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Exact analytical solutions for elliptical vortices of the shallow-water equations

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ABSTRACT

It has long been known that the shallow-water equations in a rotating framework admit exact solutions for which the velocity components are linear and the height field is quadratic in the coordinate variables. The resulting 12 coefficients are, in general, functions of time and are governed by a set of nonlinear, coupled, ordinary differential equations. While previous analytical solutions have been plagued by a reduced number of degrees of freedom, the most general solution presented here contains 10 arbitrary constants of integration. These degrees of freedom can be sorted as follows: 4 are accounted for by inertial oscillations of the vortex' center of mass, 2 by an average circular structure (depth and radius), 2 by a circular pulsation mode (amplitude and phase), and 2 by 1 of 2 possible elliptical rotation modes (amplitude and phase). The complete solution with 12 degrees of freedom would be achieved if these 2 elliptical modes could be incorporated simultaneously. Unfortunately, the solution that includes the nonlinear interaction between the elliptical rotation modes has yet to be found. The present solution sheds some light on the behavior of time-dependent elliptical warm-core rings as observed in the ocean. It can be used as the basis for further theoretical investigations and testing of numerical models.

1. Introduction

The set of shallow-water equations has long been applied to rotating fluids. In particular its reduced-gravity equivalent that describes the motion of an upper layer sitting above a slightly heavier, quiescent layer has proven to be a practical tool to model various upper-ocean phenomena. This set of equations is nonetheless severely limited for, mostly, it does exclude the baroclinic-instability mechanism. The thrust of the present contribution is the quest for exact analytical solutions in a closed form, for such solutions are often used as stepping stones toward further analytical investigations and can also be used to test the accuracy of numerical models. As such solutions are rare and quite difficult to obtain, the choice is made to limit the search within the context of the shallow-water equations where the Coriolis parameter is a constant.

Since Goldsbrough (1930), it has been known, forgotten and rediscovered that the shallow-water equations, with inclusion of rotation, admit a special class of solutions for which the velocity components are linear functions and the height field is a quadratic expression in the horizontal coordinate variables. Perhaps this discovery dates back to Kirchhoff (Kirchhoff, 1876; see also Lamb, 1932, §159) who, for the 2-dimensional Euler equations, found an elliptical-vortex solution for which the velocity components are linear functions and the pressure field is a quadratic expression of the coordinate variables. This solution is known as the Kirchhoff vortex.

After Goldsbrough, Ball (1963, 1965) took advantage of this solution for the spatial structure to study oscillatory modes in shallow, rotating basins with reference to tides. Ball discovered that the inertial oscillation of the center-of-mass is uncoupled from the detailed motion around it, and that a circular pulsation, during the cycle of which contraction and expansion alternate, can be subtracted from more general oscillations. However, he fell short of establishing the analytical solution of that latter pulsation and of finding any solution for non negligible ellipticity.

As the interest in tides vanished (with the advent of computers which solved the problem once and for all), Ball's work became obsolete and forgotten. Many years later, Thacker (1981) found a partial solution to the pulsation mode. (1 of 4 degrees of freedom for the circular solution is missing.) With oceanic warm-core rings in mind, Cushman-Roisin (1984) found an exact analytical solution for an elliptical vortex that steadily rotates without deformation. This solution, called the Rodon, was further analyzed and placed in context by Cushman-Roisin et al. (1985). The conclusions of that latter work are two-fold: (i) elliptical vortices have only 3 modes of oscillations: 2 are elliptical rotations (one superinertial, one subinertial) and the 3rd is a circular pulsation (inertial); (ii) results bear resemblance to actual oceanic warm-core ring behavior.

The philosophy adopted here in the quest for more general solutions is to proceed not by elimination of mathematical variables but rather by careful and systematic elimination of physical modes one by one. The final recombination of the individual modes will then yield a new exact analytical solution that contains all previously known circular- and elliptical-vortex solutions of the shallow-water equations. The organization of the remainder of the article is as follows. Section 2 is devoted to the elimination of the spatial structure and the reduction of the problem to temporal quadrature. The motion of the center of mass is resolved and eliminated from the equations in Section 3. Section 4 treats the equilibrium solutions. Section 5 elucidates a circular pulsation mode and proposes to eliminate the latter through a change of variables and functions. Section 6 establishes the solution for an elliptical rotation mode. In summary, the solution is reconstructed in Section 7 and further discussed in Section 8.

2. Equations and solution for spatial structure

The shallow-water equations, also commonly called reduced-gravity equations in oceanography, are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -g\frac{\partial h}{\partial x}, \tag{1a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -g\frac{\partial h}{\partial y}, \tag{1b}$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{1c}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y},$$

(u,v) are the horizontal velocity components along the (x, y) coordinate axes, h is the depth of the active fluid layer, f the Coriolis parameter (taken here as uniform but not necessarily as positive), and g the (reduced) gravity constant. As pointed out by Goldsbrough (1930) and reported and/or rediscovered by several investigators since then, a depth field quadratic in x and y associated with velocity components linear in x and y is an exact solution of the governing equations. Let us write

$$h = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F, \tag{2a}$$

$$u = U_0 + U_1 x + U_2 y \tag{2b}$$

$$v = V_0 + V_1 x + V_2 y, \tag{2c}$$

where all 12 coefficients, A to V_2, are functions of time only. Implementation of (2a–c) in eqs. (1a–c) yields 12 ordinary differential equations for these coefficients:

$$\dot{A} = -(3U_1 + V_2)A - 2V_1 B, \tag{3}$$

$$\dot{B} = -U_2 A - 2(U_1 + V_2)B - V_1 C, \tag{4}$$

$$\dot{C} = -2U_2 B - (U_1 + 3V_2)C, \tag{5}$$

$$\dot{U}_1 = U_1^2 - U_2 V_1 + fV_1 - 2gA, \tag{6}$$

$$\dot{V}_1 = -U_1 V_1 - V_1 V_2 - fU_1 - 2gB, \tag{7}$$

$$\dot{U}_2 = -U_1 U_2 - U_2 V_2 + fV_2 - 2gB, \tag{8}$$

$$\dot{V}_2 = -U_2 V_1 - V_2^2 - fU_2 - 2gC, \tag{9}$$

$$\dot{D} = -U_0 A - V_0 B - (2U_1 + V_2)D - V_1 E, \tag{10}$$

$$\dot{E} = -U_0 B - V_0 C - U_2 D - (U_1 + 2V_2)E, \tag{11}$$

$$\dot{U}_0 = -U_0 V_1 - U_2 V_0 + fV_0 - 2gD, \tag{12}$$

$$\dot{V}_0 = -U_0 V_1 - V_0 V_2 - fU_0 - 2gE, \tag{13}$$

$$\dot{F} = -2U_0 D - V_0 E - (U_1 + V_2)F, \tag{14}$$

where an upper dot denotes a temporal derivative. This system of equations is nonlinear and coupled. Its most general solution ought to have 12 arbitrary constants of integration (degrees of freedom), which are related to the initial values of the 12 coefficients. Note that the set of equations is composed, in order, of a 7-by-7 subset that, in principle, can be solved first, a second 4-by-4 subset that can be solved next, and finally a single equation for F that can be solved last. The object of the rest of this article is the search for the most general solution to this system of equations and the discussion of the possible modes of oscillation. As announced in the abstract, this goal has not been fully achieved since a solution with ten rather than twelve degrees of freedom is constructed.

Before tackling this task, it should be noted that elliptical vortices with the fluid moving inside of an outcrop line ($h = 0$) correspond to the case for which the following conditions are met: (i) discriminant $\Delta = AC - B^2$ of (2a) be positive so that the h contours are ellipses, (ii) A and C be negative so that h decreases toward negative values far away, and (iii) the center depth

$$H = \frac{1}{\Delta}\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \tag{15}$$

be positive so that there exists a lump of fluid in motion within an elliptical region bordered by a (time-dependent) outcrop line ($h = 0$). It can be shown (Cushman-Roisin et al., 1985) that, if these conditions are met initially, they will continue to be met at all times.

3. Motion of center of mass: inertial oscillations

Ball (1963) showed that, independently of the spatial structure of the velocity and height fields, the center of mass revolves around an inertial, circular trajectory. If one denotes by V the volume of the fluid in motion, and X and Y the coordinates of the center of mass

$$V = \iint h\,\mathrm{d}x\,\mathrm{d}y,$$

$$X = \frac{1}{V}\iint xh\,\mathrm{d}x\,\mathrm{d}y, \qquad Y = \frac{1}{V}\iint yh\,\mathrm{d}x\,\mathrm{d}y,$$

where the domain of integration covers the region of fluid in motion, X and Y are governed by

$$\ddot{X} - f\dot{Y} = 0,$$

$$\ddot{Y} + f\dot{X} = 0.$$

The general solution

$$X = X_0 + R\sin(ft + \Phi), \tag{16a}$$

$$Y = Y_0 + R\cos(ft + \Phi), \tag{16b}$$

corresponds to a circular trajectory centered at (X_0, Y_0) and of radius R and phase Φ. This vortex translation is independent of the motion within the vortex and relative to its center of mass. As Ball pointed out, the converse is also true: the motion within the vortex is independent of the trajectory of the center of mass. This proposition allows the elimination of the inertial oscillation of the center of mass by the following translatory change of variables and functions:

$$\begin{aligned} x' &= x - X, & y' &= y - Y, & t' &= t, \\ u &= u' + \dot{X}, & v &= v' + \dot{Y}, & h &= h'. \end{aligned} \tag{17}$$

The new equations are

$$\frac{\mathrm{D}'u'}{\mathrm{D}t'} - fv' = -g\frac{\partial h'}{\partial x'}, \tag{18a}$$

$$\frac{\mathrm{D}'v'}{\mathrm{D}t'} + fu' = -g\frac{\partial h'}{\partial y'}, \tag{18b}$$

$$\frac{\mathrm{D}'h'}{\mathrm{D}t'} + h'\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0, \tag{18c}$$

where

$$\frac{\mathrm{D}'}{\mathrm{D}t'} = \frac{\partial}{\partial t'} + u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}.$$

They are similar to the original equations, but now a solution with fixed center of mass, say at $(0,0)$, is sought

$$h' = Ax'^2 + 2Bx'y' + Cy'^2 + H, \tag{19a}$$

$$u' = U_1x' + U_2y', \tag{19b}$$

$$v' = V_1x' + V_2y', \tag{19c}$$

and the system is reduced to an 8th-order problem. The 4 degrees of freedom that are eliminated correspond to the 4 constants of integration in (16a-b), i.e., X_0, Y_0, R and Φ, and to equations (10) to (13). The equations governing A to C and

U_1 and V_2 are (3) to (9) as before, while the center depth H, defined by (15), obeys

$$\dot{H} = -(U_1 + V_2)H, \tag{20}$$

which supersedes (14). Note that the constant resulting from this latter equation is but a multiplicative constant.

4. Circular equilibrium solution

Attention is now turned to the possible steady solutions of eqs. (3)–(9) and (20). Setting the time derivatives to zero, one finds that only 2 equilibrium solutions exist. One is the circular ring $[A = C = -W(f-W)/2g,\ B = 0,\ U_1 = V_2 = 0,\ U_2 = -V_1 = W,\ H = H_0]$

$$h = H_0 - \frac{1}{2g} W(f-W)(x^2+y^2),$$
$$u = Wy, \qquad v = -Wx, \tag{21}$$

for which $0 < W \leqslant f$ (Cushman-Roisin et al., 1985). The other solution corresponds to an infinite band oriented in any direction, and is not further considered here.

One notes that the circular equilibrium solution has two arbitrary constants, the center depth H_0 and a size parameter W ($W = f$ for the smallest eddy and $W = 0$ for infinite radius). The latter degree of freedom results from the similarity of the governing equations: if u, v and h functions of x, y and t are solutions, so are λu, λv, $\lambda^2 h$ functions of x/λ, y/λ and t, for any value of λ. The former degree of freedom is a consequence of the isolation of (20) from the rest of the problem and is peculiar to the linear-quadratic solution. Mathematically, these 2 degrees of freedom can be associated with a zero-frequency mode.

5. Circular pulsation mode

One now attempts to generalize the previous circular solution by letting it be time-dependent. Circular symmetry is preserved as long as $A = C$, $B = 0$, which in turn yield $U_2 = -V_1$, and $V_2 = U_1$ by use of (3) to (5). The set of equations (3)–(9) and (20) degenerates into the following 4-by-4 problem:

$$\dot{U}_1 = -U_1^2 + V_1^2 + fV_1 - 2gA,$$
$$\dot{V}_1 = -2U_1(V_1 + \tfrac{1}{2}f),$$
$$\dot{A} = -4U_1 A,$$
$$\dot{H} = -2U_1 H.$$

The most general solution to this nonlinear, coupled system is

$$U_1 = \frac{f\gamma}{2} \frac{\cos(ft'+\phi)}{1+\gamma\sin(ft'+\phi)}, \tag{22a}$$

$$V_1 = -\frac{f}{2} + \frac{f''-2W}{2} \frac{1}{1+\gamma\sin(ft'+\phi)}, \tag{22b}$$

$$A = -\frac{1}{2g} W(f''-W) \frac{1}{[1+\gamma\sin(ft'+\phi)]^2}, \tag{22c}$$

$$H = \frac{H_0}{1+\gamma\sin(ft'+\phi)}, \tag{22d}$$

where W, γ, ϕ and H_0 are four arbitrary constants of integration, and f'' is related to γ by $f'' = f(1-\gamma^2)^{1/2}$. The ranges of applicability of the free parameters are as follows: $-1 < \gamma < 1$ (with the steady solution of the previous paragraph being recovered for $\gamma = 0$), $0 \leqslant \phi < 2\pi$, and $0 < W \leqslant f''$ for depth decreasing from center to outcrop.

The reconstructed solution is

$$u' = \frac{f}{2} y' + \frac{1}{2[1+\gamma\sin(ft'+\phi)]} \times [f\gamma x'\cos(ft'+\phi) - (f''-2W)y'], \tag{23a}$$

$$v' = \frac{f}{2} x' + \frac{1}{2[1+\gamma\sin(ft'+\phi)]} \times [(f''-2W)x' + f\gamma y'\cos(ft'+\phi)], \tag{23b}$$

$$h' = \frac{H_0}{1+\gamma\sin(ft'+\phi)} - \frac{W(f''-W)}{2g} \times \frac{x'^2+y'^2}{[1+\gamma\sin(ft'+\phi)]^2}. \tag{23c}$$

Thacker (1981) proposed a less general solution for which γ was constrained to be a particular combination of H_0 and W.

The oscillation is a breathing or pulsation pattern during the cycle of which the circular vortex alternatively contracts and deepens, then expands and shallows (Fig. 1). The minimum and maximum center depths and radii are:

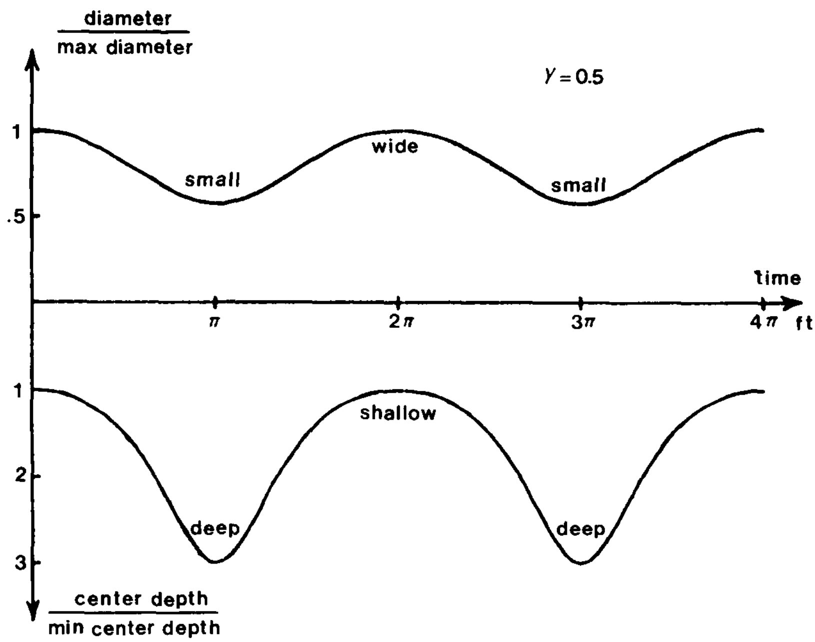


Fig. 1. Evolution of diameter and center depth of a pulsating circular eddy. Note that the period is inertial ($2\pi/f$) but that the variations are not sinusoidal. The eddy spends a relatively greater amount of time in the wide-shallow state than in the small-deep state.

$$H_{max} = \frac{H_0}{1-|\gamma|}, \qquad R_{min}^2 = (1-|\gamma|)\frac{2gH_0}{W(f''-W)},$$

$$H_{min} = \frac{H_0}{1+|\gamma|}, \qquad R_{max}^2 = (1+|\gamma|)\frac{2gH_0}{W(f''-W)}.$$

It is remarkable that, independently of the finite amplitude of the pulsation, the frequency is exactly inertial. Finally, we note that the constants W and H_0 are characteristic of a mean state (previous section) while the constants γ and ϕ are the two degrees of freedom brought by the pulsation mode, namely its amplitude and phase, respectively.

Adding these 4 degrees of freedom to the previous four by the change of variables and functions (17) yields a solution with a total of eight degrees of freedom. We now proceed to eliminate the circular pulsation mode and to look for additional modes of elliptical nature. For this purpose, a second change of variables and functions is introduced:

$$x'' = \delta^{1/2}\cos\psi\, x' + \delta^{1/2}\sin\psi\, y',$$

$$y'' = -\delta^{1/2}\sin\psi\, x' + \delta^{1/2}\cos\psi\, y',$$

$$t'' = \int_0^{t'} \delta(\tau)\,d\tau, \tag{24}$$

$$u' = \frac{1}{2}f\gamma\alpha x' + \frac{1}{2}(f - f''\delta)y' + \delta^{1/2}\cos\psi\, u'' - \delta^{1/2}\sin\psi\, v'',$$

$$v' = -\frac{1}{2}(f - f''\delta)x' + \frac{1}{2}f\gamma\alpha y' + \delta^{1/2}\sin\psi\, u'' + \delta^{1/2}\cos\psi\, v'',$$

$$h' = \delta h'',$$

where

$$\alpha = \frac{\cos(ft'+\phi)}{1+\gamma\sin(ft'+\phi)}, \qquad \delta = \frac{1}{1+\gamma\sin(ft'+\phi)},$$

$$\psi = -\frac{ft'}{2} + \frac{f''t''}{2}. \tag{25}$$

This transformation is a variation on the one proposed by Ball (1965) and corresponds to a pulsating rotation of the coordinate axes that follows the vortex pulsation mode. The transformed equations are:

$$\frac{D''u''}{Dt''} - f''v'' = -g\frac{\partial h''}{\partial x''}, \tag{26a}$$

$$\frac{D''v''}{Dt''} + f''u'' = -g\frac{\partial h''}{\partial y''}, \tag{26b}$$

$$\frac{D''h''}{Dt''} + h''\left(\frac{\partial u''}{\partial x''} + \frac{\partial v''}{\partial y''}\right) = 0, \tag{26c}$$

and are thus identical to the original governing equations. Note nonetheless that the new Coriolis parameter is f'', a reduced value that depends on the amplitude of the pulsation mode, γ.

6. Elliptical rotation mode

Since the transformed equations (26a–c) are identical to the previous ones (18a–c), one still seeks the same type of solution (19a–c) for which the 8 time-dependent coefficients, A to V_2, obey eqs. (3) to (9) and (20). But, instead of looking for a divergent, circular solution, one now looks for a non-divergent, elliptical solution. It is thus assumed that $U_1 + V_2 = 0$. Eqs. (7) and (8) then combine to yield $\dot{V}_1 - \dot{U}_2 = 0$ or $V_1 - U_2 = 2\Omega$, where Ω is an arbitrary constant. On the other side (6) plus (9) provide a diagnostic equation

$$U_1^2 + V_1(V_1 - 2\Omega) = -g(A + C) + f''\Omega. \tag{27}$$

With the help of (3) to (9), the time derivative of this latter equation implies

$$U_1(A - C) + 2(V_1 - \Omega)B = 0, \tag{28}$$

with the corollary that $U_1^2 + V_1(V_1 - \Omega)$ and $A + C$ are time-independent expressions. Noting $A = -A_0 + A'$, $C = -A_0 - A'$, where A_0 is a constant, one rewrites (27) as

$$U_1^2 + (V_1 - \Omega)^2 = 2gA_0 + \Omega(f'' + \Omega), \tag{29}$$

and the system of equations reduces to:

$$\dot{U}_1 = f''(V_1 - \Omega) - 2gA', \tag{30a}$$

$$\dot{V}_1 = -f''U_1 - 2gB, \tag{30b}$$

$$\dot{A}' = 2A_0 U_1 - 2\Omega B, \tag{30c}$$

$$\dot{B} = 2A_0(V_1 - \Omega) + 2\Omega A', \tag{30d}$$

$$\dot{H} = 0. \tag{30e}$$

This system is linear. In other words, the sole assumption of zero divergence has led to linearization of the problem. Note, however, that the system is overdetermined, for the solution of (30a–d) must also satisfy (28) and (29). As it turns out, the 4-by-4 system (30a–d) admits two oscillatory modes, but (28) and (29) require that only one mode be chosen at a time and that its amplitude be fixed by the constants Ω and A_0.

A solution of the type $\exp(i\omega t)$ is a solution of (30a–d) provided that ω be one of the 2 roots:

$$\omega = (\tfrac{1}{2}f'' - \Omega) \pm [4gA_0 + (\tfrac{1}{2}f'' + \Omega)^2]^{1/2}, \tag{31}$$

or its opposite. The solution is:

$$U_1 = Q\cos(\omega t'' + \Psi),$$

$$V_1 = \Omega - Q\sin(\omega t'' + \Psi),$$

$$A = -A_1 \sin(\omega t'' + \Psi),$$

$$B = -A_1 \cos(\omega t'' + \Psi),$$

$$H = H,$$

provided that the coefficients satisfy

$$(f'' - \omega)Q = 2gA_1,$$

$$Q^2 = \Omega(f'' + \Omega) + 2gA_0.$$

There are thus only 4 independent parameters, 2 of which must be H, the center depth and Ψ, the initial phase angle. The other two can be chosen to be A_0 and A_1, the parameters that provide the eddy's size and ellipticity. These two quantities, in combination with H, can be readily evaluated from a satellite image of an oceanic warm-core ring. Solving for ω, Ω and Q in terms of A_0, A_1 and the combination

$$\lambda = \frac{8g}{f''^2}(A_0^2 - A_1^2)^{1/2}, \tag{32}$$

one obtains

$$\omega = f'' - \tfrac{1}{2}\varepsilon f''(\sqrt{1 + \lambda} + \varepsilon\sqrt{1 - \lambda}), \tag{33}$$

$$\Omega = -\tfrac{1}{2}f'' + \tfrac{1}{4}\varepsilon f''\left(1 - \frac{8gA_0}{f''^2\lambda}\right)\sqrt{1 + \lambda} + \tfrac{1}{4}f''\left(1 + \frac{8gA_0}{f''^2\lambda}\right)\sqrt{1 - \lambda}, \tag{34}$$

$$Q = \varepsilon\frac{2gA_1}{f''\lambda}(\sqrt{1 + \lambda} - \varepsilon\sqrt{1 - \lambda}), \tag{35}$$

where $\varepsilon = +1$ or -1. The reconstructed solution is

$$u'' = -\Omega y'' + Q[x''\cos(\omega t'' + \Psi) - y''\sin(\omega t'' + \Psi)], \tag{36a}$$

$$v'' = +\Omega x'' - Q[x''\sin(\omega t'' + \Psi)$$

$$+ y'' \cos(\omega t'' + \Psi)], \tag{36b}$$

$$h'' = H - [A_0 + A_1 \sin(\omega t'' + \Psi)]x''^2$$
$$- 2A_1 \cos(\omega t'' + \Psi)x''y''$$
$$- [A_0 - A_1 \sin(\omega t'' + \Psi)]y''^2, \tag{36c}$$

and is the Rodon solution (Cushman-Roisin, 1984; Cushman-Roisin et al., 1985). It corresponds to an elliptical vortex of fixed shape, with semi-major axis $H(A_0 - |A_1|)^{-1/2}$ and semi-minor axis $H(A_0 + |A_1|)^{-1/2}$, that rotates steadily clockwise at the rate $\frac{1}{2}\omega$. This rate depends solely on the mean eddy radius $H(A_0^2 - A_1^2)^{-1/2}$ and not on its eccentricity. For discussion of additional properties, the reader is referred to either of the two references just cited.

Although the frequency ω can take one of two values given by (33), subinertial or superinertial, only one mode can be selected at a time. Otherwise (28) and (29) cannot be met. In other words, a non-divergent elliptical rotation mode can exist either alone or in conjunction with the circular pulsation mode, but if both elliptical modes are present, neither can be non-divergent. Although the single elliptical rotation mode takes a rather simple analytical solution, even in the presence of a pulsation mode, it may be that the interaction of the two elliptical rotation modes cannot be expressed in terms of known analytic functions.

Of the 4 arbitrary constants, H, A_0, A_1 and Ψ, the first 2 are not new degrees of freedom, while the last 2 are the amplitude and phase of an elliptical rotation mode, respectively. The ranges of applicability are $A_0 > 0$, $-A_0 < A_1 < A_0$ and $0 \leqslant \Psi < 2\pi$.

7. Reconstruction of the total solution

Although the previous pieces of solution are all individual solutions [simply by dropping the primes and double primes in (23) and (36)], the most general solution that emerges from the previous exercise is obtained by recombining all partial results. This solution takes the form

$$u = \tfrac{1}{2} f\gamma\alpha(x - X) + \tfrac{1}{2}(f - 2\bar{\Omega}\delta)(y - Y)$$
$$+ Q\delta[(x - X)\cos\zeta - (y - Y)\sin\zeta]$$
$$+ f(Y - Y_0), \tag{37a}$$

$$v = -\tfrac{1}{2}(f - 2\bar{\Omega}\delta)(x - X) + \tfrac{1}{2} f\gamma\alpha(y - Y)$$
$$- Q\delta[(x - X)\sin\zeta - (y - Y)\cos\zeta]$$
$$- f(X - X_0), \tag{37b}$$

$$h = H\delta - A_0\delta^2[(x - X)^2 + (y - Y)^2]$$
$$- A_1\delta^2[(x - X)^2 \sin\zeta + 2(x - X)(y - Y)$$
$$\times \cos\zeta - (y - Y)^2 \sin\zeta], \tag{37c}$$

where δ, α, ζ, X and Y are functions of time defined by

$$\delta = \frac{1}{1 + \gamma \sin(ft + \phi)}, \tag{38}$$

$$\alpha = \frac{\cos(ft + \phi)}{1 + \gamma \sin(ft + \phi)}, \tag{39}$$

$$\zeta = ft + \bar{\omega}t'' + \Psi,$$

$$t'' = \int_0^t \delta(\tau)\,d\tau = \frac{2}{f}(1 - \gamma^2)^{-1/2}$$
$$\times \tan^{-1}\left[\frac{\tan\frac{1}{2}(ft + \phi) + \gamma}{(1 - \gamma^2)^{1/2}}\right], \tag{40}$$

$$X = X_0 + R\sin(ft + \Phi), \tag{41}$$

$$Y = Y_0 + R\cos(ft + \Phi). \tag{42}$$

There are 10 arbitrary constants [X_0, Y_0, R, Φ; H, A_0; γ, ϕ; A_1, Ψ], some of which are constrained by the inequalities:

$$R \geqslant 0, \quad H > 0, \quad A_0 > 0,$$
$$-1 < \gamma < 1, \quad -A_0 < A_1 < A_0, \tag{43}$$

and from which other, dependent quantities are derived as follows

$$f'' = f(1 - \gamma^2)^{1/2}, \tag{44}$$

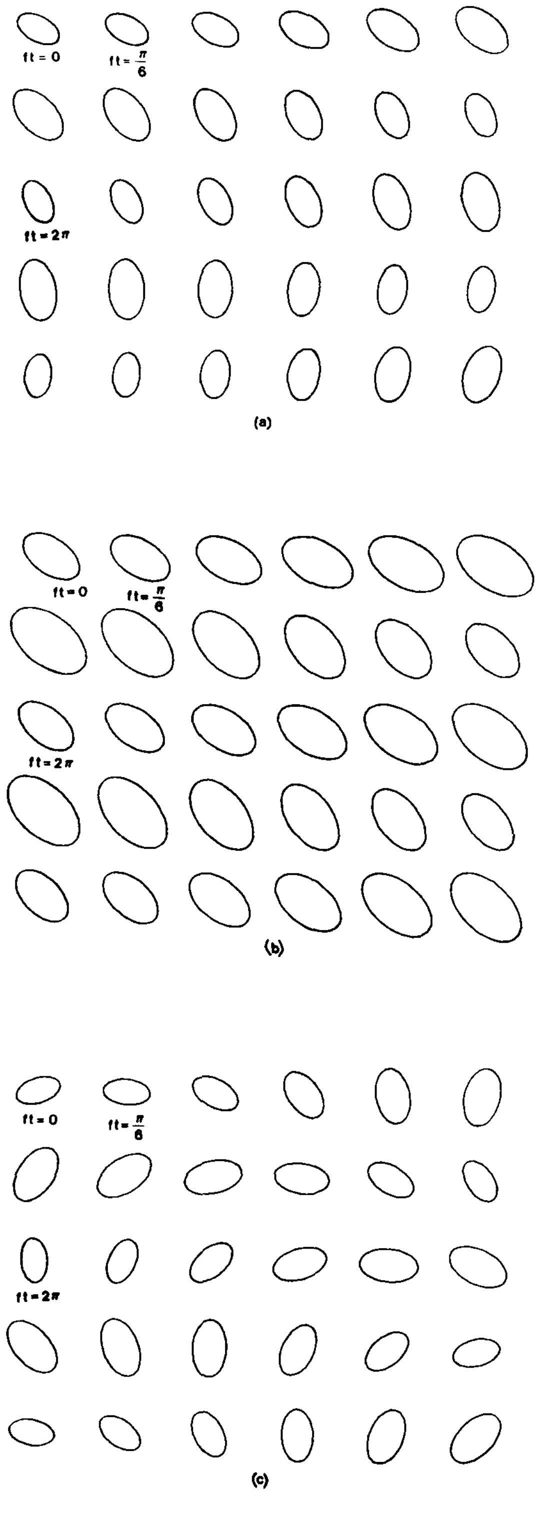
$$\lambda = \frac{8g}{f''^2}(A_0^2 - A_1^2)^{1/2}, \tag{45}$$

$$\bar{\omega} = \mp\tfrac{1}{2} f''[(1 + \lambda)^{1/2} \pm (1 - \lambda)^{1/2}], \tag{46}$$

$$\bar{\Omega} = \pm\tfrac{1}{4} f''\left(1 - \frac{8gA_0}{f''^2\lambda}\right)(1 + \lambda)^{1/2}$$
$$+ \tfrac{1}{4} f''\left(1 + \frac{8gA_0}{f''^2\lambda}\right)(1 - \lambda)^{1/2}, \tag{47}$$

$$Q = \pm\frac{2gA_1}{f''\lambda}[(1 + \gamma)^{1/2} \mp (1 - \gamma)^{1/2}] \tag{48}$$

The 10 arbitrary constants correspond to the following degrees of freedom: X_0 and Y_0 are the Cartesian coordinates of an arbitrary mean eddy



position; R and Φ are the radius and phase of the inertial circular trajectory of the center of mass around the mean position (X_0, Y_0); H and $(H/A_0)^{1/2}$ correspond to the center depth and radius of a basic circular state that translates with the center-of-mass; γ and ϕ are the amplitude and phase of the circular pulsation mode that alternatively deepens/contracts and shallows/expands the eddy about its basic state; finally, A_1 and Ψ are the amplitude and phase of an elliptical rotation mode rotating clockwise around an otherwise circular eddy. This latter rotation mode is either subinertial or superinertial depending on whether the lower or upper sign is chosen in (46) to (48). As previously mentioned in the text, the most general solution, with twelve arbitrary constants, would have 2 simultaneous elliptical modes, but such solution does not seem to take a simple, analytical form.

Several examples of the behavior that the above solution may exhibit are displayed on Fig. 2 where the temporal evolution of an elliptical, pulsating eddy is traced. Fig. 2a shows the superposition of the pulsation mode and the slow elliptical mode. One notes that the clockwise rotation of the elliptical mode is irregular, being faster when the eddy is contracted (smaller rim) and slower when the eddy is expanded (wider rim). For Fig. 2b, the size of the eddy has been increased to demonstrate that the clockwise rotation of the slow elliptical mode rapidly decreases with increasing size. A plausible explanation is as follows: for a given eccentricity, the larger the eddy, the greater the radius of curvature at the extremities compared to the radius of inertia, and the lesser the inertial tendency for a particle to overshoot the rim's curve at its point of maximum curvature. Fig. 2c displays the superposition of the pulsation mode and the fast elliptical rotation mode. The rate of rotation of this latter elliptical mode also decreases with increasing size

Fig. 2. Evolution of the rim of an elliptical, pulsating eddy. The time interval between plots in the sequence is a twelfth of the inertial period. The pulsation amplitude is $\gamma = 0.3$, the eccentricity parameter is $A_1/A_0 = 0.5$. (a) Slow elliptical mode and pulsation on a medium-size eddy ($8gA_0/f^2 = 1$). (b) Slow elliptical mode and pulsation mode on a larger eddy ($8gA_0/f^2 = 0.5$); note the imperceptible rotation of the elliptical mode. (c) Fast elliptical mode and rotation mode on a medium-size eddy ($8gA_0/f^2 = 1$).

(not shown on figure, but see Cushman-Roisin et al., 1985).

8. Discussion

The above solution can be considered as a simple model for a buoyant lens in the upper ocean (warm-core ring). The actual ocean is continuously stratified, the velocity profile is not linearly varying from the center to the rim, and the dynamic height is not quadratic. Moreover, ambient shear may be present, and the variation of the Coriolis parameter with latitude may not be negligible. However, the present solution may be a helpful tool in estimating some bulk characteristics of some oceanic warm-core rings.

For example, suppose that the eddy is sufficiently isolated from nearby currents and that one can reasonably estimate the following characteristics:

$$\text{time-average area} = \pi H(A_0^2 - A_1^2)^{-1/2},$$

$$\text{difference between maximum and minimum area} = 2\pi\gamma H(A_0^2 - A_1^2)^{-1/2},$$

$$\text{time-average center depth} = H(1-\gamma^2)^{-1/2}$$

$$\text{eccentricity} = [2|A_1|/(A_0 + |A_1|)]^{1/2},$$

then one can readily determine the four key parameters H, A_0, A_1 and γ. Formulas (44) to (48) will provide the other, dependent parameters. In particular, a prediction of the clockwise rotation of the subinertial elliptical mode can be made, as it is given by $\frac{1}{2}(f'' + \bar{\omega})$. Alternatively, a sequence of satellite images on which such a mode is observed can be used to estimate the time-average area, its pulsation in time, the eccentricity and the rotation rate of the eddy as a whole. This will lead to the determination of H, A_0, A_1, γ and the dependent parameters, without the use of any hydrographic data, such as center depth. [Moreover, as shown in Cushman-Roisin et al. (1985), since the elliptical rotation rate drops rapidly as the ratio of the eddy radius over the deformation radius increases, it is preferable to determine the deformation radius, hence the average center depth, from the elliptical rotation rate than vice versa.]

Once these key parameters are estimated, the bulk properties of the eddy (the invariants of the system) can be easily evaluated. These are (the integral covers the extent of the eddy):

$$\text{volume} = \iint h\,\mathrm{d}x\mathrm{d}y = \frac{\pi}{2} H^2(A_0^2 - A_1^2)^{-1/2}$$

angular momentum

$$= \iint [(xv - yu) + \frac{f}{2}(x^2+y^2)]\, h\,\mathrm{d}x\mathrm{d}y,$$

$$= \frac{\pi}{6} H^3(\bar{\Omega} A_0 + QA_1)(A_0^2 - A_1^2)^{-3/2},$$

$$\text{potential vorticity} = \iint \left(\frac{v_x - u_y + f}{h}\right) h\,\mathrm{d}x\mathrm{d}y$$

$$= 2\pi H\bar{\Omega}(A_0^2 - A_1^2)^{-1/2},$$

$$\text{total energy} = \iint [\tfrac{1}{2}gh + \tfrac{1}{2}(u^2+v^2)]\, h\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{\pi}{24} H^3(f^2A_0 - 2f\bar{\Omega}A_0 - 2fQA_1)(A_0^2 - A_1^2)^{-3/2}.$$

Some time-dependent quantities are:

$$\text{area} = \iint \mathrm{d}x\mathrm{d}y = \pi H(A_0^2 - A_1^2)^{-1/2} \times [1 + \gamma \sin(ft + \phi)],$$

$$\text{potential energy} = \tfrac{1}{2}g \iint h^2\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{\pi}{6} gH^3(A_0^2 - A_1^2)^{-1/2}[1 + \gamma\sin(ft+\phi)]^{-1},$$

$$\text{moment of inertia} = \iint (x^2+y^2)\, h\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{\pi}{6} H^3 A_0(A_0^2 - A_1^2)^{-3/2}[1 + \gamma\sin(ft+\phi)].$$

The kinetic energy is most easily derived by subtraction of the potential energy from the total energy.

Elliptical eddies can be considered as circular eddies with the addition of an azimuthal mode number 2 (two lobes around the periphery). It is easy to conceive that more general states corresponding to modes 1, 3, 4, 5, ... may exist. If they exist, they are the natural analogs of the V-states of the Euler equations (Deem and Zabusky, 1978). But here again, analytical solutions may be precluded.

Although the inclusion of the interaction between the elliptical modes and the extension to other azimuthal modes are the two generalizations that first come to mind, they are fraught with considerable mathematical complexities and render the solution presented here somewhat of an oddity. In the author's mind, the mathematical simplicity of the solution presented here is an accident of physics. The solution can, however, be useful as a model of oceanic warm-core rings or other lens-like eddies, as a basic state for further investigation such as an infinitesimal-amplitude stability study and an accuracy test for nonlinear numerical models using the shallow-water equations.

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