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Finite-amplitude stability of a Rossby wave in a channel

By JOHN W. PRZYBYLOWICZ and ARTHUR Z. LOESCH, *Department of Atmospheric Science, State University of New York at Albany, Albany, NY 12222 USA*

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ABSTRACT

The linear stability characteristics and life cycle, in the presence of weak nonlinearities, of an inviscid Rossby wave are examined in a mid-latitude channel. Two cases are investigated: one in which the superimposed disturbance is purely wavelike and a second in which it includes a zonal flow component.

The linear analysis reveals that in the first case, the disturbance is resolved into two decoupled sets of modes. The most unstable set as a function of wavenumber is then determined by obtaining the curves of marginal stability for each set. The second case is characterized by a single unstable set.

The finite amplitude asymptotic analysis of the first case reveals that, when the nonlinearities are stabilizing, the system exhibits an amplitude vacillation on a slow time scale. It also reveals that the most severe meridional truncation is valid for the longer zonal scales and does not affect the mean field at any order. If an additional meridional mode is included, the resulting truncation becomes valid for the shorter zonal scales and alters the mean field at second order. In the second case, the nonlinearities are always stabilizing.

1. Introduction

A glance at any weather map reveals that large amplitude waves are always present. Thus, it is questionable as to whether the large scale mid-latitude atmospheric flow is realistically represented as a zonal flow in its unperturbed state. One would always expect a zonally nonuniform flow to be present because of the diversion of the zonal flow by mountain ranges and the contrast in heating of land and sea which constitutes permanent large scale forcings on the flow.

The horizontal shear of a wavelike pattern introduces the possibility of a Rayleigh, or inflection point, type instability which is a form of barotropic instability. Thus, the evolution and structure of a large amplitude wave, which may have been generated by baroclinic instability or boundary forcing, may be subsequently altered by this type of barotropic instability.

The first attempt to study the stability characteristics of a fully developed wave state analytically was made by Lorenz (1972). Lorenz considered the classic zonally propagating Rossby wave as a basic state upon which he placed a wavelike disturbance. He indeed found that the Rossby wave is unstable provided its amplitude is large enough. This result is in contrast to horizontally sheared zonal flows found in the middle latitudes which tend to be barotropically stable. Thus, one result of Lorenz's study is the increased importance now given to barotropic processes in the general circulation of the mid-latitudes.

Gill (1974) generalized Lorenz's study to a Rossby wave propagating in an arbitrary horizontal direction. His objective was to more clearly establish the nature of the instability. He found the determining fact to be a parameter M, which is the ratio of the relative vorticity gradient to the planetary vorticity gradient and is directly proportional to the amplitude of the basic state wave. When M is large ($M \geqslant O(1)$), the instability is of a Rayleigh type and when M is small the instability is of a resonant triad type. Lorenz chose a value $M = 2$ and thus considered a Rayleigh type instability.

A finite amplitude extension of the Lorenz problem was performed by Loesch (1978) and Deininger and Loesch (1982). By including weak nonlinearities they obtained a finite amplitude behavior consisting of an oscillatory exchange of energy between the basic state Rossby wave and the disturbance field.

In a similar fashion, Deininger (1982) performed a finite amplitude extension of Gill's stability problem and also obtained a wave amplitude vacillation.

All the wave stability analyses cited above have been performed on an infinite β-plane, which yields propagation characteristics more appropriate for an ocean. The mid-latitude mid-tropospheric wave propagation is more realistically modelled by an east-west oriented channel whose width corresponds to that of the baroclinic zone. The goal of this study is to determine both the linear stability characteristics and the subsequent finite amplitude evolution of a wavelike basic state and superimposed disturbance in a mid-latitude channel. First the linear stability characteristics of a single Rossby wave are examined. The amplitude of the Rossby wave is taken large enough so that any instability that is found is of a Rayleigh type. Consequently, the problem dealt with here is that of the classic inflection point instability but applied to a wavelike basic state instead of a zonal flow.

For generality, the stability characteristics of the wavelike basic state are examined for two kinds of superimposed disturbances: (i) a disturbance of purely wavelike structure and (ii) a wavelike disturbance that includes a zonal flow as part of its spectrum.

The first case is a pure wave-wave interaction problem for the most severe truncation but includes weak wave-zonal flow interaction for less severe meridional truncations.

The second case involves both wave-wave and wave-zonal flow interactions. The main difference between the two cases is that, in the first, the zonal flow is generated as a second order effect (if it appears at all) while in the second the zonal flow is present to lowest order in the disturbance.

The life cycle of the basic state wave and disturbance field is then investigated by considering finite amplitude versions of the wave-wave and wave-zonal flow problems cited above. By assuming the initial supercriticality to be small we restrict the growth rates to be correspondingly small. We can thus balance the weak initial instability by introducing compensating weak nonlinearities.

2. The model

The nondimensional vorticity equation and boundary conditions governing the barotropic motion of a quasigeostrophic, inviscid, homogeneous fluid confined to a mid-latitude channel on a β-plane with flat upper and lower horizontal boundaries, are

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} + J(\psi, \nabla^2\psi) = 0 \tag{2.1}$$

$$\frac{\partial\psi}{\partial x} = 0 \qquad @\ y = 0, 1 \tag{2.2a}$$

$$\int_{-X}^{X} \frac{\partial^2\psi}{\partial t\,\partial y}\,dx = 0 \qquad @\ y = 0, 1. \tag{2.2b}$$

For correspondence with motion around a latitude circle, cyclic continuity in the downstream (zonal) direction is also assumed. The symbols above have their usual meanings (Lorenz (1972), Loesch (1978), and Deininger (1982), hereafter referred to as DE). The symbol J represents the Jacobian operator and X is the zonal wavelength of the flow field.

The horizontal coordinates, x and y, are scaled with L, the horizontal velocity components with U and the time with L/U. The scale for the geostrophic streamfunction, ψ, is then UL. L is taken as the width of the channel which encompasses the mid-latitude baroclinic zone. β is the planetary vorticity factor taken to be of O(1) and given by

$$\beta = \beta'\frac{L^2}{U},$$

where β' is the dimensional northward gradient of the Coriolis parameter, f.

An *exact* solution of (2.1) and (2.2) is a Rossby wave embedded in a uniform zonal current given by

$$\psi = \psi_0 = -Uy + A\sin\pi y\cos\theta, \tag{2.3}$$

where

$$\theta = N(x - ct), \tag{2.4}$$

$$c = U - \beta/\alpha^2, \tag{2.5}$$

$$\alpha^2 = N^2 + \pi^2. \tag{2.6}$$

Eq. (2.3) constitutes a wavelike basic state whose stability and subsequent weakly nonlinear evolution are examined here.

A stability analysis of (2.3) is performed by letting

$$\psi = \psi_0 + \psi', \tag{2.7}$$

where ψ' represents a small but not necessarily infinitesimal disturbance. Substitution of (2.7) into (2.1) and (2.2) yields the following nonlinear problem for ψ'

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} - A\left[\pi\cos\pi y\cos\theta\frac{\partial}{\partial x} + N\sin\pi y\sin\theta\frac{\partial}{\partial y}\right](\nabla^2+\alpha^2)\psi' + J(\psi',\nabla^2\psi') = 0, \tag{2.8}$$

$$\frac{\partial\psi'}{\partial x} = 0 \qquad @\ y = 0, 1, \tag{2.9a}$$

$$\int_{-X}^{X}\frac{\partial^2\psi'}{\partial t\,\partial y}\,\mathrm{d}x = 0 \qquad @\ y = 0, 1. \tag{2.9b}$$

when ψ' is infinitesimal, (2.8) can be linearized by neglecting the last term which represents the self-advection of the disturbance field.

The resulting equation still contains periodic coefficients in x and y due to the wavelike structure of the basic state (2.3) and thus requires its solutions to be specified in terms of infinite series. A solution which satisfies boundary condition (2.9a) can be taken to be

$$\psi' = \mathrm{e}^{-i\sigma t}\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty} X_{k,l}\sin l\pi y\,\mathrm{e}^{i\theta_k} + * \tag{2.10}$$

where

$$\theta_k = \theta_0 + k\theta \tag{2.11}$$

$$\theta_0 = M(x - c_0 t), \tag{2.12}$$

provided $M \neq 0$. The symbol $*$ represents the complex conjugate. In view of the constraint on M and cyclic continuity, the solution (2.10) satisfies (2.9b), and is *purely wavelike* in structure.

Several truncations of (2.10) have been considered. The most severe truncation consists of the zonal modes $k = -1, 0, 1$ and the meridional modes $l = 1, 2$ and is a channel analogue of the Lorenz (1972) truncation. As is shown later, this truncation does *not* alter the zonal flow at any order. To allow for zonal flow changes at finite amplitude a second less severe meridional truncation $l = 1, 2, 3$ was also considered. In addition, to assess the convergence of the linear problem, two still less severe truncations have been considered. They consist of the modes $k = \mp 2, \mp 1, 0$; $l = 1, 2, 3$ and $k = \mp 3, \mp 2, \mp 1, 0$; $l = 1, 2, 3, 4$. The linear stability analysis and its finite amplitude extension of the $M \neq 0$ case are carried out in Subsections 3.1 and 4.1, respectively. Only the equations for the most severe truncation are shown in these sections.

Any assumed solution which includes a zonal flow must have the meridional structrure of the x independent part of its spectrum described by $\cos l\pi y$ rather $\sin l\pi y$ in order to satisfy (2.9b). In particular, when $M = 0$ in (2.10) this type of solution has the form

$$\psi' = \mathrm{e}^{-i\sigma t}\left\{\sum_{l=1}^{\infty}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} X_{k,l}\sin l\pi y\,\mathrm{e}^{i\theta_k} + \sum_{\substack{l=1\\k=0}}^{\infty} X_{0,l}\cos l\pi y\right\} + *, \tag{2.13}$$

where

$$\theta_k = k\theta = kN(x - ct). \tag{2.14}$$

It must be understood, however, that (2.13) is not the limit of (2.10) as $M \to 0$; rather it is a distinct solution of (2.8) and (2.9a, b). Its linear stability characteristics and finite amplitude behavior are examined in Subsections 3.2 and 4.2, respectively. Only the most severe truncation, as defined for the $M \neq 0$ case, is considered in the analysis of the $M = 0$ case.

3. Linear stability analysis

3.1. *The $M \neq 0$ case*

Formulation (2.11) of the kth disturbance mode in x follows Gill (1974). We designate the basic state Rossby wave as the primary wave and the disturbance mode whose wavenumber vector is (M, π) as the secondary wave. In the secondary wave the gravest meridional mode is chosen to *maximize* energy transfer. The general stability problem posed by the linearized version of (2.8) and (2.10) through (2.12) depends on the zonal wavenumbers N and M taken as continuous variables. The phase speed of the secondary wave, c_0, is as yet an unspecified parameter.

Substitution of (2.10) into the linearized version of (2.8) yields an infinite set of homogeneous algebraic equations each connecting 5 adjacent disturbance modes. The relationship among the amplitudes of (2.10) is

$$\frac{\pi}{4}[M+(k-l)N]a_{k-1,l-1}AY_{k-1,l-1} + \frac{\pi}{4}[M+(k+l)N]a_{k-1,l+1}AY_{k-1,l+1} + (\sigma+\delta_{k,l})Y_{k,l} + \frac{\pi}{4}[M+(k+l)N]a_{k+1,l-1}AY_{k+1,l-1} + \frac{\pi}{4}[M+(k-l)N]a_{k+1,l+1}AY_{k+1,l+1} = 0, \tag{3.1.1}$$

where

$$\alpha^2_{k,l} = (M+kN)^2 + l^2\pi^2, \tag{3.1.2a}$$

$$a_{k,l} = \frac{\alpha^2_{k,l}-\alpha^2}{\alpha^2_{k,l}}, \tag{3.1.2b}$$

$$\delta_{k,l} = Mc_0 + kNc - (M+kN)\left(U - \frac{\beta}{\alpha^2_{k,l}}\right), \tag{3.1.2c}$$

$$Y_{k,l} = \alpha^2_{k,l}X_{k,l}. \tag{3.1.2d}$$

Solutions of (3.1.1) are obtained by a truncated expansion technique analogous to Lorenz (1972) but applied here in both zonal and meridional directions.

The most severe truncation ($k=-1,0,1$; $l=1,2$) yields six equations which reveal that amplitudes $Y_{-1,2}$, $Y_{0,1}$, and $Y_{1,2}$ are *decoupled* from amplitudes $Y_{-1,1}$, $Y_{0,2}$, and $Y_{1,1}$. This decoupling between the set whose sum of indices is odd (henceforth designated as set I) and the set whose sum of indices is even (henceforth designated as set II) occurs irrespective of truncation level. Each set represents a distinct stability problem with its own eigenvalue σ_J; $J=1,2$ for sets I and II, respectively. For the most severe truncation the dispersion relation is

$$\sigma_J^3 + (\delta_{0,J}+\delta_{-1,L}+\delta_{1,L})\sigma_J^2 + \Big\{\delta_{0,J}(\delta_{-1,L}+\delta_{1,L}) + \delta_{-1,L}\delta_{1,L} - \frac{\pi^2}{16}A^2a_{0,J}[a_{-1,L}(M+(-2)^{J-1}N)^2 + a_{1,L}(M-(-2)^{J-1}N)^2]\Big\}\sigma_J + \delta_{0,J}\delta_{-1,L}\delta_{1,L} - \frac{\pi^2}{16}A^2a_{0,J}[a_{-1,L}\delta_{1,L}(M+(-2)^{J-1}N)^2 + a_{1,L}\delta_{-1,L}(M-(-2)^{J-1}N)^2] = 0, \tag{3.1.3}$$

where the index L is given by $L=J+1$ for set I and $L=J-1$ for set II.

From (3.1.3), for instability to occur, the secondary zonal wavenumber, M, must satisfy certain bounds defined below while the primary wave amplitude, A, must exceed a certain critical value, A_{c_J}.

Analysis of the coefficients of the equation for A_{c_J} obtained from (3.1.3) by setting its discriminant to zero, shows that they are independent of the zonal current U and the secondary phase speed c_0. To simplify the algebra, we choose $c_0 = U - \beta/\alpha^2_{0,1}$ for set I and $c_0 = U - \beta/\alpha^2_{0,2}$ for set II. Then the critical amplitude of the basic state wave is defined by

$$4b_2^3A_c^6 + [12b_1b_2^2 - a^2b_2^2 - 18ab_2c_2 + 27c_2^2]A_c^4 + [12b_1^2b_2 - 2a^2b_1b_2 - 18ab_1c_2 + 4a^3c_2]A_c^2 + 4b_1^3 - a^2b_1^2 = 0 \tag{3.1.4}$$

where

$$a = \delta_{-1,L} + \delta_{1,L} \tag{3.1.5a}$$

$$b_1 = \delta_{-1,L}\delta_{1,L} \tag{3.1.5b}$$

$$b_2 = -\frac{\pi^2}{16}a_{0,J}[(M+(-2)^{J-1}N)^2a_{-1,L} + (M-(-2)^{J-1}N)^2a_{1,L}] \tag{3.1.5c}$$

$$c_2 = -\frac{\pi^2}{16}a_{0,J}[(M+(-2)^{J-1}N)^2a_{-1,L}\delta_{1,L} + (M-(-2)^{J-1}N)^2a_{1,L}\delta_{-1,L}], \tag{3.1.5d}$$

and where the subscript on A_c has been dropped.

The requirement of the Fjortoft (1953) necessary condition for instability that at least one disturbance mode be upscale from the primary wave is satisfied by mode (M, π) in set I whenever $M < N$ and by mode $(M - N, \pi)$ in set II whenever $M < 2N$. When $N \geqslant 6$ the additional modes $(M - N, 2\pi)$ in set I and $(M, 2\pi)$ in set II are also upscale from the primary wave.

The real positive roots of (3.1.4) for each set have been examined as a function of the secondary wavenumber M, for fixed integer values of the primary zonal wavenumber N, in the range $2 \leqslant N \leqslant 10$. The wavenumber M was taken as a continuous variable in the range $0 < M < 2N$. Since, from (3.1.4), A_c is related linearly to β, β was fixed at 1.0.

The marginal stability curves reveal some unusual characteristics. For $N \geqslant 6$, there exist points at which the marginal curves for both sets asymptotically approach infinity. For $N < 6$, no such behavior is evident. Thus the graphs for $N = 5$ and $N = 6$ for each set are chosen to illustrate the fundamental characteristics of each type of behavior. They are shown in Figs. 1 and 2.

When $N \geqslant 6$ for set I and for all N for set II, there are regions of M space where there exist both upper and lower critical curves and the instability is confined between the curves. Regions in which only a single curve exists are interpreted in the usual manner, e.g., Loesch (1978). The sharply defined local minima in the marginal curves are actually points at which A_c vanishes and triad resonance occurs.

In general, the linearly most unstable set is that corresponding to the smallest value of A_c, for a given M. Set II dominates the instability for very long primary waves ($N = 2$) progressively being replaced by set I as N increases. Set I dominates for short primary waves ($N \geqslant 8$). When $N \leqslant 5$, set II is the only unstable set in the interval $N < M < 2N$. This largely accounts for the tendency of set II to dominate the instability for small N.

To assess the accuracy of the most severe truncation in M space for each set, the marginal curves corresponding to two less severe truncations for $M = 5$ and 6 were obtained. The first consists of 15 modes ($k = \mp 2, \mp 1, 0$; $l = 1, 2, 3$), while the second consists of 28 modes ($k = \mp 3, \mp 2, \mp 1, 0$; $l = 1, 2, 3, 4$).

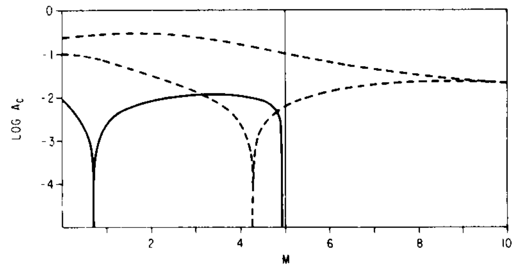

Fig. 1. The marginal stability curves for the 3 mode truncation of Set I (solid curve) and Set II (dashed curve), $N = 5$. The critical amplitude of the basic state wave is plotted on a logarithmic scale to the base 10 against the zonal wavenumber, M, of the secondary wave.

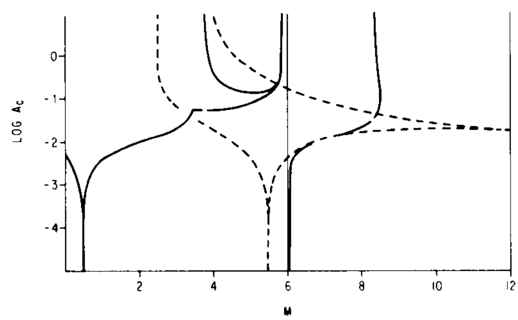

Fig. 2. Same as Fig. 1 but $N = 6$.

The comparison of the marginal curves for these three truncations, relevant to the M domain of the most severe truncation, is summarized in Table 1. Figs. 3 and 4 also provide a visual comparison for set I, $N = 5$. For the most severe truncation the threshold of instability is defined by a single curve. However, for the less severe truncations there appear first one or more narrow bands of very weak instability separated by narrow bands of stability. The threshold curve is taken to lie above this banded structure.

Since the degree of the dispersion relation is 8(7) for the first less severe truncation and 14 for the second, there can be as many as 4(3) and 7 unstable eigenmodes, respectively, existing simultaneously in the system. At most only 2 were actually found and the intervals over which they exist in M space are indicated in Table 1. They represent disturbance modes which are identical in spatial structure but differ in phase speed.

Table 1. *Summary of the linear stability characteristics of the various truncations*

(a) $N = 5$

	Set I			Set II		
	3 modes	8 modes	14 modes	3 modes	7 modes	14 modes
Location of resonance points (minima) in M space	0.7 5.0	0.7 5.0	0.7 5.0	4.3	4.3 5.7	4.3 5.7
Interval in M of agreement of marginal curves (to within $0.20A_c$) with 3 modes	—	0–4.6	0–4.7	—	3.8–4.6	3.8–4.6
Interval(s) in M containing additional unstable eigenmodes	—	none	none	—	none	none

(b) $N = 6$

	Set I			Set II		
	3 modes	8 modes	14 modes	3 modes	7 modes	14 modes
Location of resonance points (minima) in M space	0.5 6.0	0.5 6.0	0.5 6.0	5.5	5.5 6.5	5.5 6.5
Interval in M of agreement of marginal curves (to within $0.20A_c$) with 3 modes	—	0–2.1	0–2.7	—	5.0–5.8	5.0–5.8
Interval(s) in M containing additional unstable eigenmodes	—	none	3.5–5.7 6.3–8.5	—	none	9.6–10.3

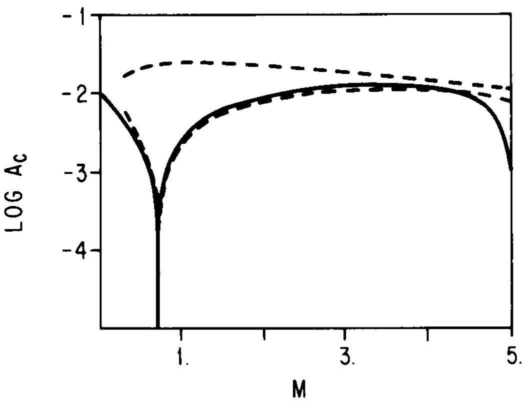

Fig. 3. The marginal stability curves for the 3 mode truncation of Set I (solid curve) and the 8 mode truncation of Set I (dashed curve), $N = 5$.

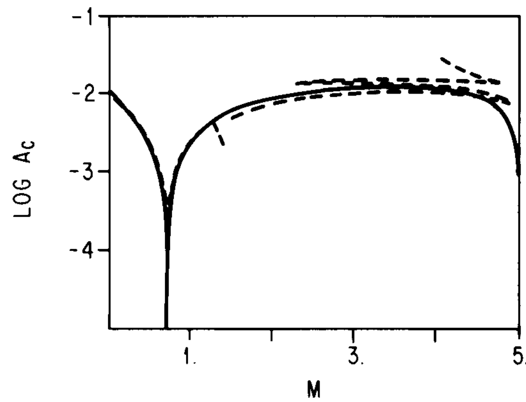

Fig. 4. Same as Fig. 3 but the dashed curve represents the 14 mode truncation.

The marginal curves of the three truncations are in close agreement (i.e., $\Delta A_c/A_c \leqslant 0.2$, where ΔA_c is the difference in the A_c values for 2 different truncations) for both sets I and II in the neighborhood of the resonance points. In addition, there is close agreement for set I for small M, the interval of close agreement decreasing with increasing N. The interval of close agreement in M is overall much smaller for set II.

A search was also made for values of A_c common to both sets. Such values exist but correspond to noninteger values of M, and are therefore precluded by the zonal cyclic continuity condition. The restriction on M to integers also precludes triad resonance and allows consideration of only the linearly most unstable set.

On the neutral curve defined by (3.1.4) (i.e., for the most severe truncation), we can define the following recursion relations

$$X_{-1,L} = C_{-1,L} X_{0,J}, \tag{3.1.6a}$$

$$X_{1,L} = C_{1,L} X_{0,J}, \tag{3.1.6b}$$

where

$$C_{-1,L}=\frac{-\pi(M+(-2)^{J-1}N)a_{0,J}\alpha_{0,J}^2A_c}{4(\sigma+\delta_{-1,L})\alpha_{-1,L}^2}, \quad (3.1.7a)$$

$$C_{1,L}=\frac{-\pi(M-(-2)^{J-1}N)a_{0,J}\alpha_{0,J}^2A_c}{4(\sigma+\delta_{1,L})\alpha_{1,L}^2}. \quad (3.1.7b)$$

and where $J=1$ and $L=2$ for set I and $J=2$ and $L=1$ for set II.

In anticipation of the finite amplitude analysis, the order of magnitude of the initial linear growth rate when the system is only slightly unstable is determined next. The growth rate for each set is obtained from (3.1.3) but with the simplification $\delta_{0,J}=0$ due to our choice of c_0. Let

$$A=A_c+\Delta \quad (3.1.8)$$

where

$$\Delta \ll A_c. \quad (3.1.9)$$

To lowest order, using (3.1.8) and (3.1.9), we obtain

$$\sigma_i=O(|\Delta|^{1/2}). \quad (3.1.10)$$

Therefore, the initial growth rate is identical in its small parameter dependency to that of Pedlosky (1970), Loesch (1978), and DE.

3.2. The $M=0$ case

An important special case, which arises when a part of the disturbance field is a zonal flow and which is given by the solution (2.13), is examined next. In this case, the wave part of the disturbance field ($k\neq 0$) is characterized by the x and y structure of the basic wave and its harmonics.

Both the linear and nonlinear analyses for the $M=0$ case will be based on the most severe truncation of the $M\neq 0$ case so the results at each stage can be directly compared. With $M=0$, the following parameters are defined:

$$\alpha_{k,l}^2=k^2N^2+l^2\pi^2, \quad (3.2.1)$$

$$\delta_{k,l}=kN\left[c-\left(U-\frac{\beta}{\alpha_{k,l}^2}\right)\right]. \quad (3.2.2)$$

The parameters c, $a_{k,l}$, and $Y_{k,l}$ are defined as before. Note that

$$\alpha_{-k,l}^2=\alpha_{k,l}^2, \quad (3.2.3)$$

$$\delta_{-k,l}=-\delta_{k,l}, \quad (3.2.4)$$

and $\delta_{0,l}=0$ for all l. We also have

$$a_{-1,1}=a_{1,1}=0 \quad (3.2.5)$$

which simplifies the analysis as shown below.

Substitution of (2.13) into the linearized version of (2.8) yields a recursion relation of the same form as (3.1.1). However, the zonal flow-basic wave interaction produces wave modes with meridional structure proportional to cosine rather than sine and the disturbance-basic wave interaction produces zonal flow modes proportional to sine rather than cosine. We determine the projection of a given cosine (sine) structure on a given sine (cosine) structure by expanding the cosine (sine) term in a sine (cosine) series over the range [0, 1]. From the expansion, the term which enters the general recursion relation is the one whose meridional structure is identical to that of the other terms. For the most severe truncation the result is a set of six homogeneous algebraic equations analogous to those obtained in the $M\neq 0$ case.

As a result of (3.2.5) the equation for $(k,l)=(0,1)$ is

$$Y_{0,1}=0. \quad (3.2.6)$$

This further implies

$$Y_{-1,1}=0, \quad (3.2.7)$$

$$Y_{1,1}=0. \quad (3.2.8)$$

Thus, in contrast to the $M\neq 0$ case, the modes $(k,l)=(1,2)$ and $(-1,2)$ are linked to the mode $(0,2)$ instead of $(0,1)$ and the distinction between sets I and II is absent in this case.

For a nontrivial solution the following relation must be satisfied

$$\sigma(\sigma^2-\delta_{1,2}^2)-\frac{32\cdot 22}{(21)^2}A^2a_{0,2}a_{1,2}=0. \quad (3.2.9)$$

Therefore, on the marginal stability curve the basic state wave amplitude, A, assumes a critical value, A_c, given by

$$A_c=\frac{21}{8N}\left[\frac{-\delta_{1,2}^2}{11a_{0,2}a_{1,2}}\right]^{1/2}, \quad (3.2.10)$$

provided $a_{0,2}<0$. The last condition is satisfied when $N^2>3\pi^2$ and is really a consequence of Fjortoft's theorem that at least one mode, $(0,2\pi)$, be upscale of the basic wave. The marginal curve corresponding to (3.2.10) is shown in Fig. 5. Note

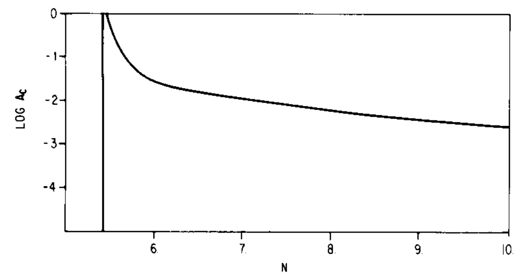

Fig. 5. The marginal stability curve for the 3 mode truncation of the $M = 0$ case. The abscissa is the zonal wavenumber, N, of the primary (basic state) wave.

that $A_c \to \infty$ as $N \to \sqrt{3}\pi$, it decreases monotonically with increasing N and $A_c \to 0$ as $N \to \infty$. Except in the vicinity of the resonance points for the $M \neq 0$ case, shorter basic state waves ($N \geqslant 8$) are more unstable to a disturbance with $M = 0$ structure than to one with $M \neq 0$. For intermediate basic state waves ($\sqrt{3}\pi < N < 8$) the situation is not as clear and depends on the wavenumber M. For long basic state waves ($N < \sqrt{3}\pi$) instability is only possible when $M \neq 0$.

On the marginal curve we have the recursion relations

$$X_{-1,2} = C_{-1,2} X_{0,2} \tag{3.2.11a}$$

$$X_{1,2} = C_{1,2} X_{0,2} \tag{3.2.11b}$$

where

$$C_{-1,2} = C_{1,2} = -\frac{16}{21} \frac{NA_c a_{0,2} \alpha_{0,2}^2}{\delta_{1,2} \alpha_{1,2}^2}. \tag{3.2.12}$$

Finally, to complete the linear stability analysis, we observe that the growth rate corresponding to a small supercriticality Δ, where $\Delta \ll A_c$, is given by

$$\sigma_i = \frac{32 \cdot 44}{(21)^2} N[-a_{0,2} a_{1,2} A_c]^{1/2} |\Delta|^{1/2} \tag{3.2.13}$$

and is of the same order as in the $M \neq 0$ development.

4. Finite amplitude analysis

4.1. The $M \neq 0$ case

On the basis of (3.1.10) we define a slow time scale, T, by

$$T = |\Delta|^{1/2} t. \tag{4.1.1}$$

This also implies that the appropriate asymptotic expansion of the total streamfunction field is

$$\begin{aligned} \psi &= \psi_0 + \psi' \\ &= \psi_0 + |\Delta|^{1/2} \psi^{(1)} + |\Delta| \psi^{(2)} + |\Delta|^{3/2} \psi^{(3)} + \dots . \end{aligned} \tag{4.1.2}$$

We follow DE in the manner of specifying both amplitude and phase changes in the wavefields. Changes in the phase of the primary wave will be handled by introducing a slightly stretched fast time, t_*, defined by

$$\frac{\partial t_*}{\partial t} = 1 + W(T), \tag{4.1.3}$$

where $W(T)$ is the stretching function and, in view of (4.1.1) has the expansion

$$W(T) = |\Delta|^{1/2} W_1(T) + |\Delta| W_2(T) + |\Delta|^{3/2} W_3(T) + \dots . \tag{4.1.4}$$

The phases of the primary wave and the k,lth disturbance mode are given, respectively, in stretched time by

$$\theta = N(x - c_* t_*) \tag{4.1.5}$$

$$\theta_k = \hat{\theta}_0 + k\theta = \theta_0 - \sigma t_* + k\theta. \tag{4.1.6}$$

With the time derivative modified by (4.1.1) and (4.1.4), the governing equation (2.1) is now given by

$$[1 + |\Delta|^{1/2} W_1 + |\Delta| W_2 + \dots] \frac{\partial}{\partial t_*} \nabla^2 \psi + |\Delta|^{1/2} \frac{\partial}{\partial T} \nabla^2 \psi + \beta \frac{\partial \psi}{\partial x} + J(\psi, \nabla^2 \psi) = 0. \tag{4.1.7}$$

The substitution of (4.1.2) and (3.1.8) into (4.1.7) yields a sequence of problems in the $\psi^{(n)}$, $n = 0, 1, 2, 3, \dots$, which is identical in form to DE and so only the results at each order will be shown.

Revisiting the O(1) problem in stretched time, t_*, reveals that (4.1.7), to this order, implies the same dispersion relation (2.5) for the primary wave as in ordinary time.

The removal of the resonant inhomogeneities proportional to $\sin\theta$ and $\cos\theta$ at $O(|\Delta|^{1/2})$ yields

$$W_1 = 0 \tag{4.1.8}$$

$$\frac{dA_c}{dT} = 0. \tag{4.1.9}$$

Thus to $O(|\Delta|^{1/2})$ there are no phase or amplitude changes evident in the primary wave.

The removal of the resonant terms proportional to $\sin\theta$ and $\cos\theta$ at $O(|\Delta|)$ yields, respectively,

$$W_2(T) = \gamma_{1,J}|X_{0,J}|^2, \tag{4.1.10}$$

where

$$\begin{aligned}\gamma_{1,J} = \frac{\pi}{Nc\alpha^2 A_c}&[(M+(-2)^{J-1}N)\\ &\times(\alpha_{0,J}^2-\alpha_{-1,L}^2)C_{-1,L}\\ &-(M-(-2)^{J-1}N)\\ &\times(\alpha_{0,J}^2-\alpha_{1,L}^2)C_{1,L}],\end{aligned} \tag{4.1.11}$$

and

$$\frac{dA_R^{(1)}}{dT} = 0. \tag{4.1.12}$$

According to (4.1.10) the self interaction of the disturbance field alters the frequency of the basic state wave. Eq. (4.1.12) permits us to set $A_R^{(1)}$ to zero so that, to $O(|\Delta|)$, there are no amplitude changes in the basic state wave.

The phase shifted solution obtained at this order is given by

$$X_{-1,L}^{(2)} = -i\frac{C_{-1,L}}{\sigma+\delta_{-1,L}}\frac{dX_{0,J}}{dT} \tag{4.1.13a}$$

$$X_{1,L}^{(2)} = -i\frac{C_{1,L}}{\sigma+\delta_{1,L}}\frac{dX_{0,J}}{dT}, \tag{4.1.13b}$$

where the arbitrary $X_{0,J}^{(2)}$ has been set to zero.

The self interaction of the modes within each set also generates forced solutions. To first approximation they are

$$\begin{aligned}\psi_f^{(2)} = F_{-1,1,J}^{(2)}\sin\pi y\, e^{i\bar\theta_{-1}} + \varepsilon_{2,J}F_{0,2,2}^{(2)}\sin 2\pi y\, e^{i\bar\theta_0}\\ + F_{1,1,J}^{(2)}\sin\pi y\, e^{i\bar\theta_1} + *\end{aligned} \tag{4.1.14}$$

where

$$\varepsilon_{2,J} = 0,1 \quad \text{if } J = 1,2$$

and

$$\psi_{R,f}^{(2)} = A_{R,f}^{(2)}\sin 2L\pi y\cos 2\theta \tag{4.1.15}$$

where

$$\bar\theta_k = 2\hat\theta_0 + k\theta. \tag{4.1.16}$$

A third subscript is added to the forced mode amplitudes denoting the set under consideration. The amplitudes of the forced solutions satisfy the relations

$$F_{-1,1,J}^{(2)} = \varepsilon_{2,J}\bar C_{-1,1}F_{0,2,2}^{(2)} + \bar D_{-1,1}X_{0,J}^2, \tag{4.1.17a}$$

$$F_{1,1,J}^{(2)} = \varepsilon_{2,J}\bar C_{1,1}F_{0,2,2}^{(2)} + \bar D_{1,1}X_{0,J}^2, \tag{4.1.17b}$$

$$F_{0,2,2}^{(2)} = f_{0,2}X_{0,2}^2, \tag{4.1.17c}$$

$$A_{R,f}^{(2)} = -\frac{LM\pi(\alpha_{1,L}^2-\alpha_{-1,L}^2)C_{-1,L}C_{1,L}}{[c-(U-\beta/\alpha_{f,J}^2)]N\alpha_{f,J}^2}|X_{0,J}|^2. \tag{4.1.17d}$$

The barred parameters are defined in the appendix.

The removal of the resonant terms proportional to $\cos\theta$ and $\sin\theta$ at $O(|\Delta|^{3/2})$ yields

$$\frac{dA_R^{(2)}}{dT} = -\gamma_{2,J}\frac{d}{dT}|X_{0,J}|^2, \tag{4.1.18}$$

where

$$\begin{aligned}\gamma_{2,J} = \frac{\pi}{2\alpha^2}\Bigg[&(M+(-2)^{J-1}N)\frac{(\alpha_{-1,L}^2-\alpha_{0,J}^2)C_{-1,L}}{\sigma+\delta_{-1,L}}\\ &+(M-(-2)^{J-1}N)\frac{(\alpha_{1,L}^2-\alpha_{0,J}^2)C_{1,L}}{\sigma+\delta_{1,L}}\Bigg]\end{aligned} \tag{4.1.19}$$

and

$$W_3 = 0. \tag{4.1.20}$$

The amplitude changes of the basic state wave on the long time scale, T, given by (4.1.18), are due to the interaction of the $O(|\Delta|^{1/2})$ free disturbance and the $O(|\Delta|)$ phase shifted disturbance. Eq. (4.1.20) is obtained by taking $X_{0,J}$ to be real and shows that at $O(|\Delta|^{3/2})$ there are no phase changes in the basic state wave.

The remaining resonant terms at this order have the same spatial and fast time structure as the homogeneous solution on the marginal curve. Their removal yields the equation governing the long-time variation of the disturbance field.

Eliminating

$$\frac{dX_{-1,L}^{(2)}}{dT},\quad \frac{dX_{1,L}^{(2)}}{dT},\quad F_{-1,1,J}^{(2)},\quad F_{1,1,J}^{(2)},\quad F_{0,2,2}^{(2)},\quad A_{R,f}^{(2)}$$

by using (4.1.13) and (4.1.17) respectively, and letting $X_{0,J} = X$, we obtain the following equation for the disturbance amplitude:

$$\frac{d^2X}{dT^2} - \frac{\Delta}{|\Delta|}\sigma_i^2 X - \sigma_i^2 XA_R^{(2)} + \gamma_4 XW_2 + \gamma_5 X^3 = 0. \tag{4.1.21a}$$

This equation, together with (4.1.10) and (4.1.18), i.e.,

$$W_2 = \gamma_1 X^2, \tag{4.1.21b}$$

$$\frac{\mathrm{d}A_{\mathrm{R}}^{(2)}}{\mathrm{d}T} = -\gamma_2 \frac{\mathrm{d}X^2}{\mathrm{d}T}, \tag{4.1.21c}$$

constitute a closed set for X, $A_{\mathrm{R}}^{(2)}$, and W_2. These variables are amplitudes of fields which depend on x so that *there is no change in the zonal flow on the long time T for the most severe truncation.*

The equation for the disturbance field (4.1.21a) is second order in long time, T. It consists of a term linear in X, which represents exponential growth, and three nonlinear terms which represent the feedback of the basic state wave amplitude and phase corrections and that of the forced solutions onto the disturbance. The coefficients of (4.1.21a) are defined in the appendix.

Eq. (4.1.21a) can be further simplified if $A_{\mathrm{R}}^{(2)}$ and W_2 are eliminated. Substituting from (4.1.21b, c), we obtain

$$\frac{\mathrm{d}^2 X}{\mathrm{d}T^2} = C_1 X + C_2 X^3, \tag{4.1.22}$$

where

$$C_1 = \sigma_i^2 \left[\frac{\Delta}{|\Delta|} + \gamma_2 X^2(0) \right], \tag{4.1.23a}$$

$$C_2 = -[\sigma_i^2 \gamma_2 + \gamma_4 \gamma_1 + \gamma_5]. \tag{4.1.23b}$$

The amplitude equations (4.1.21a–c) are identical in form to those of DE (eq. 4.20a–c) and thus may be qualitatively interpreted in the same manner. When the nonlinearities are stabilizing they produce a perpetual amplitude vacillation. However, the equation for the disturbance field, (4.1.21a), is nonlinearly unstable for large regions of M-space and thus, in these regions, the most severe truncation is not a valid approximation. The extent of this instability in wavenumber space is shown in Table 2.

To allow for a zonal flow change without excessively complicating the algebra, a slightly less severe meridional truncation which includes the mode $l = 3$ was also considered. The added mode permits a weak alteration of the zonal flow at $O(|\Delta|)$ and thus an additional nonlinear feedback on the disturbance field at $O(|\Delta|^{3/2})$.

The regions of validity of this truncation in wavenumber space are shown in Table 3. The nonlinear instability has now shifted to smaller values of zonal wavenumber M so that this truncation yields the correct behavior for the shorter zonal scales.

Thus the two truncations considered in the finite amplitude analysis of the $M \neq 0$ case seem to complement each other. The truncation consisting of zonal modes $k = -1, 0, 1$ and meridional modes $l = 1, 2$ appears to be valid for the longer zonal scales while the truncation consisting of the additional meridional mode $l = 3$ appears to be valid for the shorter scales.

Due to the prohibitive algebra, the finite amplitude analyses of still less severe truncations

Table 2. *Table indicating the extent of nonlinear instability in wavenumber (M) space for the truncation $k = -1, 0, 1$; $l = 1, 2$*

Primary wavenumber N	Secondary wavenumber M																		
2	1		$\boxed{3}$																
3	1	2		$\boxed{4}$	$\boxed{5}$														
4	1	2	3		$\boxed{5}$	$\boxed{6}$	$\boxed{7}$												
5	1	2	3	4		6	$\boxed{7}$	$\boxed{8}$	$\boxed{9}$										
6	1	2	3	$\boxed{4}$	5		$\boxed{7}$	$\boxed{8}$	$\boxed{9}$	$\boxed{10}$	$\boxed{11}$								
7	1	2	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	6		8	$\boxed{9}$	$\boxed{10}$	$\boxed{11}$	$\boxed{12}$	$\boxed{13}$						
8	1	2	$\boxed{3}$	$\boxed{4}$	5	6	7		9	10	$\boxed{11}$	$\boxed{12}$	$\boxed{13}$	$\boxed{14}$	$\boxed{15}$				
9	1	2	$\boxed{3}$	$\boxed{4}$	5	6	7	8		10	11	12	$\boxed{13}$	$\boxed{14}$	$\boxed{15}$	$\boxed{16}$	$\boxed{17}$		
10	1	2	$\boxed{3}$	4	5	6	7	8	9		11	12	13	$\boxed{14}$	$\boxed{15}$	$\boxed{16}$	$\boxed{17}$	$\boxed{18}$	$\boxed{19}$

Boxes represent nonlinear instability.

Table 3. *Table indicating the extent of nonlinear instability in wavenumber (M) space for the truncation $k = -1, 0, 1$; $l = 1, 2, 3$*

Primary wavenumber N	Secondary wavenumber M																
2	1																
3	$\boxed{1}$	2		4													
4	1	$\boxed{2}$	$\boxed{3}$		5	$\boxed{6}$											
5	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	4		6	$\boxed{7}$										
6	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	5		7	$\boxed{8}$			$\boxed{11}$						
7	$\boxed{1}$	$\boxed{2}$	3	4	5	6		8	9	10	$\boxed{11}$	$\boxed{12}$	$\boxed{13}$				
8	1	2	$\boxed{3}$	4	5	6	7		9	10	11	12	13	$\boxed{14}$			
9	$\boxed{1}$	$\boxed{2}$	3		5	6	7	$\boxed{8}$		10	11	12	13	14	15		
10	1	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	6	7	8	9		11	12	13	14	15	16	17

Boxes represent nonlinear instability.

are very difficult. Yet, as the foregoing results indicate, to achieve finite amplitude stability, a further relaxation of truncation is required. The only practical alternative is a fully numerical treatment of this case. In contrast, as is shown next, in the case $M = 0$, finite amplitude stability is obtained at the most severe truncation.

4.2. *The $M = 0$ case*

In view of (3.2.13) the nonlinear analysis for the case $M = 0$ proceeds as in $M \neq 0$ case. Thus, we need only compare the results obtained at the various orders with those of the previous case.

At $O(|\Delta|^{1/2})$ we obtain

$$W_1 = 0, \tag{4.2.1}$$

$$A_c = \text{constant}, \tag{4.2.2}$$

as before.

At higher orders, in contrast to $M \neq 0$ case, the self-interaction of the free disturbance field produces forced advections which only have the structure of the free disturbance. Therefore, no formal introduction of forced solutions is required at any order.

The removal of the resonant term proportional to $\sin\theta$ and $\cos\theta$ at $O(|\Delta|)$ yields

$$W_2 = \gamma_1 X_{0,2}^2 \tag{4.2.3}$$

where

$$\gamma_1 = \frac{64\cdot 32}{63}\frac{(\alpha_{1,2}^2 - \alpha_{0,2}^2)\,C_{1,2}}{\alpha^2 c A_c}, \tag{4.2.4}$$

and

$$A^{(1)} = 0, \tag{4.2.5}$$

which are analogous to (4.1.10) and (4.1.12).

The phase-shifted solution is given by

$$X_{-1,2}^{(2)} = \frac{iC_{1,2}}{\delta_{1,2}}\frac{\mathrm{d}X_{0,2}}{\mathrm{d}T}, \tag{4.2.6a}$$

$$X_{1,2}^{(2)} = -\frac{iC_{1,2}}{\delta_{1,2}}\frac{\mathrm{d}X_{0,2}}{\mathrm{d}T}, \tag{4.2.6b}$$

where the arbitrary $X_{0,2}^{(2)}$ is set to zero.

The removal of the term resonant in $\cos\theta$ and $\sin\theta$ at $O(|\Delta|^{3/2})$ yields, respectively,

$$\frac{\mathrm{d}A^{(2)}}{\mathrm{d}T} = -\gamma_2\frac{\mathrm{d}X_{0,2}^{(2)}}{\mathrm{d}T}, \tag{4.2.7}$$

where

$$\gamma_2 = \frac{64\cdot 16}{63}N\frac{(\alpha_{1,2}^2 - \alpha_{0,2}^2)\,C_{1,2}}{\alpha^2\delta_{1,2}} \tag{4.2.8}$$

and

$$W_3 = 0, \tag{4.2.9}$$

which are the same as (4.1.18) and (4.1.20).

The remaining inhomogeneities at this order are nonresonant. They give rise to an $O(|\Delta|^{3/2})$ solution which is phase shifted with respect to the $O(|\Delta|)$ solution but which represents only an amplitude correction to the $O(|\Delta|^{1/2})$ solution. This is the fundamental difference between the analysis of the $M = 0$ case and the $M \neq 0$ case.

We must, therefore, continue to the *fourth* order to obtain the amplitude equation which governs the long time evolution of the disturbance field.

With the arbitrary $X_{0,2}^{(3)}$ set to zero, the $O(|\Delta|^{3/2})$ phase shifted solution is given by

$$X_{-1,2}^{(3)} = \frac{i}{\delta_{1,2}} \frac{dX_{-1,2}^{(2)}}{dT} - \frac{\Delta}{|\Delta|} \frac{16}{21} \frac{N(\alpha^2 - \alpha_{0,2}^2)}{\delta_{1,2}\alpha_{1,2}^2} X_{0,2} - \frac{16}{21} \frac{N(\alpha^2 - \alpha_{0,2}^2)}{\delta_{1,2}\alpha_{1,2}^2} A^{(2)} X_{0,2} - \frac{NcC_{1,2}}{\delta_{1,2}} W_2 X_{0,2}, \tag{4.2.10a}$$

$$X_{1,2}^{(3)} = X_{-1,2}^{(3)}. \tag{4.2.10b}$$

The removal of the terms resonant in $\sin\theta$ and $\cos\theta$ at $O(|\Delta|^2)$ yields, respectively,

$$W_4 = -\frac{64\cdot 32}{63} \frac{(\alpha_{1,2}^2 - \alpha_{0,2}^2) C_{1,2}}{\alpha^2 c A_c} \left\{ \frac{C_{1,2}}{2\delta_{1,2}} \frac{dX_{0,2}^2}{dT} + \frac{16}{21} \frac{N(\alpha^2 - \alpha_{0,2}^2)}{\delta_{1,2}\alpha_{1,2}^2} (1 + \gamma_2 X_{0,2}^{(2)}(0)) X_{0,2}^2 - \left[\frac{16}{21} \frac{N(\alpha^2 - \alpha_{0,2}^2)}{\delta_{1,2}\alpha_{1,2}^2} \gamma_2 - \frac{NcC_{1,2}}{\delta_{1,2}} \gamma_1 \right] X_{0,2}^4 \right\}, \tag{4.2.11}$$

$$A^{(3)} = 0. \tag{4.2.12}$$

The removal of the terms resonant with the free disturbance structure yields an amplitude equation for the disturbance field which is of third order in time. Upon eliminating $X_{\mp 1,2}^{(2)}$ by (4.2.6a, b) and $X_{\mp 1,2}^{(3)}$ by (4.2.10a, b) and $A^{(2)}$ and W_2 by (4.2.7) and (4.2.3) respectively, the equation is then in terms of $X_{0,2}$ only. Its similarity to its counterpart in the $M \neq 0$ case, (4.1.22), can be seen by direct integration, taking as an initial condition the exponential growth predicted by linear theory:

$$\frac{d^2 X_{0,2}}{dT^2}(0) = \frac{18}{35} \frac{N^2(N^2 - 3\pi^2)}{N^2 + 4\pi^2} A_c X_{0,2}(0) = 0. \tag{4.2.13}$$

The final form of the amplitude equation, with $X_{0,2}$ now denoted by X, is

$$\frac{d^2 X}{dT^2} = \frac{18}{35} \frac{N^2(N^2 - 3\pi^2)}{N^2 + 4\pi^2} \times \left[6\frac{\Delta}{|\Delta|} + \frac{64\cdot 8\cdot 16}{21} \frac{N^2}{N^2 + \pi^2} X^2(0) \right] X - \frac{2\cdot(64)^2\cdot 232}{63\cdot 35\cdot 9} \frac{N^4(N^2 - 3\pi^2)}{(N^2 + \pi^2)(N^2 + 4\pi^2)} X^3 + \frac{2\cdot(64)^2\cdot 232}{63\cdot 35\cdot 9} \times \frac{N^4(N^2 - 3\pi^2)}{(N^2 + \pi^2)(N^2 + 4\pi^2)} X^3(0). \tag{4.2.14}$$

The coefficient of the nonlinear term in (4.2.14) is negative definite for $N^2 > 3\pi^2$. Since this condition is always satisfied for linear instability, the nonlinearity is *always* stabilizing. This is in contrast to the result obtained for the $M \neq 0$ case. The corresponding quantity, C_2, in (4.1.22) becomes destabilizing for larger values of M. However, when M is small and in particular as $M \to 0$, C_2 becomes negative. Thus the result for $M = 0$ is consistent with that of the most severe truncation of $M \neq 0$ in the limit as M becomes small.

5. Summary and conclusions

We have investigated the linear and finite amplitude stability characteristics of a wavelike basic state in a mid-latitude channel. The basic state consists of a large amplitude barotropic Rossby wave. Two types of imposed disturbance, denoted as the $M \neq 0$ case and the $M = 0$ case, are considered.

The $M \neq 0$ case consists of a purely wavelike disturbance. Several truncations of this case are considered. The most severe truncation consists of the modes $k = -1, 0, 1$; $l = 1, 2$. To allow for zonal flow changes at finite amplitude, which are not permitted by this truncation, a less severe meridional truncation consisting of $l = 1, 2, 3$ is also considered. In addition, to assess the convergence of the linear problem, still less severe truncations are considered.

The wave-wave interactions produced above, at most, generate a zonal flow at $O(|\Delta|)$. It is of physical importance to investigate a *stronger* involvement of the zonal flow in the nonlinear dynamics. This is achieved by considering the $M = 0$ case. With $M = 0$, the secondary wave takes the form of a horizontally sheared zonal current. Because of the finite amplitude stability of this case, only the most severe truncation, as defined for the $M \neq 0$ case, is considered.

By imposing truncated versions of either type of disturbance upon the basic flow, we indeed find the Rossby wave to be unstable provided its amplitude exceeds a certain critical value, A_c, which is dependent on wavenumber.

The linear stability analysis of the $M \neq 0$ case reveals that, because of the channel configuration, any given truncation is divided into *two* decoupled sets of modes, each with its own stability characteristics. By imposing cyclic continuity in the zonal direction, the zonal wavenumbers of both the primary and secondary waves are restricted to integers. This constraint precludes resonant interactions and allows for consideration of only the linearly most unstable set in the finite amplitude analysis.

The instability so obtained is thus of a Rayleigh type and shows the importance of barotropic instability, in addition to the universally accepted mechanism of baroclinic instability, in the evolution of large-scale waves in the mid-latitudes. This is perhaps the most fundamentally important result derived from considering a wavelike basic state as opposed to zonally uniform basic states which tend to be barotropically stable.

The most severe truncation of the disturbance, consisting of three zonal modes and two meridional modes, is admittedly very severe. However, by obtaining the curves of marginal stability for the less severe truncations, $k = \mp 2, \mp 1, 0$; $l = 1, 2, 3$ and $k = \mp 3, \mp 2, \mp, 0$; $l = 1, 2, 3, 4$, the validity of the most severe truncation in the linear stability analysis for small wavenumber M for set I and small A_c (near resonance condition) for either set is established.

A linear stability analysis of the $M = 0$ case is also performed in a manner similar to the $M \neq 0$ case. However, the most severe truncation of the disturbance field as defined above, reduces to 3 modes instead of 6 and the distinction between the two sets of modes found in the $M \neq 0$ case vanishes. In further contrast to the $M \neq 0$ case, Fjortoft's necessary condition for instability for the modes involved requires that the basic state wave be restricted to shorter zonal scales ($N \geqslant 6$). Away from the resonance points of the $M \neq 0$ case, the threshold of instability is *lower* for the $M = 0$ case, for $N \geqslant 8$. Thus, on the basis of the results obtained here, nonresonant barotropic energy transfers occur among waves for all zonal scales while the reinforcement of the zonal flow occurs only for intermediate and short waves ($N \geqslant 6$).

The finite amplitude analysis of the most severe truncation of the $M \neq 0$ case yields, for smaller values of M for which the nonlinear coupling coefficient, C_2, is negative, a long-time evolution of the system characterized by a vacillatory exchange of energy between the basic Rossby wave and the disturbance. This wave-wave nonresonant interaction implies that energy can be transferred directly from an equivalent barotropic system, say an occluded low of quasigeostrophic scale, to another wavelike disturbance and not necessarily to the zonal flow as implied by previous results on the barotropic stability of zonal currents. Thus large-scale waves of small amplitude in mid-latitudes can grow not only because of baroclinic instability, as is generally accepted, but also as a result of the barotropic instability of preexisting large amplitude waves.

The unexpectedly widespread occurrence of destabilizing (i.e. positive) values of the nonlinear coupling coefficient C_2 in M-space in the $M \neq 0$ case requires a relaxation of the truncation scheme. Unfortunately, the algebra quickly becomes prohibitive and one must resort to a numerical treatment. However, the problem is still analytically tractable with the inclusion of the meridional mode $l = 3$. For this truncation, amplitude vacillation is found for the larger values of M and nonlinear instability now arises for the smaller values of M. Thus, it appears that the most severe meridional truncation $l = 1, 2$ is more adequate for the longer zonal scales while the less severe meridional truncation is more adequate for the shorter scales. It is evident that, over much of the wavenumber domain, very severely truncated wavefields do not describe either the linear or the nonlinear dynamics of the $M \neq 0$ case well.

The analysis of the $M = 0$ case is also concluded by a finite amplitude extension of the linear stability problem for small supercriticality. In contrast to the $M \neq 0$ case, a phase shifted solution is generated at third order instead of an amplitude equation. Closure of the problem is obtained at fourth order. Provided Fjortoft's instability condition is met, the nonlinear coupling coefficient is *always* stabilizing (negative) and a vacillatory exchange of energy results between the basic state wave and the disturbance. Thus, in the $M = 0$ case, the region of nonlinear stabilization coincides with the region of linear instability. The nonlinear stability is consistent, as a limiting case, with the behavior of the most severe truncation of the $M \neq 0$ case for small M. The results of this case are also consistent with the observational studies, e.g., Saltzman (1970), which find energy transfer from quasigeostrophic barotropic waves (e.g., an occluded low) directly to the zonal flow.

6. Appendix

The parameters connected with the amplitudes of the forced modes are defined below.

$$\bar{C}_{-1,1} = \frac{-1}{(2\sigma + \bar{\delta}_{-1,1})\bar{\alpha}^2_{-1,1}}\left[\frac{\pi}{4}(2M - 2N)\bar{a}_{0,2}\bar{\alpha}^2_{0,2}A_c\right], \tag{A1}$$

$$\bar{C}_{1,1} = \frac{-1}{(2\sigma + \bar{\delta}_{1,1})\bar{\alpha}^2_{1,1}}\left[\frac{\pi}{4}(2M + 2N)\bar{a}_{0,2}\bar{\alpha}^2_{0,2}A_c\right], \tag{A2}$$

$$\bar{D}_{-1,1} = \frac{(-1)^{J-1}}{(2\sigma + \bar{\delta}_{-1,1})\bar{\alpha}^2_{-1,1}}\left[\frac{\pi}{2}C_{-1,L}(\alpha^2_{0,J} - \alpha^2_{-1,L})(3M - JN)\right], \tag{A3}$$

$$\bar{D}_{1,1} = \frac{(-1)^{J-1}}{(2\sigma + \bar{\delta}_{1,1})\bar{\alpha}^2_{1,1}}\left[\frac{\pi}{2}C_{1,L}(\alpha^2_{0,J} - \alpha^2_{1,L})(3M + JN)\right], \tag{A4}$$

$$\begin{aligned} f_{0,2} = \frac{1}{D\bar{\alpha}^2_{0,2}}\Big[& -\frac{\pi}{2}2N(\alpha^2_{1,1} - \alpha^2_{-1,1})C_{-1,1}C_{1,1}(2\sigma + \bar{\delta}_{-1,1})(2\sigma + \bar{\delta}_{1,1}) \\ & + \frac{\pi^2}{8}(2M - 2N)(3M - 2N)A_c\bar{a}_{-1,1}(\alpha^2_{0,2} - \alpha^2_{-1,1})C_{-1,1}(2\sigma + \bar{\delta}_{1,1}) \\ & - \frac{\pi^2}{8}(2M + 2N)(3M + 2N)A_c\bar{a}_{1,1}(\alpha^2_{1,1} - \alpha^2_{0,2})C_{1,1}(2\sigma + \bar{\delta}_{-1,1})\Big], \end{aligned} \tag{A5}$$

$$\begin{aligned} D = & -\frac{\pi^2}{16}(2M - 2N)^2A_c^2\bar{a}_{-1,1}\bar{a}_{0,2}(2\sigma + \bar{\delta}_{1,1}) + (2\sigma + \bar{\delta}_{-1,1})(2\sigma + \bar{\delta}_{0,2})(2\sigma + \bar{\delta}_{1,1}) \\ & - \frac{\pi^2}{16}(2M + 2N)^2A_c^2\bar{a}_{1,1}\bar{a}_{0,2}(2\sigma + \bar{\delta}_{-1,1}), \end{aligned} \tag{A6}$$

$$\bar{\delta}_{k,l} = 2Mc_0 + kNc - (2M + kN)\left(U - \frac{\beta}{\bar{\alpha}^2_{k,l}}\right), \tag{A7}$$

$$\bar{\alpha}^2_{k,l} = (2M + kN)^2 + l^2\pi^2, \tag{A8}$$

$$\bar{a}_{k,l} = \frac{\bar{\alpha}^2_{k,l} - \alpha^2}{\bar{\alpha}^2_{k,l}}, \tag{A9}$$

$$\alpha^2_{f,J} = 4N^2 + \frac{16}{J^2}\pi^2. \tag{A10}$$

The coefficients of the amplitude equation (4.1.21a) which governs the long time behavior of the disturbance field are defined below.

$$\sigma_i^2 = \frac{2}{A_c}\left[\frac{(M + (-2)^{J-1}N)^2 a_{-1,L}}{\sigma + \delta_{-1,L}} + \frac{(M - (-2)^{J-1}N)^2 a_{1,L}}{\sigma + \delta_{1,L}}\right]$$

$$\times\left[\frac{(M+(-2)^{J-1}N)^2 a_{-1,L}}{(\sigma+\delta_{-1,L})^3}+\frac{(M-(-2)^{J-1}N)^2 a_{1,L}}{(\sigma+\delta_{1,L})^3}\right]^{-1}, \tag{A11}$$

$$\gamma_4 = Nc\left[\frac{(M-(-2)^{J-1}N)^2 a_{1,L}}{(\sigma+\delta_{1,L})^2}-\frac{(M+(-2)^{J-1}N)^2 a_{-1,L}}{(\sigma+\delta_{-1,L})^2}\right]$$

$$\times\left[\frac{(M+(-2)^{J-1}N)^2 a_{-1,L}}{(\sigma+\delta_{-1,L})^3}+\frac{(M-(-2)^{J-1}N)^2 a_{1,L}}{(\sigma+\delta_{1,L})^3}\right]^{-1}, \tag{A12}$$

$$\gamma_5 = \left[\frac{(M+(-2)^{J-1}N)^2 a_{-1,L}}{(\sigma+\delta_{-1,L})^3}+\frac{(M-(-2)^{J-1}N)^2 a_{1,L}}{(\sigma+\delta_{1,L})^3}\right]^{-1}$$

$$\times\left\{(-1)^{J-1}\left[\frac{2(M+(-2)^{J-1}N)(3M-JN)(\alpha^2_{-1,L}-\alpha^2_{0,J})(\alpha^2-\bar{\alpha}^2_{-1,1})}{\alpha^2_{-1,L}(\alpha^2_{0,J}-\alpha^2)(\sigma+\delta_{-1,L})A_c}\,f_{-1,J}\right.\right.$$

$$-\frac{\varepsilon_{2,J}\pi N(M+2N)(M-2N)(\bar{\alpha}_{0,2}-\alpha^2)(\alpha^2_{1,1}-\alpha^2_{-1,1})}{\alpha^2_{-1,1}\alpha^2_{1,1}(\sigma+\delta_{-1,L})(\sigma+\delta_{1,L})}\,f_{0,2}$$

$$\left.+\frac{2(M-(-2)^{J-1}N)(3M+JN)(\alpha^2_{1,L}-\alpha^2_{0,J})(\alpha^2-\bar{\alpha}^2_{1,1})}{\alpha^2_{1,L}(\alpha^2_{0,J}-\alpha^2)(\sigma+\delta_{1,L})A_c}\,f_{1,J}\right]$$

$$+\frac{\pi^4}{16J^2}\,\frac{2M^2(M+(-2)^{J-1}N)^2(M-(-2)^{J-1}N)^2A_c^2a^2_{0,J}\alpha^4_{0,J}(\alpha^2_{-1,L}-\alpha^2_{1,L})}{(\alpha^2_{f,J}N(c-U)+\beta N)(\sigma+\delta_{-1,L})^2(\sigma+\delta_{1,L})^2\alpha^4_{-1,L}\alpha^4_{1,L}}$$

$$\left.\times\left[(\alpha^2_{-1,L}-\alpha^2)(\alpha^2_{f,J}-\alpha^2_{1,J})+(\alpha^2_{1,L}-\alpha^2)(\alpha^2_{f,J}-\alpha^2_{-1,L})\right]\right\}, \tag{A13}$$

$$f_{-1,J}=\frac{1}{(2\sigma+\bar{\delta}_{-1,1})\bar{\alpha}^2_{-1,1}}\left[-\varepsilon_{2,J}\frac{\pi}{4}(2M-2N)A_c(\bar{\alpha}^2_{0,2}-\alpha^2)\,f_{0,2}\right.$$

$$\left.+(-1)^J\frac{\pi}{2}(3M-JN)(\alpha^2_{-1,L}-\alpha^2_{0,J})C_{-1,L}\right], \tag{A14}$$

$$f_{1,J}=\frac{1}{(2\sigma+\bar{\delta}_{1,1})\bar{\alpha}^2_{1,1}}\left[-\varepsilon_{2,J}\frac{\pi}{4}(2M+2N)A_c(\bar{\alpha}^2_{0,2}-\alpha^2)\,f_{0,2}\right.$$

$$\left.+(-1)^J\frac{\pi}{2}(3M+JN)(\alpha^2_{1,L}-\alpha^2_{0,J})C_{1,L}\right]. \tag{A15}$$

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