

The spatial statistics of random geostrophic modes and first-guess errors

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ABSTRACT

It is hypothesized that a data assimilation system with excellent forecast and analysis components and frequent access to good data should have first-guess errors similar to an ensemble of random slow modes with equipartition of energy. The covariance statistics of such a system on a constant f -plane are calculated and shown to have interesting features. Of these, the overall increase with elevation of wind and geopotential errors, the size of temperature variances for given height variances, the increase with elevation of the horizontal velocity scale, and the sharper correlation in the vertical for wind than geopotential, are also found in the empirical studies of first-guess errors over North America by Hollingsworth and Lönnberg. Two major differences are (a) the presence of a maximum wind and geopotential error at the tropopause in the empirical data, and (b) the empirical horizontal spectra for velocity peak at the largest accessible wave length (5000 km) while the theoretical spectra peak at the shortest wave length (1000 km). If it is assumed that basic conditions of the hypothesis are satisfied, the ECMWF forecast and analysis methods must introduce relatively large errors at long wave lengths. A more believable conclusion is that the assumption of “frequent access to good data” is invalidated by data voids outside of the test network.

Other results agree with the Hollingsworth–Lönnberg study in that the separable mathematical models used to represent correlations in operational practice misrepresent some important features. The results also suggest that temperature errors in a forecast of high vertical resolution will be underestimated by evaluation of thickness errors between the standard pressure levels.

1. Background and motivation

The analysis of the variables in a numerical model into normal modes of the linearized equations has become a valuable technique in numerical weather prediction. Machenhauer (1976) and Baer (1977) have shown how the initial conditions for a numerical prediction can be adjusted to contain less meteorological “noise” by adjusting the amplitudes and phases of the “fast” gravity modes to values that are forced by non-linear interactions between the “slow” Rossby modes. Although it is now well-known that this simple picture must be modified in the tropics, this approach has made it possible to extend earlier ideas about the suppression of meteorological noise (Hinkelman, 1951; Phillips, 1960) with consider-

able ease and flexibility to large-scale numerical models. Analysis procedures in extratropical latitudes are also related to the concept of slow modes. These procedures generally assume a geostrophic relation between the changes that are made by data to the wind and geopotential fields presented by the short-period forecast that is used as a “first guess” for the analysis.

Ghil et al. (1981) have examined the application of Kalman filtering to meteorological data assimilation. This procedure undoubtedly has much to offer since it attempts to predict the evolution of the first-guess error covariance matrix with time, and allows also for spatial inhomogeneity of the data. It requires considerable computation, however, much more than the forecast itself. It is therefore likely that the current practice of using a fixed error

covariance matrix derived from a past data base will continue for some time. The present paper is an attempt to understand some of the features present in such a matrix under idealized conditions.

It is instructive to first consider the case of a data assimilation system that operates with little data, but with an excellent forecast model that incorporates all important physical processes in an accurate (but not perfect) manner and with only small numerical approximations. Under these conditions one would expect the first-order time means of the errors to be very small. But second-order moments of the errors would be large, because the phase of individual eddies would eventually become completely erroneous. The spatial structure of the forecast errors would be very similar to those present in the true eddy fields, mirroring the large vertical scales typical of synoptic patterns and the dominant cyclone horizontal scale of several thousands of kilometers.

Suppose that this assimilation system with an excellent forecast model and analysis method is now given access to excellent data at frequent intervals. If the frequency of data insertion is sufficiently often compared to the growth rate of instabilities in the atmosphere (or model), the forecast errors would have little time in which to develop patterns resembling those of true fields. The forecast field would now always be close to the true field. The small forecast errors would be due to a host of small effects such as minor numerical approximations, slight inadequacies in the representation of sub-grid scale processes and radiation, plus errors in the analysis system. Under these conditions it seems reasonable to expect that the errors would, in some sense, be *random**. However, it would probably be wrong to expect that they would be characterized by zero correlations between errors at neighboring grid points, since the data assimilation intervals, although frequent, are still longer than the time for an influence to travel between grid points of the model during the forecast part of the assimilation system. It seems more logical to phrase this randomness in terms of the amplitudes of the normal modes of the forecast model.

* Exceptions can be imagined. For example, temperature errors in the bottom layer might be kept especially small by the controlling effect of accurately known sea-surface water temperatures.

Suppose, then, that each realization of a field of forecast error in this idealized assimilation system is projected onto a set of orthonormal modes that either are, or closely approximate, the natural modes of the linearized forecast equations. Let $P(q, j)$ represent the amplitude coefficient for mode q in realization j . We might expect to begin with that randomness would result in the expected value over the ensemble of j -realizations of a product $P(q_1, j) \times P(q_2, j)$ to vanish unless q_1 was the same as q_2 . Furthermore, if there are as many sources of error for one mode as for another, we might expect that in an average over the ensemble, the amplitude of each mode, if expressed in the natural measure of energy, would be the same for all modes. The errors will also be exchanged between wave numbers through advection by the spatially non-uniform true fields, and thereby tend to attain an equipartition of energy.

In extratropical latitudes, however, it is necessary to recognize that all modes do not have the same status; the fast gravity modes do not have an independent existence, but are primarily forced by non-linear interactions between the slow Rossby modes. This interaction is mainly a quadratic one. If G_q denotes a gravity mode coefficient, we may write this symbolically as

$$G_q = \epsilon[I_{12q} R_1 R_2 + I_{13q} R_1 R_3 + \dots], \quad (1.1)$$

R_i is the amplitude of a slow mode, and I_{12q} , etc., are interaction coefficients. ϵ is the Rossby number. It is small compared to unity, while R_i and the interaction coefficients are of order unity. This relation holds for the true fields, and, since the forecast model is excellent, for the model fields also. Thus, if (1.1) is the statement for the true fields, the forecast fields, denoted by a prime, will satisfy the same relation,

$$G'_q = \epsilon[I_{12q} R'_1 R'_2 + I_{13q} R'_1 R'_3 + \dots], \quad (1.2)$$

since the forecast and initialization procedures are reasonably accurate. If we now express the forecast Rossby mode amplitude R'_i as equal to R_i plus the error δ_i , we easily find that

$$G'_q = G_q + \epsilon[I_{12q}(R_1 \delta_2 + R_2 \delta_1 + \delta_1 \delta_2) + \dots]. \quad (1.3)$$

The error in this fast mode is then of order

$$G'_q - G_q = \epsilon I R \delta. \quad (1.4)$$

This is smaller than the error δ in the slow mode.

The constraint of balance in both model and atmosphere leads therefore to the result that the error field, just like the true and forecast fields, *consists mostly of slow modes*.

In Section 2, these general statistical thoughts will be stated more precisely for the simple case of a cyclically continuous Cartesian region with a constant Coriolis parameter. This will be specialized later in the paper to match the conditions under which Hollingsworth and Lönnerberg (1986—referred to hereafter as HL) and Lönnerberg and Hollingsworth (1986—referred to hereafter as LH) have evaluated the first-guess errors of the European Center for Medium-range Weather Forecasting (ECMWF) grid-point model for wind (HL) and geopotential (LH). Their evaluation of the errors was done with radiosonde data over North America, which, since it is a region of relatively dense, homogeneous, and accurate data would seem to have some possibility of satisfying the conditions under which the above hypothesis of randomness has been formulated.

2. Covariance of random geostrophic modes

We consider a square, cyclically continuous Cartesian region of size $0 \leq x \leq L$, $0 \leq y \leq L$. In the appendix, a set of orthogonal vertical eigenfunctions is derived for a simplified finite-difference vertical structure of a resting model atmosphere. In the horizontal we use trigonometric functions $\exp(i\psi)$, where

$$\psi = ax + by. \quad (2.1)$$

a and b are discretized horizontal wave numbers defined by

$$a = 2\pi m_a/L, \quad b = 2\pi m_b/L, \quad (2.2)$$

in which m_a and m_b take on the integer values 0, 1, 2, ..., M . The horizontal resolution is now set by the sizes of L and M , and these will have considerable influence on the length scales characterizing the covariance functions of interest. The symbol $n = 1, \dots, N$ will denote the vertical wave number.

The geopotential component of mode (a, b, n) will, at location x, y on vertical level k , have a value

$$\phi'(x, y, k; a, b, n) = P_{abn} A_{kn} \operatorname{Re}(\exp i\psi). \quad (2.3)$$

P_{abn} is the energy normalization factor for the geostrophic slow modes, as defined in eq (A38):

$$P_{abn} = \left[\frac{a^2 + b^2}{f^2} + \frac{1}{gh_n} \right]^{-1/2} \quad (2.4)$$

f is the constant Coriolis parameter, g is gravity, and h_n is the equivalent depth for vertical mode n . A_{kn} is the geopotential eigenvector derived in the appendix. The u and v velocity components for this mode are obtained by inserting the factors $(-ib/f)$ and (ia/f) in front of $\exp(i\psi)$. For simplicity, let κ denote the wave-number dependence on (a, b, n) and let r denote the dependence on location (x, y, k) . The j th realization of the geopotential is given by

$$\phi_j(r) = \operatorname{Re} \sum_{\kappa} X_j(\kappa) \phi'(\kappa, r). \quad (2.6)$$

X , a function of both the wave number and realization j , is a complex number expressing the amplitude and phase of the geopotential from each mode in this realization.

The covariance over the j -ensembles of $\phi_1 = \phi$ at $r = r_1$ with $\phi_2 = \phi$ at $r = r_2$ is then defined by the expected value, denoted by angular brackets, of the product

$$\langle \phi_j(r_1) \phi_j(r_2) \rangle = \left\langle \left[\operatorname{Re} \sum_{\kappa} X_j \phi'_1 \right] \left[\operatorname{Re} \sum_{\kappa'} X_j \phi'_2 \right] \right\rangle \quad (2.7)$$

We must therefore specify the expected values of four products:

$$\begin{aligned} X(\kappa) X(\kappa'), & \quad X^*(\kappa) X^*(\kappa'), \\ X^*(\kappa) X(\kappa'), & \quad X(\kappa) X^*(\kappa'). \end{aligned} \quad (2.8)$$

Independence of the individual modes means that these vanish unless κ and κ' are the same. This simplifies (2.7) to

$$\begin{aligned} 4 \langle \phi(r_1) \phi(r_2) \rangle &= \sum_{\kappa} [\phi'_1 \phi'_2 \langle XX \rangle \\ &+ \phi'_1{}^* \phi'_2{}^* \langle X^* X^* \rangle + (\phi'_1 \phi'_2{}^* + \phi'_1{}^* \phi'_2) \langle XX^* \rangle] \end{aligned} \quad (2.9)$$

(The κ dependence is suppressed for simplicity.)

If we express X as $|X|$ times $\exp(i\theta)$, in which both X and θ depend on κ and j , the required four product terms become

$$\begin{aligned}\langle XX \rangle &= \langle |X|^2 \exp(2i\theta) \rangle, \\ \langle X^* X^* \rangle &= \langle |X|^2 \exp(-2i\theta) \rangle, \\ \langle XX^* \rangle &= \langle |X|^2 \rangle.\end{aligned}\quad (2.10)$$

In the first of these we can express $|X|^2$ and $\exp 2i\theta$ as ensemble means and a deviation from that mean. We first apply a condition that the ensemble mean of $\exp(2i\theta)$ is zero—there is no preferred phase for any mode. We then apply a condition that there is no correlation between the deviations of $|X|^2$ about its mean and the deviations of $\exp(2i\theta)$ about its (zero) mean. (In other words for each mode there is no preferred horizontal phase in which to attain its maximum or minimum amplitude.) This leads to the result that

$$\langle XX \rangle = 0. \quad (2.12)$$

The same reasoning will eliminate the ensemble average of $X^* X^*$, so that only $|X|^2$ contributes:

$$\langle \phi(r_1) \phi(r_2) \rangle = \frac{1}{2} \sum_{\kappa} \langle |X|^2 \rangle (\phi'_1 \phi'^*_2 + \phi'^*_1 \phi'_2). \quad (2.13)$$

According to the definition (2.3) for ϕ' , we need consider only the complex aspects of $\exp(i\psi_1)$ and $\exp(i\psi_2)$ in continuing with (2.13). The first result is

$$\begin{aligned}\langle \phi(r_1) \phi(r_2) \rangle &= \frac{1}{2} \sum_{\kappa} \langle |X|^2 \rangle P^2 A_{k_1 n} A_{k_2 n} \\ &\times \cos(\psi_1 - \psi_2).\end{aligned}\quad (2.14)$$

The expressions for the covariance of u or of v are the same, except that the geostrophic factors b^2/f^2 or a^2/f^2 must be inserted inside the sum.

The energy considerations at the end of the appendix now require that $\langle |X|^2 \rangle$, at most a function only of the wave number, must be the same for all wave numbers if there is to be equipartition of energy in the ensemble. We can also specialize to the case of r_1 located at $x = y = 0$:

$$\begin{aligned}\langle \phi(0, 0, k_1) \phi(x, y, k_2) \rangle \\ = \frac{1}{2} |X|^2 \sum_n A_{k_1 n} A_{k_2 n} \\ \sum_b \sum_a P^2(a, b, n) \cos(ax + by).\end{aligned}$$

At this point it becomes necessary to specify the range of a and b . The simplest procedure is to take advantage of the fact that we shall evaluate only covariances of a variable with itself, as is already implicit in the above equation. If the resolution is set by an upper limit M to the integers m_a and m_b , it is sufficient to let m_a range from 0 to M if we have m_b range from $-M$ to $+M$. This accounts for all directions of the horizontal wave number vector (a, b). For this set of variances, the sum over b from $-M$ to $+M$ can be replaced by the sum from 0 to $+M$ if we also replace $\cos(ax + by)$ by the product $\cos(ax)$ times $\cos(by)$. The final formula for the geopotential covariance is then

$$\begin{aligned}\langle \phi(0, 0, k_1) \phi(x, y, k_2) \rangle \\ = \text{CON} \sum_n A_{k_1 n} A_{k_2 n} \\ \sum_b \sum_a P^2(a, b, n) \cos(ax) \cos(by).\end{aligned}\quad (2.15)$$

CON is a constant. Covariances for (u, u) and (v, v) are obtained by inserting the factors $(b/f)^2$ or $(a/f)^2$ in front of the cosine terms.

A formula for temperature covariance can be derived by first defining the temperature at the pressure midway between the levels at which geopotential is defined. For $k = 2, 3, \dots, K$ this is accomplished by

$$T_k = 2p_k(\phi_k - \phi_{k-1})/R(p_{k-1} - p_{k+1}). \quad (2.16)$$

The covariance $\langle T(0, 0, k_1) T(x, y, k_2) \rangle$ is then identical with (2.15) except that the geopotential eigenvector A_{kn} must be replaced by τ_{kn} :

$$\tau_{kn} = 2p_k(A_{kn} - A_{k-1n})/R(p_{k-1} - p_{k+1}) \quad (2.17)$$

for $k = k_1$ and for $k = k_2$.

The *variance* of the variables is obtained by setting x and y equal to 0, and setting k_1 and k_2 equal to the value of k at which the variance is desired. It depends only on the model level.

3. Statistics of an idealized model

The statistics presented in this section are calculated at the model levels. Table 1 lists the different ensembles that will be described in this section. In

Table 1. *Parameters defining the different ensembles described in Section 3; blanks indicate that the value used was the same as that of the basic ensemble A*

Ensemble	L (km)	M	f (s^{-1})	T (K)	δ_p
A (basic)	20,000	48	1.03×10^{-4}	standard atmosphere	ECMWF
B				254	
C					uniform
D			0.52×10^{-4}		
E			1.55×10^{-4}		
F	10,000				
G	10,000	24			
H		24			

Section 3, an attempt will be made to compare features with those found by Hollingsworth and Lönnberg; this will require modifying the vertical and the horizontal resolution at which statistics are calculated.

The vertical structure used in the basic ensemble will be patterned after that of the ECMWF model, with 15 layers bounded by the pressure surfaces of 1000, 980, 940, 880, 800, 720, 630, 540, 450, 370, 300, 230, 160, 100, 50, and 0 mb. Geopotential and winds are defined in the middle of these layers, while temperatures are defined at the pressures centered between the geopotential levels. When the basic state temperatures are obtained from the US Standard Atmosphere, the equivalent depths of the modes varied from 9521 meters for $n = 1$ (the "external mode") to 0.3 meters for $n = 15$. The resulting eigenvectors A_{kn} (k increases upward) show the usual variation with k and n that is found in such computations, with large values at large k for small n and at small k for large n . (This will turn out to have a profound effect on the horizontal length scales.)

The basic ensemble (A of Table 1) was done with a value of $1.03 \times 10^{-4} s^{-1}$ for the Coriolis parameter. The horizontal resolution for the basic computations was set by the values

L = side of periodic region = 20,000 km,

M = integer for the largest horizontal
wave number = 48.

20,000 km for the dimension of the square region was selected as a compromise between a distance of 10,000 km from pole to equator and a circumference of about 30,000 km for the latitude

circle of 45° . The shortest one-dimensional wave length is then

$$L(\min) = 20,000/48 = 417 \text{ km.}$$

This is consistent with the horizontal space increment of 1.875 degrees of latitude (206 km) that was used in the ECMWF grid-point model. However, it is useful to note at this time that if an assimilation system contains significant smoothing operations on the shorter wave-lengths, either in the forecast steps or in the output procedures, it cannot be considered as providing a first guess on as fine a scale as its forecast grid would imply. A final point to remember is that the horizontal resolution that can be obtained in empirical determinations of first-guess error structure is limited by the sample size and by the distance between the radiosondes that are used for the verification.

The constant "CON" in the covariance formula (2.15) is arbitrary. Its value will affect the variances, but not the correlations. For definiteness it has been set always to a value that will result in $(13 \text{ meter})^2$ for the variance of height (= geopotential/gravity) at 500 mb in each ensemble. This value was found by LH for the ECMWF system over North America in winter.

Table 2 lists the square root of the variances (denoted by σ) for height, the u velocity component, and the temperature in the basic ensemble. The height and velocity values first decrease slightly with elevation and then increase, whereas that for temperature decreases steadily with elevation. The value for height increases by a factor of 2.1 from bottom to top, whereas that for u increases by a factor of only 1.5.

Table 2. Square root of the variances (σ) of height, velocity component u , and temperature, together with the horizontal microscale L_f associated with each of them, from the basic ensemble A ; values for temperature are at levels one half increment lower

Pressure (mb)	Height		Velocity		Temperature	
	σ (m)	L_f (km)	σ (m/s)	L_f (km)	σ (deg)	L_f (km)
25	30.7	159	13.0	80	1.3	140
75	26.4	140	12.8	80	1.9	124
130	22.3	131	11.5	79	2.2	116
195	18.8	123	10.3	78	2.4	106
265	16.4	116	9.5	76	2.5	96
335	15.0	114	8.9	76	2.6	94
410	13.9	114	8.2	75	2.5	93
495	13.0	115	7.6	75	2.6	92
585	12.5	116	7.3	75	2.8	91
675	12.1	117	7.0	75	3.1	89
760	12.0	117	6.9	74	3.4	88
840	12.0	120	6.7	75	3.9	86
910	12.5	119	7.1	75	5.1	84
960	13.4	114	7.9	75	7.1	82
990	14.4	108	8.9	74	**	**

An explanation for this is to be found in the concept of "microscale" that HL have introduced from the theory of turbulence. This is the length L_f computed from a two-dimensional correlation function ρ by the operation

$$L_f^2 = -\rho / (\partial^2 \rho / \partial x^2 + \partial^2 \rho / \partial y^2), \quad (3.1)$$

in which both numerator and denominator are evaluated at the origin. L_f reflects the rate at which ρ initially decreases from its value of 1 at the origin. (Note the discussion later following eq (4.4).) For the present ensemble this computation (for the geopotential field) is given by

$$L_f^2(k) = \frac{\sum_n A_{kn}^2 \sum_{a,b} P^2(a, b, n)}{\sum_n A_{kn}^2 \sum_{a,b} (a^2 + b^2) P^2(a, b, n)}. \quad (3.2)$$

Values of L_f for the three variables are also listed in Table 2. (Omission of wave number $a, b = 0, 0$ decreased L_f for geopotential by only 2 to 4 km and left that for u unchanged.) The geostrophic relation between the height and velocity in these slow mode ensembles is summarized by the following relations among the variances.

$$\text{var}(u) = f^{-2} \text{CON} \sum_n A_{kn}^2 \sum_{a,b} b^2 P^2(a, b, n)$$

$$\text{var}(v) = f^{-2} \text{CON} \sum_n A_{kn}^2 \sum_{a,b} a^2 P^2(a, b, n). \quad (3.3)$$

These enable one to show that L_f^2 for the height correlation is proportional to the ratio of the height variance to the geostrophic velocity variance:

$$L_f^2(\text{height}) = \frac{\text{var}(\phi)}{f^2[\text{var}(v) + \text{var}(u)]} = \frac{g^2 \text{var}(\text{height})}{2f^2 \text{var}(u)} \quad (3.4)$$

(See also eq (15) in Buell, 1954.) The differences in the rate of increase with elevation of the variance of height and velocity is therefore associated with a change to larger values of L_f for height in the upper layers of the model. L_f for velocity, on the other hand, is almost uniform with elevation.

This upward increase of L_f for height can only enter through the behaviour of A_{kn} and the variation of the equivalent depth h_n in the energy factor P contained in (3.2). The numerical working can be exposed by considering a simplified numerical example with only 2 levels and two horizontal wave numbers, by assigning properties to A_{kn} that represent an extreme version of the dependence of eigenvector amplitude dependence on k and n referred to earlier and that seems to be typical of all numerical models. In doing so it is worthwhile to introduce a vertical wave number μ , defined by $\mu^2 = f^2/gh$;

$$\text{mode } n = 1: \quad gh_1 = f^2/\mu_1^2;$$

$$A_{11} = 0 \text{ and } A_{21} = 1.$$

$$\text{mode } n = 2: \quad gh_2 = f^2/\mu_2^2;$$

$$A_{12} = 1 \text{ and } A_{22} = 0.$$

μ_2^2 is greater than μ_1^2 . For the horizontal wave numbers, we take

$$a_1^2 + b_1^2 = \kappa^2,$$

$$a_2^2 + b_2^2 = r\kappa^2 (r \neq 1).$$

The sums in (3.2) for $k = 1$ and for $k = 2$ reduce to

$$k = 1: \quad L_f^2 = \frac{1}{\kappa^2} \left[\frac{(1+r)\kappa^2 + 2\mu_2^2}{2r\kappa^2 + (1+r)\mu_2^2} \right].$$

$$k = 2: \quad L_f^2 = \frac{1}{\kappa^2} \left[\frac{(1+r)\kappa^2 + 2\mu_1^2}{2r\kappa^2 + (1+r)\mu_1^2} \right].$$

The factor in brackets decreases with increasing μ^2 . Since μ_2^2 exceeds μ_1^2 , L_f at level $k = 2$ will exceed L_f at level $k = 1$.

The powerful influence of A_{kn} in setting L_f means that it is important to verify that this behaviour of A_{kn} is not an accidental property of the normal modes used in the basic ensemble A. Therefore two new ensembles were defined (See Table 1.) In ensemble B, the basic state temperatures for the ECMWF layers were set at a uniform value of 254 K instead of the values given by the US Standard Atmosphere. In ensemble C, the ECMWF layer structure was changed to one of 15 layers of equal pressure thickness, with the basic state temperatures from the US Standard Atmosphere. The three different distributions of L_f for height are presented in Table 3. These large changes in the basic state parameters of the model make only modest changes in A_{kn} and, as shown in Table 3, only modest changes in L_f . (The details of the variation with elevation of the variances of height and velocity were also similar.) It therefore seems that this subtle, but potentially significant property of the modal eigenvectors in a numerical model, can only be a function of the density decrease with elevation in the model and/or the difference in the vertical boundary conditions at the ground and top of the model.

A second set of variations about the basic ensemble was made to examine the effect of changing the uniform Coriolis parameter. The possibility here is that this parameter might have a profound effect on the horizontal length scales, similar to the results of Balgovind et al. (1983). Meteorologists are accustomed to think of the non-dimensional number

$$Lf/HN \quad (3.5)$$

(L and H are horizontal and vertical length scales, N is the buoyancy frequency) as having a typical value of one in the atmosphere, and associate this with baroclinic instability and with the adjustment problems first formulated by Prandtl and by Rossby. Our ensemble, however, is not meant to represent real flow patterns, but rather the small differences between the real flow patterns and their representation in an assimilation system. We therefore should not expect (3.5) to have much

Table 3. Variation with pressure of the horizontal microscale L_f for height, for the base ensemble A, the ensemble B with isothermal basic state, and ensemble C with uniform thickness of the layers

Pressure (A, B) (mb)	A (basic) (km)	B ($T = 254$) (km)	Pressure (C) (mb)	C (uniform layers) (km)
25	159	162	33	159
75	140	142	100	138
130	131	134	167	127
195	123	130	233	118
265	116	124	300	114
335	114	120	367	113
410	114	120	433	112
495	115	119	500	112
585	116	118	567	112
675	117	117	633	113
760	117	115	700	115
840	120	117	767	116
910	119	116	833	118
960	114	113	900	119
990	108	105	967	114

significance in this ensemble. Eq. (3.4) nonetheless shows that f will play a rôle in the relation between the height variance, the velocity variance, and the microscale. But it is quite conceivable that a parametric change of f would merely change the ratio of the two variances in (3.4) and yield little or no change in L_f .

Ensembles D, A, and E in Table 1 are identical except that the Coriolis parameter varies by respectively 1, 2, and 3 times $0.52 \times 10^{-6} \text{ s}^{-1}$. The corresponding values for L_f for height at 495 mb are 128, 116, and 112 km. This is a much smaller variation of L_f with f than would be obtained by holding H and N constant in (3.5).

A much stronger effect on the microscale is set by the horizontal resolution defined by the choices of the side of the square domain (L) and the number (M) of waves allowed. Ensembles F, A, G, and H define a sequence (see Table 1) in which the minimum horizontal wavelength ($= L/M$) has the values 208, 417, 417, and 833 km. The four corresponding values of L_f for height at 495 mbs for these ensembles are 64, 115, 121, and 227 km. These bear ratios of 0.31, 0.28, 0.29 and 0.27 to the values of L/M for the ensemble in question. L/M therefore appears to be the dominant factor in

setting the value of the horizontal microscale for geopotential in this type of ensemble:

$$L_r(\text{geopotential}) = 0.3(L/M) \quad (3.6)$$

This numerical relation is not unreasonable. For example, the horizontal dependence of first-guess height error correlations is often modelled by a Gaussian profile of the form $\exp(-d^2/s^2)$, where d is the horizontal distance between two points. The microscale definition (3.1) for such a function gives

$$L_r(\text{Gauss}) = s/2. \quad (3.7)$$

At the National Meteorological Center in Washington, for example, experience based on forecasts with a 30-mode spherical harmonic model has led to the use of the value 700 km for s . From the numerical relations derived above for the present ensemble, we find $L_r = 700/2 = 350$ km, followed by $L/M = L_r/0.3 = 1200$ km. This is surprisingly close to the minimum wave length of 1333 km characterizing a 30-mode spherical harmonic model, although the closeness of this agreement is undoubtedly a coincidence.

An interesting aspect of Table 2 is the fact that the temperature has a horizontal microscale that is definitely shorter than that for height. The smaller scale for u , on the other hand, is not surprising since the correlation for u with itself in the horizontal is proportional to the second horizontal derivative of the correlation of height with itself. But since T is the vertical derivative of height, this reasoning will not explain the reduced L_r for T . This reduced L_r for T does demonstrate that *in this ensemble, certain aspects of the height-height correlation structure cannot be modelled by the common device of a product of two functions:*

$$\begin{aligned} \text{corr}(\phi_1, \phi_2) &= \text{function of } (r_2 - r_1) \\ &\times \text{function of } (P_2 - P_1). \end{aligned} \quad (3.8)$$

The separability of this product results in equal values of L_r for height and temperature and a parallel variation with elevation of L_r for height and geostrophic wind. (See, for example, the formulas in the appendix of the 1979 paper by Bergman.) It is therefore unsatisfactory for ensemble A.

The isomorphism between the relation of T to a vertical derivative of geopotential and of u to a horizontal derivative of geopotential suggests that the auto-correlation *in the vertical* for u should show the same effect that T exhibits *in the*

Table 4. *Distribution with pressure of the correlation in the vertical between geopotential at two levels and between u at two levels*

Reference level:	910 mb		495 mb		195 mb	
	geo.	vel.	geo.	vel.	geo.	vel.
Variable:						
25	0.04	0.00	0.04	0.00	0.05	0.00
75	0.06	0.00	0.06	0.00	0.09	0.00
130	0.08	0.00	0.09	0.00	0.21	0.04
195	0.12	0.00	0.14	0.01	1.00	1.00
265	0.15	0.00	0.22	0.02	0.28	0.07
335	0.18	0.01	0.30	0.05	0.19	0.02
410	0.22	0.01	0.48	0.21	0.16	0.01
495	0.27	0.02	1.00	1.00	0.14	0.00
585	0.32	0.04	0.52	0.24	0.13	0.00
675	0.41	0.09	0.37	0.08	0.13	0.00
760	0.53	0.20	0.32	0.04	0.13	0.00
840	0.73	0.48	0.29	0.02	0.12	0.00
910	1.00	1.00	0.27	0.02	0.12	0.00
960	0.82	0.65	0.24	0.01	0.11	0.00
990	0.74	0.53	0.23	0.01	0.10	0.00

horizontal with respect to the geopotential, namely that the correlation distance in the vertical for u should be smaller than that for geopotential. Table 4 lists the distribution with elevation of the correlation coefficients for geopotential with itself and for u with itself for three reference points at 910, 495, and 195 mb. The sharper character of the u correlation structure is very marked, and again illustrates the shortcoming of the simplified representation (3.8). (In Section 4, it will be shown that the reason for the sharper character of the u auto-correlation is that (2.15), when interpreted so as to define the vertical spectra for u and geopotential, results in relatively more power at high vertical wave numbers for u than it does for geopotential.)

Fig. 1 shows the horizontal correlation of geopotential with itself at the 495-mb level for ensemble A, in the first quadrant of the x - y plane. It is symmetric in azimuth about the x - and y -axes and about the axes $x = y$ and $x = -y$. It does not possess the complete azimuthal symmetry that it is customary to assume in empirically determined distributions. This incomplete azimuthal symmetry is presumably a consequence of the square region in a, b space that is being used here in defining the ensemble. (In Section 4, a circular region in a, b

.21	.20	.18	.17	.16	.16	.16	.15	.14	.14
.19	.19	.19	.18	.17	.17	.16	.16	.15	.14
.19	.19	.20	.20	.19	.18	.17	.16	.16	.15
.24	.24	.22	.20	.20	.19	.18	.17	.16	.16
.29	.27	.24	.22	.21	.20	.19	.18	.17	.16
.27	.27	.26	.24	.23	.21	.20	.19	.17	.16
.28	.29	.29	.28	.24	.22	.20	.20	.18	.17
.49	.45	.36	.29	.26	.24	.22	.20	.19	.18
.83	.70	.45	.29	.27	.27	.24	.19	.19	.20
1.00	.83	.49	.28	.27	.29	.24	.19	.19	.21

Fig. 1. Correlation of geopotential on the model level 495 mb for ensemble A. Values are shown at intervals of 100 km in x (abscissa) and y (ordinate).

space will be used, and the result for geopotential will become very symmetrical.)

Fig. 2 shows the correlation coefficient at 495 mb from the same ensemble for the u velocity component. The correlations decrease more rapidly along the y -axis—transverse to the direction of u —than they do along the x -axis. HL illustrate this as a well-known feature of horizontally non-divergent flow.

.13	.09	.02	-.02	-.01	.01	.00	-.01	-.00	.01
-.06	-.04	-.02	.00	-.00	-.01	-.00	.00	-.00	-.00
-.19	-.14	-.04	.02	.00	-.02	-.01	.01	.00	-.01
.03	.02	-.00	-.01	-.01	.00	.00	-.00	-.00	.00
.24	.16	.02	-.04	-.01	.02	.01	-.01	-.00	.02
-.08	-.07	-.04	-.02	-.00	-.00	-.00	.00	.00	-.00
-.58	-.42	-.12	.05	.01	-.05	-.02	.02	.01	-.03
-.40	-.28	-.05	.06	.02	-.03	-.01	.02	.01	-.02
.48	.38	.13	.00	.01	.05	.03	-.01	.00	.03
1.00	.74	.24	-.04	.01	.10	.05	-.03	-.00	.05

Fig. 2. Correlation of the x -component of velocity on the model level 495 mb for ensemble A. Values are shown at intervals of 100 km in x (abscissa) and y (ordinate).

4. A comparison with the ECMWF first-guess errors

In attempting to explore differences between the covariance properties of this idealized slow mode ensemble and the empirically determined results of HL and LH, it is necessary to pay attention to the numerical methods used. An important aspect is that of resolution. In the vertical, for example, the temperature errors are evaluated in LH by the error in geopotential thickness between the standard pressure levels, where the forecast geopotential was first interpolated from the ECMWF model levels to the standard levels. This difference can be allowed for by changing the eigenvector A_{kn} in (2.15) for model level k to an interpolated eigenvector A_{sn} for standard level s :

$$A_{sn} = C_{1s} A_{k_1n} + C_{2s} A_{k_2n} \quad (4.1)$$

in which the standard pressure s lies between model layers k_1 and k_2 , and C_{1s} and C_{2s} are interpolation factors. (The same procedure will produce velocities at standard level s .) The formula for T_{sn} is then identical in form to (2.17) with k replaced by s .

In the bottom part of the atmosphere, the 500-mb surface is the 4th standard level, whereas in the model used here, it is the 8th level that is close to 500 mb. The coarser vertical resolution of the standard levels will primarily affect the temperature variances. A similar but more complicated effect of resolution can occur in the horizontal for the wind. Here the effect enters in the idealized ensemble through the effect of L_f of the range of horizontal wave numbers allowed in the covariance sum (2.15). It occurs in the HL analysis through the number of terms that they are able to allow in their Bessel function expansion of horizontal covariance functions. This last effect is illustrated clearly in Fig. 1 of HL, in which the horizontal correlation of the total wind difference (forecast minus observed) is plotted as a function of radial distance for various resolutions of the Bessel series. These curves have a value near, but not at, the origin that increases from 0.3 for 6 terms in the expansion to 0.75 for 14 terms in the expansion. (HL take 10 terms.) According to A. Hollingsworth (personal communication) a similar, but much smaller effect occurs for geopotential. Let this correlation value near the origin be denoted by ρ^0 . The partitioning of the variance $(v(\text{fcst}) - v(\text{obs}))^2 = d^2$ into

forecast error variance $(fc)^2$ and observational error variance $(ob)^2$ is then given by

$$(fc)^2 = \rho^0 d^2, \quad (ob)^2 = (1 - \rho^0) d^2, \quad (4.2)$$

if it is assumed that observational errors are uncorrelated with forecast errors and that observational errors at different stations are also uncorrelated. Clearly the wave number resolution of empirically derived correlation functions can be an important matter.

Horizontal resolution of ensemble A is also an important matter in attempting to compare it with the HL and LH results. The spatial resolution of ensemble A has until now been based on the resolution of the ECMWF grid-point model. However the following unknown effects would have to be determined in deciding on theoretical grounds how ensemble A should be modified.

(a) *The ECMWF assimilation system*

- (1) The resolution of the analyses is affected by
 - (a) The data spacing (an average of about 400 kms for the United States radiosondes).
 - (b) The resolution contained in the horizontal correlation model used in the ECMWF optimum interpolation analyses.
- (2) The resolution in the forecast first guess may be less than that of the grid-point separation if low-pass filtering (e.g., horizontal viscosity) is included in the forecast equations.
- (3) Smoothing may enter in interpolating the first guess from grid points to observation locations.

(b) *The calculations in HL and LH*

- (1) Observation spacing will set an upper limit to the resolution of the modelled horizontal correlation functions.
- (2) Interpolation to station locations may again involve smoothing.
- (3) The number of station pairs may be too small, especially at distances less than the average distance between radiosondes.
- (4) Deviations of the station-pair covariances from the assumed conditions of homogeneity and radial symmetry could confuse the estimation.

It does not seem practical to analyze all of the above possibilities to derive a theoretical a priori definition of how ensemble A should be modified to match the results of HL and LH with respect to horizontal resolution. A practical means of resolving this ambiguity was therefore necessary for the present study. The statistics for ensemble A were calculated with the formula (2.15) for geopotential and the corresponding ones for wind and temperature but ignoring those terms for which the horizontal squared wave number $a^2 + b^2$ exceeded a critical value κ_c^2 . This filters out horizontal wave numbers greater than κ_c . The value of the constant CON was in each case adjusted to result in a value of (13 meters)² for the height variance at 495 mb, in conformity with the HL results for 500 mb. A value for κ_c of

$$\kappa_c = 2\pi/1000 \text{ km} \quad (4.3)$$

was found to produce a variance of u at 495 mb of $(2.85 \text{ m/s})^2$, which is one half the value of $(4 \text{ m/s})^2$ reported by HL for the total wind forecast error variance $(u^2 + v^2)$ at 500 mb. Wave number $a, b = 0, 0$ was also omitted to match the usage of HL and LH. Of the $49 \times 49 = 2401$ horizontal wave numbers present in ensemble A, this procedure dropped 2067, located (approximately) in the circular region $m_a^2 + m_b^2 > 21^2$, where $a = 2\pi m_a/L$ and $b = 2\pi m_b/L$. Although this truncation seems like a large number of waves to omit, it is only slightly more severe than omitting all x - and all y -wave lengths shorter than 4 grid increments, and in this context may not seem excessive. It does, however, increase the microscale for geopotential appreciably: the unfiltered ensemble A had a u variance (Table 2) of $(7.6 \text{ m/s})^2$. This procedure, although crude, seems to be a reasonable practical method of imposing on ensemble A the basically unknown filtering details implicit in the ECMWF procedures and in the HL and LH computations. The horizontally filtered ensemble A with statistics computed at the model levels will be referred to as ensemble A'. The same filtered ensemble, but with statistics computed at standard levels will, for convenience, be referred to as ensemble A'' (although it is the same ensemble as A'). The variance of height and wind were, as functions of pressure, reasonably alike in A' and A'', but, as we shall see, the temperature variances differ significantly.

4.1. Variance of geopotential and wind

Fig. 3 shows the rms value of the height variance at standard levels from ensemble A'' and those from LH. (The values at 500 mb are of course equal.) The two curves agree in confining major changes to the levels above 700 mb. A major difference is that LH has a pronounced maximum at the tropopause level that is missing in A''. The latter increases more in the stratosphere than does LH, also. Fig. 4 shows the corresponding curves for the rms of the total wind variance $u^2 + v^2$. The HL curve again shows the tropopause maximum, and, in distinction to the A'' curve, does not have a continued increase into the stratosphere*.

4.2. Horizontal scale of height and wind

Ensemble A'' has almost the same increase with height on Fig. 3 and 4, whereas in the top of the atmosphere, HL and LH have less of an increase for wind than height. We must therefore expect almost uniform values of L_r in ensemble A'' for geopotential, and an increase of L_r with height for HL. This is verified in Fig. 5. (The somewhat smaller L_r for HL than for A'' at 500 mb is due to the fact that the HL wind errors used to set κ_c in (4.3) include the total rotational wind error, not only the geostrophic error.) This behaviour of L_r for the geopotential of A'' is in contrast to that of the unfiltered ensemble A (Table 2). It is presumably due to the smaller range in horizontal wave numbers that can now participate in the selection process that was explored in the paragraph following eq. (3.4).

A more detailed examination of this discrepancy is possible. We begin with the significance of the microscale L_r as defined by (3.1). A correlation coefficient ρ must equal one and have a zero derivative at the origin. The leading terms in a Taylor series expansion of an azimuthally symmetric correlation coefficient must therefore be

$$\rho' = 1 - O(r^2) = 1 - 0.5(r/L_r)^2. \quad (4.4)$$

* It is natural to explain the tropopause maximum in HL and LH as due to the maximum wind and geopotential gradients at that height. Additional possibilities are a maximum advection of large errors from the Pacific with the strong westerlies at the tropopause, and the presence of large truncation errors at this height. (The vertical resolution in the forecast model may be relatively coarse compared to the rapid changes in vertical structure near the jet stream.)

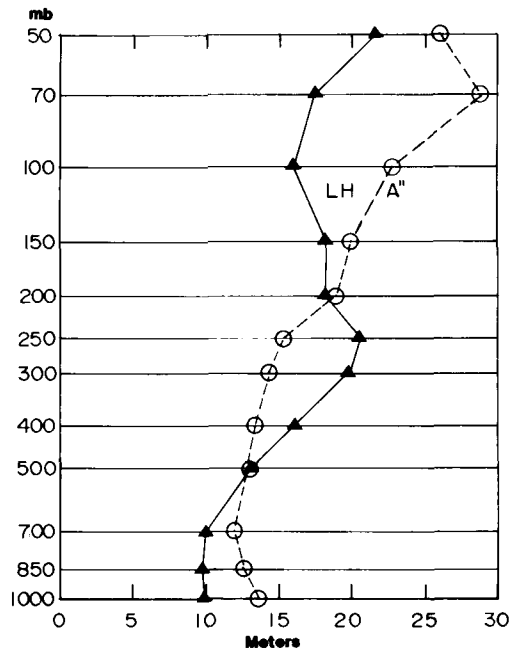


Fig. 3. Distribution of square root of height variance as determined for the ECMWF forecast errors by LH (solid line) and for ensemble A'' (dashed line).

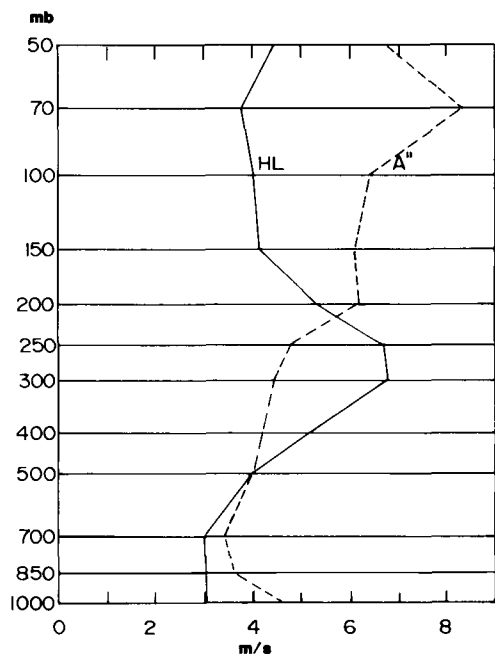


Fig. 4. Distribution of square root of total wind variance ($u^2 + v^2$) as determined for the ECMWF forecast errors by HL (solid line) and for ensemble A'' (dashed line).

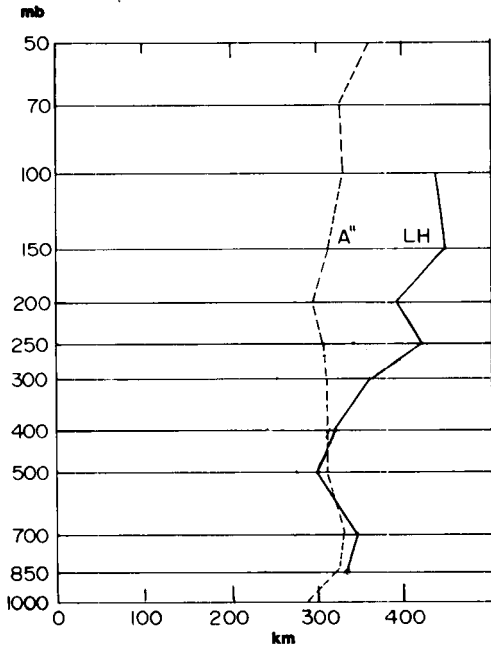


Fig. 5. Variation of the microscale for geopotential as determined by LH for the ECMWF forecast errors (solid line) and for ensemble A'' (dashed line).

The geopotential at 495 mb in ensemble A' has $L_r = 319.47$ km. The corresponding values of ρ' from (4.4) are 1, 0.90, 0.61, 0.12, and -0.57 at r values of 0, 100, 200, 300, and 400 km. These differ appreciably from the complete values of 1, 0.97, 0.90, 0.79, and 0.66 that are shown on Fig. 10. As such they emphasize that the *microscale*, as its name implies, describes the correlation function only at very short separations. (L_r is therefore most sensitive to the highest wave numbers. This may explain the difference between A and A'' with respect to the increase of L_r for geopotential with elevation.)

4.3. Horizontal spectra of the errors

A more complete description is afforded by the horizontal spectrum at different levels. Eq. (2.15), with x and $y = 0$, and $k_1 = k_2 = k$ defines the variance of geopotential at level k . If the horizontal wave number a, b is replaced by its polar components, $a = \kappa \cos \theta$ and $b = \kappa \sin \theta$, this variance may be expressed as

$$\text{var}(\phi) = \sum_{a,b} \sum_n (f^2 \text{CON}) A_{kn}^2 [\kappa^2 + \mu_n^2]^{-1}, \quad (4.5)$$

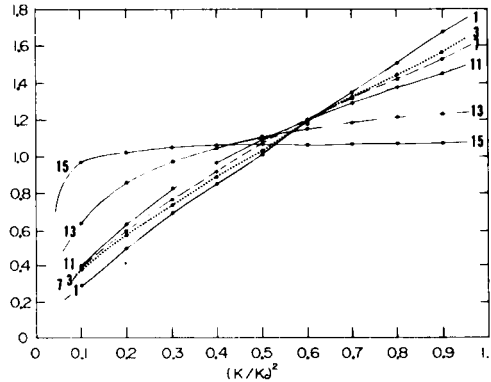


Fig. 6. Horizontal spectra of total wind at a representative sampling of the 15 model levels ($k = 1, 3, 7, 11, 13, 15$). Abscissa is $(\kappa/\kappa_c)^2$, where $\kappa_c = 2\pi(10^6 \text{ m})^{-1}$. Each curve is normalized to have an area of one.

where $\mu_n^2 = f^2/g h_n$. This may be converted into a formula for the spectral density by reinterpreting the sum over a, b as an integral with area element $da db = \kappa d\theta d\kappa$. The result is

$$\text{var}(\phi) = \int d\theta \int I(\phi) \kappa d\kappa \quad (4.6)$$

where the limits are $0 \leq \theta \leq \pi/2$ for θ and $0 \leq \kappa \leq \kappa_c$ for κ . $I(\phi)$ is the radial spectral density:

$$I(\phi) = f^2 \text{CON} \sum_n A_{kn}^2 [\kappa^2 + \mu_n^2]^{-1}. \quad (4.7)$$

This definition makes $I(\phi)$ independent of azimuth, but as shown by Fig. 10, this symmetry is almost achieved in the discrete a, b representation by the restriction $a^2 + b^2 \leq \kappa_c^2$. To get this symmetry in velocity, it is necessary to consider the variance for the total wind, $v^2 = u^2 + v^2$, instead of that for a single component. The formula for $I(v)$ is obtained by adding the variances for u and v :

$$I(v) = \text{CON} \sum_n A_{kn}^2 \kappa^2 [\kappa^2 + \mu_n^2]^{-1}. \quad (4.8)$$

Fig. 6 shows $I(v)$ for a representative sample of levels from ensemble A' as a function of κ^2/κ_c^2 . To emphasize the change in shape of the spectra with elevation, at each level the values of I have been

* Recall that the spectra are, by definition, discrete and finite. Therefore an increase in amplitude up to wave-number κ_c is permissible. These are not spectra of real atmospheric fields!

multiplied by a constant so that the area under each curve from $\kappa^2 = 0$ to κ_c^2 is one. There is a clear tendency for longer wave lengths to receive increased emphasis at higher elevations. HL present discrete spectral estimates for the non-divergent wind error at 9 standard levels (See Fig. 14 in HL). At all levels they find a maximum at *low* wave numbers, in contrast to the high wavenumber maximum on Fig. 6. The smallest non-zero wave number accessible to HL is $(789,000 \text{ m})^{-1}$, about 20% of κ_c . This may be too large to experience the tendency toward zero amplitude at zero wave number that must occur in spectra of geostrophic (or non-divergent) wind, and which is clearly evident in Fig. 6. However, the HL data do show, superimposed on the low wavenumber maximum, a pronounced tendency for the larger wavelengths to be more dominant at higher than at lower elevations. Fig. 6 agrees with this part of the HL results.

The spectrum $I(\phi)$ is more difficult to display because at all levels it is dominated by small wavenumbers. At each level $I(\phi)$ —after the normalization—begins at a large value (~ 10) at $\kappa = 0$, and drops rapidly to a small value (0.2) well before κ_c is reached. The 15 averages of the pair of values at $(\kappa/\kappa_c)^2 = 0.0$ and 0.1 range from 4.3 to 7.0 with no systematic dependence on elevation. These results are consistent with LH, who state: "The change in the character of the spectra between the stratosphere and the troposphere is much more marked for the rotational wind field than it is for the height field."

4.4. Variance of temperature

Fig. 7 compares A'' and LH with respect to temperature variance, where this has been computed from the geopotential differences between standard levels. Here the principal difference is a somewhat larger value for A''. This implies that the ECMWF forecast errors contain less amplitude in the high-order internal modes than does A''.

A different consideration may put the LH and A'' curves on Fig. 7 in a more informative context. If one were given only the LH height variances of Fig. 3, it is possible to set lower and upper bounds to the LH temperature variances by assuming that the height errors at neighboring standard levels are correlated with one another with a correlation of +1 or -1, respectively. The bounds so defined are given by the formulas

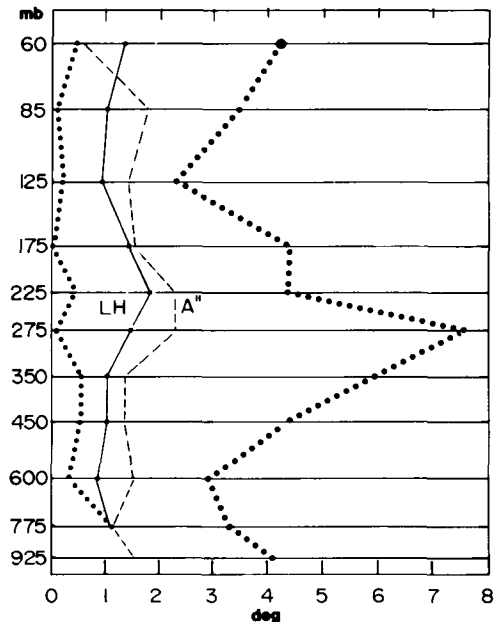


Fig. 7. Distribution of square root of temperature variance as determined by LH for ECMWF forecast errors (solid curve) and for ensemble A'' (dashed line). The dotted curves show the minimum and maximum values consistent with the LH height variances.

$$\text{minimum rms } T = \frac{g|z_s - z_{s+1}|}{R \ln(p_s/p_{s+1})}, \quad (4.9)$$

$$\text{maximum rms } T = \frac{g(z_s + z_{s+1})}{R \ln(p_s/p_{s+1})}, \quad (4.10)$$

where z_s is the rms height variance at level s . These have been plotted as the dotted curves on Fig. 7. They suggest that A'' has come surprisingly close to obtaining the LH temperature variances in spite of its failure to do well on the height variances.

The process of computing temperature variances from the standard-level geopotential values, as discussed at the beginning of this section, can modify the temperature variances from those contained in the model itself. The effect of this on ensemble A is shown in Fig. 8, containing the rms temperature variance for ensembles A' and A''. The greater roughness of the A'' curve is presumably due to the very local nature of the interpolation formula (4.1). More significantly, however, in the bottom of the atmosphere, where the forecast model layers are much closer together than

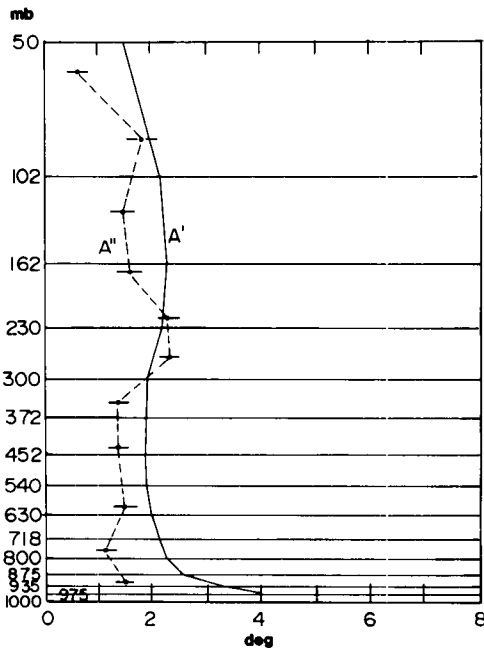


Fig. 8. Square root of temperature variance from ensemble A' (solid line, statistics evaluated at model levels), and from ensemble A'' (dashed line, statistics evaluated at standard pressure surfaces).

are the standard levels, A' has a much larger temperature variance; at the bottom layer, A' has a value of almost 4 degrees compared to about 2 degrees for the standard level results. The implication of Fig. 8 is that under these circumstances, the temperature forecast errors in the layers of a forecast model may be significantly larger than the results computed from standard level geopotentials. This could have important consequences for analysis systems that directly analyse temperature (or geopotential) at the levels of the forecast model.

4.5. Vertical correlation of height and of wind

HL and LH discuss the behaviour in the vertical of the autocorrelation of wind and the autocorrelation of height, and find it convenient to consider that the logarithm of the correlation behaves as a function of $|dZ|$, where dZ is the logarithm of the pressure ratio of two levels: $dZ_{12} = \ln(p_1/p_2)$.

$$\text{correlation} = \exp(\text{function of } |dZ|). \quad (4.11)$$

Their experimental results for height and wind are

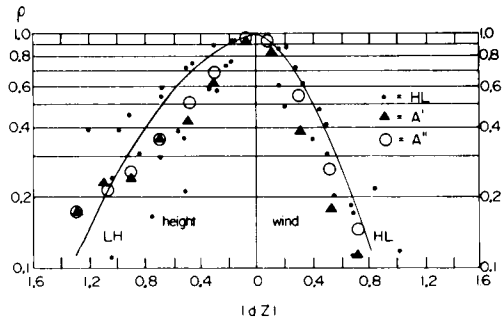


Fig. 9. Autocorrelation of height (left half) and of wind (right half) as a function of $|dZ|$. Small dots and curves are for the ECMWF results. Open circles and black triangles are for ensemble A' and A''.

shown on the left and right halves of Fig. 9, together with their curves of best fit. The striking aspect of their results is the more rapid fall-off of correlation for wind than for geopotential. This was pointed out earlier as a feature of ensemble A (Table 4). For a comparison with results of HL and LH, values of the vertical correlations in ensemble A', and separately in ensemble A'', were organized according to intervals of 0 to 0.2, 0.2 to 0.4, etc., for $|dZ|$ from all pairs of levels. The averages of the correlations so stratified are plotted on Fig. 9. The fall-off appears to be sharper than that obtained by HL and LH for both geopotential and wind. This is consistent with the comparison of temperature covariances discussed earlier. Another feature of interest is the sharper fall-off for ensemble A' than for ensemble A''. This may have implications for multivariate analyses of data at model levels that are closer together than are the standard levels. But the principal conclusion from Fig. 9 is the fact that the theoretical ensembles mirror closely the significant feature of the ECMWF first-guess error statistics that the wind error has a sharper vertical correlation structure than does the geopotential error.

The sharper vertical correlation structure for wind than for geopotential is a direct consequence of the equipartition of energy in this hypothetical ensemble. This equipartition of energy is expressed in (2.15) by the factor

$$P^2 = f^2/(a^2 + b^2 + \mu_n^2),$$

where $\mu_n^2 = f^2/gh_n$ plays the role of a vertical wave number squared. In (2.15), with $x = y = 0$, the sum

Table 5. *Vertical power spectrum as a function of vertical mode number n for geopotential and for a velocity component; values for n greater than one are expressed as fractions of the value for $n = 1$ in each case*

n	Geopotential	Velocity component
1	5.771×10^5	1.639×10^2
2	0.767	0.966
3	0.522	0.873
4	0.366	0.747
5	0.278	0.636
6	0.196	0.498
7	0.125	0.348
8	0.078	0.229
9	0.053	0.161
10	0.040	0.123
11	0.030	0.095
12	0.021	0.066
13	0.013	0.042
14	0.006	0.020
15	0.002	0.005

over a and b is a function only of n , representing the *vertical power spectrum* averaged over all horizontal wave numbers. At small values of $a^2 + b^2$, P^2 behaves as gh_n , and small vertical wave lengths (large n) are suppressed. At large values of $a^2 + b^2$, however, P^2 will vary less with n . Consider now the case of the vertical spectrum for a wind component, say v , where P^2 will be multiplied by a^2/f^2 . This means that in the sum over a and b , greater weight will now be given to the contributions of terms at large values of $a^2 + b^2$, i.e., where small vertical wavelengths are *not* suppressed. The result is a vertical spectrum for wind that has relatively more amplitude in the short vertical wavelengths than does that for geopotential. Table 5 contains a tabulation of this vertical spectrum for geopotential and for u , for ensemble A' (which is the same for ensemble A''). The result of this difference in the vertical spectra is the sharper drop-off of correlation with vertical separation for velocity than for height.

4.6. Horizontal correlation maps

Figs. 10 and 11 show the horizontal correlations for geopotential and for u at 495 mb in ensemble A'. Comparison with the corresponding Figs. 1 and 2 for ensemble A shows clearly the presence of a

.26	.20	.14	.10	.08	.07	.09	.11	.13	.15
.29	.21	.14	.08	.05	.04	.05	.08	.11	.13
.34	.25	.16	.09	.04	.01	.02	.04	.08	.11
.42	.32	.22	.12	.05	.01	.00	.02	.05	.09
.53	.43	.31	.19	.10	.04	.01	.01	.04	.07
.66	.55	.43	.30	.18	.10	.05	.04	.05	.08
.79	.69	.56	.43	.30	.19	.12	.09	.08	.10
.90	.82	.70	.56	.43	.31	.22	.16	.14	.14
.97	.92	.82	.60	.55	.43	.32	.25	.21	.20
1.00	.97	.90	.79	.66	.53	.42	.34	.29	.26

Fig. 10. Autocorrelation of geopotential on the 495-mb model level for ensemble A'. Values are shown at intervals of 100 km in x (abscissa) and y (ordinate).

-.37	-.33	-.25	-.15	-.05	.05	.14	.21	.24	.25
-.53	-.52	-.48	-.40	-.30	-.18	-.07	.03	.11	.15
-.59	-.63	-.63	-.60	-.52	-.42	-.30	-.18	-.08	-.00
-.52	-.62	-.68	-.69	-.66	-.59	-.50	-.38	-.27	-.17
-.32	-.46	-.57	-.65	-.68	-.66	-.60	-.52	-.41	-.31
-.01	-.18	-.34	-.46	-.55	-.59	-.59	-.54	-.47	-.39
.35	.18	.00	-.16	-.30	-.40	-.45	-.46	-.43	-.37
.68	.54	.37	.19	.02	-.12	-.22	-.27	-.29	-.27
.91	.82	.69	.53	.36	.20	.07	-.03	-.09	-.12
1.00	.97	.89	.77	.63	.47	.33	.21	.12	.07

Fig. 11. Autocorrelation of x -velocity component on the 495-mb model level for ensemble A'. Values are shown at intervals of 100 km in x (abscissa) and y (ordinate).

larger horizontal scale. This is especially marked for velocity along the x - and y -axes.

Fig. 12 shows the horizontal correlation for temperature at 540 mb in ensemble A'. This pattern has a horizontal scale that is smaller than that for geopotential on Fig. 10. L_r for temperature (not shown) parallels that of geopotential on Fig. 5, but with a value smaller by 40 km. (The horizontal scale of the ECMWF temperature errors is not discussed in LH.)

-.06	-.14	-.20	-.23	-.23	-.19	-.12	-.05	.02	.07
-.07	-.17	-.26	-.32	-.33	-.30	-.23	-.14	-.05	.02
-.02	-.15	-.27	-.36	-.41	-.39	-.33	-.24	-.14	-.05
.09	-.07	-.23	-.36	-.43	-.45	-.42	-.33	-.23	-.12
.25	.07	-.11	-.28	-.40	-.45	-.45	-.39	-.30	-.19
.44	.27	.06	-.14	-.30	-.40	-.43	-.41	-.33	-.23
.65	.48	.28	.06	-.14	-.28	-.36	-.36	-.32	-.23
.83	.69	.50	.28	.06	-.11	-.23	-.27	-.26	-.20
.96	.86	.69	.48	.27	.07	-.07	-.15	-.17	-.14
1.00	.96	.83	.65	.44	.25	.09	-.02	-.07	-.06

Fig. 12. Autocorrelation of temperature on 540-mb model level for ensemble A'. Values are shown at intervals of 100 km in x (abscissa) and y (ordinate).

5. Conclusions

The covariance statistics of an ensemble of slow modes, with equipartition of energy, have interesting features. Variances of geopotential and wind increase with elevation and are not sensitive to vertical resolution or to reasonable changes in the basic state temperature profile. The horizontal microscale for geopotential is capable of increasing with elevation whereas that for velocity is almost uniform with elevation. Parametric change of the Coriolis parameter, in an ensemble for which it is a constant, has only a small effect on the horizontal scale, for fractional changes (0.5–1.5) of the parameter about typical middle-latitude values.

For comparison with the ECMWF errors, the basic ensemble was modified by setting the overall amplitude to agree with the ECMWF values for height at 500 mb, filtering out all wave lengths shorter than 1000 km, and interpolating geopotential and wind to standard pressure surfaces. The following common characteristics were then found between the theoretical ensemble and the ECMWF errors.

(a) Variances of height and wind are generally larger at high than at low elevations.

(b) The temperature variances (when computed at standard levels) are somewhat larger than the ECMWF values, indicating less amplitude of the

higher-order internal modes in the ECMWF errors. The size of the temperature variances are about the same, however, when judged against the extreme values possible from the ECMWF height variances.

(c) The spectrum of the wind field shows an increasing dominance of long wave lengths at higher elevations.

(d) The correlations in the vertical for wind are sharper than those for geopotential in both systems.

The major differences between the theoretical ensemble and the ECMWF statistics are the following.

(e) The ECMWF height and wind errors have a pronounced maximum at the tropopause; this maximum is missing in the theoretical ensemble. The decrease of the ECMWF wind errors above 250 mb is associated with a significant increase with elevation of the horizontal microscale for geopotential in the ECMWF errors.

(f) The horizontal spectrum of the ECMWF wind errors has at all levels a maximum at the *longest* wave length accessible to HL (about 5000 km). The theoretical ensemble had a wind maximum at the *shortest* wave lengths.

We have assumed that the North American region treated by HL and LH is large enough to satisfy the data availability required by the hypothesis of this paper. In this case, items e and f above lead to the conclusion that the ECMWF *forecast and analysis methods* produce large errors in wave lengths longer than several thousand kilometers. (The better agreement for temperature variances than for height variances, and the broader vertical correlation scale present in the ECMWF covariances suggest further that these errors are mostly in the low-order vertical modes.)

Alternatively, it is possible that the North American region is not large enough to satisfy the data availability requirement of the hypothesis. The conclusion in this case differs from the former by ascribing the source of the error at long wave lengths to *lack of data in the surrounding regions* rather than to weaknesses in the forecast or analysis methods. This is undoubtedly the more likely of the two alternatives.

The theoretical ensemble also suggests the following additional points that are not included in the HL and LH studies.

(g) First-guess errors for temperature at model levels can be significantly underestimated if these levels are closer together than standard levels and

the errors are evaluated from geopotential thicknesses between standard levels.

(h) The horizontal scale for temperature correlations can be smaller than that for geopotential. In the theoretical ensemble this is a mathematical counterpart to item d above.

The present paper has emphasized the properties of an idealized ensemble that has been defined with a minimum of fitting to observed values of short-term forecast errors. As such, it may be useful as a guide to properties to be expected in empirically determined error statistics from a good data assimilation system. But it may be most useful by directing attention to those aspects of real first-guess error fields that differ from the idealized ensemble; as symptoms of systematic errors in a data assimilation system these presumably merit explanation and, where possible, correction. It must be remembered, however, that the hypothesis only concerns first-guess errors that occur in an extratropical region where data is excellent. There is little reason to expect the properties of the present equal energy ensemble to be reflected, for example, in forecast errors over the Pacific Ocean or the tropics.

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6. Appendix

We consider a square region in Cartesian coordinates x, y , with cyclic periodic boundary conditions at $x = 0, L$ and $y = 0, L$. The vertical coordinate will be pressure, on which we impose a finite-difference structure as follows. The K layers will be separated by the pressures p_k , with k increasing upward, and each layer will have a pressure thickness Δ_k .

$$p_{k+1} = p_k - \Delta_k. \quad (\text{A1})$$

p_1 is at the ground and p_{K+1} is at the top of the atmosphere where $p = 0$. The K values of Δ_k

therefore add up to p_1 . At these pressures p_k we define the vertical pressure velocity ω

$$\omega_k = dp/dt, \quad (\text{A2})$$

with ω equal to zero at $k = K + 1$:

$$\omega_{K+1} = 0. \quad (\text{A3})$$

Horizontal velocities v_k and geopotential ϕ_k are defined at the midpoints of the K layers, between p_k and p_{k+1} . The linearized equations of motion and continuity are written as

$$\partial v_k / \partial t - f \mathbf{k} \times v_k = -\nabla \phi_k, \quad (\text{A4})$$

$$\nabla \cdot v_k = (\omega_{k+1} - \omega_k) / \Delta_k. \quad (\text{A5})$$

At the $K - 1$ interface pressures $p_2, p_3, \dots, p_K$, we write the second law of thermodynamics as

$$\partial(\phi_k - \phi_{k-1}) / \partial t = \omega_k / S_k. \quad (\text{A6})$$

S_k is the reciprocal of a static stability factor:

$$1/S_k = \frac{\Delta_k + \Delta_{k+1}}{2} \frac{R^2 \bar{T}_k}{g} \left(\frac{d\bar{T}}{dz} + \frac{g}{c_p} \right) \frac{1}{p_k^2}. \quad (\text{A7})$$

$\bar{T}_k$ and $(d\bar{T}/dz)_k$ represent the temperature distribution assigned to the basic state at these $K - 1$ pressures. At the level ground we express the linearized boundary condition of zero vertical velocity by an uncentered finite-difference expression:

$$\omega_1 = \bar{p}_1 \partial \phi / \partial t = \bar{p}_1 \partial \phi_1 / \partial t. \quad (\text{A8})$$

These equations have separable solutions of the form

$$v_k = V'(x, y, t) A_k, \quad (\text{A9})$$

$$\phi_k = \Phi'(x, y, t) A_k, \quad (\text{A10})$$

$$\omega_k = \Omega'(x, y, t) B_k. \quad (\text{A11})$$

Insertion of these into (A4)–(A8), with a separation constant of c^2 (equal to gravity times the "equivalent depth"), results in the usual shallow water equations,

$$\partial V' / \partial t - f \mathbf{k} \times V' = -\nabla \Phi', \quad (\text{A12})$$

$$\partial \Phi' / \partial t = -c^2 \nabla \cdot V' = c^2 \Omega', \quad (\text{A13})$$

and the following finite difference relation between A_k and B_k :

$$\text{for } k = 1, K: \quad A_k = (B_k - B_{k+1})/\Delta_k, \quad (\text{A.14})$$

$$\text{for } k = 2, K: \quad B_k = c^2 S_k(A_k - A_{k-1}). \quad (\text{A.15})$$

The boundary condition (A3) at $p = 0$ is

$$B_{K+1} = 0, \quad (\text{A.16})$$

and the boundary condition at p_1 is

$$B_1 = \bar{\rho} c^2 A_1 \quad (\text{A.17})$$

Eq. (A14)–(A17) may be arranged into a symmetric system of homogeneous equations for $k = 1, K$ by defining

$$\delta_k = (\Delta_k)^{1/2}, \quad (\text{A.18})$$

$$Z_k = \delta_k A_k, \quad (\text{A.19})$$

$$\alpha_1 = 0, \quad (\text{A.20})$$

$$\alpha_k (k = 2, K) = S_k / (\delta_k \delta_{k-1}), \quad (\text{A.21})$$

$$\alpha_{K+1} = 0. \quad (\text{A.22})$$

The following coefficients, K in number, are also necessary. For $k = 1$,

$$\gamma_1 = (\bar{\rho}_1 / \Delta_1) + \alpha_2 \delta_2 / \delta_1, \quad (\text{A.23})$$

and, for $k = 2, K$

$$\gamma_k = (\delta_{k-1} \alpha_k + \delta_{k+1} \alpha_{k+1}) / \delta_k. \quad (\text{A.24})$$

The resulting symmetric system is, for $k = 1, K$:

$$\alpha_k Z_{k-1} - \gamma_k Z_k + \alpha_{k+1} Z_{k+1} = -(1/c^2) Z_k. \quad (\text{A.25})$$

This defines K eigenvectors Z_{kn} , each associated with its own value of c^2 . The eigenvectors are orthonormal

$$\sum_k Z_{kn} Z_{km} = \delta_{nm} \quad (\text{A.26})$$

where δ_{nm} is the Kronecker delta.

The vertical structure derived here is simpler than that obtained by a strict linearization of the sigma system equations used in numerical weather prediction. (The latter approach, for example, will

not provide an orthogonal system.) The simplicity is considered permissible in view of the simplifications that will be introduced now in the horizontal description of the modes.

The horizontal equations (A12)–(A13) will possess, in the cyclically periodic region, separable solutions of the form $\exp i(ax + by - vt)$, where a and b independently assume the discrete values

$$a \text{ and } b = m(2\pi/L), \quad m = \text{integer}. \quad (\text{A.27})$$

and the frequency v has the well-known solutions

$$v = 0, \quad (\text{A.28})$$

$$v = \pm [f^2 + c^2(a^2 + b^2)]^{1/2}, \quad (\text{A.29})$$

for geostrophic and gravity modes.

The three-dimensional velocity and geopotential fields of a single mode take the form

$$v_k(x, y, t) = A_k \operatorname{Re} V \exp i(ax + by - vt), \quad (\text{A.30})$$

$$\phi_k(x, y, t) = A_k \operatorname{Re} \Phi \exp i(ax + by - vt), \quad (\text{A.31})$$

subject to the requirement that if U and V denote the x - and y -components of V , they are related to the geopotential amplitude Φ by the following relations:

for the geostrophic modes:

$$(U, V) = i(-b, a)\Phi/f; \quad (\text{A.32})$$

for the gravity modes:

$$(U, V) = (ifb + va, -ifa + vb)\Phi/v^2. \quad (\text{A.33})$$

It remains to choose a convenient normalization for each mode. Our purpose is satisfied best by normalizing the energy in each mode. The linear equations possess the energy integral $\partial E / \partial t = 0$, where

$$E = \frac{1}{2} \iint \left[\sum_{k=1}^K \Delta_k v_k^2 + \sum_{k=2}^K S_k (\phi_k - \phi_{k-1})^2 + \bar{\rho}_1 \phi_1^2 \right] dx dy. \quad (\text{A.34})$$

Restatement of (A25) in terms of A_k and S_k , followed by multiplication with A_k and summation over k produces the equality

$$\sum_{k=2}^K S_k (A_k - A_{k-1})^2 + \bar{\rho}_1 A_1^2 = 1/c^2. \quad (\text{A.35})$$

This, plus (A30)–(A33), enables E for each mode to be related to its $|\Phi|^2$ value.

$$E(\text{geostrophic}) = \frac{L^2}{2} \left[\frac{a^2 + b^2}{f^2} + \frac{1}{c^2} \right] |\Phi|^2, \quad (\text{A36})$$

$$E(\text{gravity}) = \frac{L^2}{2} \left[\frac{1}{c^2} \left(2 - \frac{f^4}{v^4} \right) \right] |\Phi|^2. \quad (\text{A37})$$

For the geostrophic modes of concern to us, we therefore define the normalization of the modes by

specifying the geopotential amplitude factor Φ for mode (a, b, n) to have the following form:

$$\Phi_{\text{geo}} = X \left[\frac{a^2 + b^2}{f^2} + \frac{1}{c_n^2} \right]^{-1/2} \quad (\text{A38})$$

The factor X is complex and has the dimensions of a velocity. It will vary for this mode in different realizations of flow patterns consisting of many modes, and in any one realization X may differ between modes. However, the value of $|X|^2$ for a mode in any single realization will be equal to a constant times the energy of that mode in that realization, and the constant is the same for all modes and all realizations.

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