^2} = 6$$

If now we compare the result obtained for the minimum of thrust and that obtained for the minimum of energy flux, we find that the second criterion gives smaller critical values of Π.

At the beginning of this study we performed an analogous comparison for the hydraulics of open channels and obtained opposite results. It must be added, however, that the two cases are not comparable, among other things because in the case of open channels we had assumed that the velocity curves remained similar during the variations.

18. Supplementary remarks on surface currents

One can also ask the same question as for the minimum of flux energy: Do we have to let the vertical displacements of the free surface be completely arbitrary during the variations or shall we subject them to some such conditions as the constancy of total head,

$$z_1 + \frac{u_1^2}{2g} = d \tag{12}$$

Perhaps it will be useful to investigate here a little more completely the modifications that such a statement of problem imposes on the critical state.

The variations of δz_0 of the lower interface always remains arbitrary so that the velocity u_0 must be zero because of the expression of the variation of the thrust (section 15). The general expression (10) of the velocity shows that the position of the maximum of the velocity is given by $z = -\alpha$. Taking into account that $u = 0$ for $z = z_0$ and integrating this equation we obtain

$$u = \frac{1}{4 q_0}\,\frac{\Delta\varrho}{\varrho}\, g\,[z_0^2 - z^2 + 2\alpha(z_0 - z)] \tag{13}$$

We still have the condition

$$\int_{z_0}^{z_1} u\,dz = q_0$$

and, as it was observed already, z_1 is extremely small and the free surface can be assumed to coincide with $z = 0$ so that $\int_{z_0}^{0} u dz = q_0$.

This condition can be immediately written in dimensionless form. Calling h the height of the current ($h = -z_0$) and Π the parameter $(\Delta\varrho/\varrho)\ (gh^3/q^2)$,

$$\frac{1}{\Pi} = \frac{1}{6} - \frac{\alpha}{4h}$$

We must observe that up to now we have not used the surface condition; if we designate by $m = \alpha/h$ the relative depth of the velocity maximum, we will have in the most general manner and for all surface conditions

$$\frac{1}{\Pi} = \frac{1}{6} - \frac{m}{4} \tag{14}$$

In particular, if the maximum is at the surface we find for the critical parameter the value $\Pi = 6$ previously computed.

Let us consider now Eq. (11). Noting that the velocity is zero for z_0, we have

$$0 = \varrho_0 g\alpha - p_m$$

and this relation becomes

$$\varrho u^2 = p + \varrho g z - g\alpha(\varrho_0 - \varrho)$$

In particular, on the free surface where $p = 0$

$$\frac{u_1^2}{g} = z_1 - \frac{\Delta\varrho}{\varrho}\,\alpha$$

Equation 12 becomes, by introducing the preceding value of z_1,

$$\frac{u_1^2}{2g} = \frac{1}{3}\left(d - \frac{\Delta\varrho}{\varrho}\,\alpha\right)$$

and in its dimensionless form

$$\frac{u_1^2}{2gh} = \frac{1}{3}\left(\frac{d}{h} - \frac{\Delta\varrho}{\varrho}\,m\right)$$

But the velocity at the surface is given by Eq. (13) in which we can again assume that $z_1 = 0$; thus,

$$u_1 = \Pi\,\frac{q_0}{4\,h}\,(1-2\,m)$$

We can note here (an obvious property of the parabola) that if $m > 1/2$ the velocity at the surface becomes negative; we have to exclude such solutions from a physical viewpoint, because they correspond to an instable gradient of the density and the appearance of densities foreign to the problem; m can then vary from $-\infty$ to $1/2$, but beyond that, the mathematical solution has no physical significance.

Writing now the imposed condition

$$\frac{u_1^2}{2gh} = \frac{1}{3}\left(\frac{d}{h} - \frac{\Delta\varrho}{\varrho}\,m\right) = \frac{\Pi^2\,q_0^2}{32\,gh^3}\,(1-2\,m)^2$$

and by calling X the ratio $\varrho d/h\Delta\varrho$ we get

$$\frac{1}{3}\,(X - m) = \frac{\Pi}{32}\,(1-2\,m)^2 \qquad (15)$$

The elimination of m between Eqs. (14) and (15) leads to

$$X = \frac{\Pi}{96} + \frac{2}{\Pi} + \frac{1}{6}$$

We must note, however, that the auxiliary quantity X contains the unknown h sought. We shall separate the given quantity and the unknown by setting

$$K = \sqrt[3]{\left(\frac{\varrho}{\Delta\varrho}\right)^2 \frac{gd^3}{q_0^2}}$$

which gives

$$X = \frac{K}{\sqrt[3]{\Pi}}$$

We have then in final form

$$K = \sqrt[3]{\Pi}\left[\frac{\Pi}{96} + \frac{2}{\Pi} + \frac{1}{6}\right]$$

to which we must add the relation

$$m = \frac{2}{3} - \frac{4}{\Pi}$$

as giving a clear picture of the velocity distributions in relation to the relative depth of the maximum.

As K is not very expressive, we can write it also in terms of the height and velocity of an initial state of uniform velocity and constant density gradient. It can easily be shown that in this case $z_1 = (1/2)\,(\Delta\varrho/\varrho)\,h_0$ and as $u_1^2/2\,g = q_0^2/2\,g\;h_0^2$, we obtain by setting $\Pi_0 = (\Delta\varrho/\varrho)\,(g\,h_0^3/q_0^2)$ the form

$$K = \frac{1}{2}\,\sqrt[3]{\Pi_0}\left(1 + \frac{1}{\Pi_0}\right)$$

Thus, the critical value of Π depends this time on Π_0, and one could interpret the preceding results as a tendency of an initial state Π_0 toward a (corresponding) critical state characterized by Π.

The study of the case where the level of the free surface is imposed

$$z_1 = d$$

leads to similar results. If we introduce again a parameter $K = \sqrt[3]{(\varrho/\Delta\varrho)^2\,(gd^3/q_0^2)}$ the critical value of Π is given by

$$K = \sqrt[3]{\Pi}\left(\frac{\Pi}{144} + \frac{1}{3}\right)$$

and K is related to a critical state of constant velocity and constant density gradient by

$$K = \frac{1}{2}\,\sqrt[3]{\Pi_0}$$

Summary and perspective

As has been shown in the foregoing pages, the momentum principle and the energy principle of mechanics permit two different integrals to be written for the cross section of a gravity current of constant width. Because of external shear on the one hand and energy loss on the other, both integrals must decrease in magnitude in the direction of flow. Moreover, if we consider all the possible states which the steady flow of such a current can take, each of the integrals is found to have a minimum magnitude corresponding to a certain critical thickness of the current and a certain critical velocity distribution. The critical states indicated by the two principles are not identical, although qualitatively comparable.

The expression "all the possible states" requires particular attention in the case of surface currents. If some condition restricts the vertical displacement at the free surface, the critical state will be modified by this condition. In general it will result in a critical velocity distribution in which the maximum is reached at some distance from the surface. If this condition does not exist, however, the maximum will occur at the surface.

The reader might well ask what precise physical significance these critical regimes have. They appear to play the same role as a road sign on a highway, in that they provide a sense of direction — a prediction as to a future stage — but unfortunately without any indication as to the distance which remains to be traversed before that stage is reached. In the early stage ,however, it is perhaps, the local effects of the surroundings which actually govern the velocity distribution rather than the perception of a low-ebb limit beyond which further travel is impossible. In other words, the critical stage may be only a hypothetical terminus — at least for the assumed nature of the current (very small curvature, hydrostatic pressure distribution, etc.).

In open-channel flow, nevertheless, the critical stage is generally reached at such a catastrophic change in boundary conditions as an abrupt drop, and the effect of this change continues upstream in the form of a backwater curve. Not only may there well be analogous catastrophes in the case of gravity currents, but the critical regime may then also be an element of more general "backwater" effects. This hypothesis remains to be elaborated, just as we still have to verify the existing theoretical results through additional experimental evidence from nature and from the laboratory.

A further question as to marine currents arises in connection with their freedom of lateral change as well as vertical. The answer to how and why lateral equilibrium is established must await further progress in the theory. Perhaps one will be tempted to conclude from these remarks that what is known is little in comparison with what remains to be learned — but is this not the general situation in all science?

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