Tellus (1986), *38A*, 162–169

Deep and shallow quasi-geostrophic flow over mountains

By PETER R. BANNON, *Department of the Geophysical Sciences, The University of Chicago, 5734 S. Ellis Avenue, Chicago, IL 60637, USA*

(Manuscript received February 12; in final form May 21, 1985)

ABSTRACT

The effect of a finite density scale height on steady quasi-geostrophic flow over an isolated mountain is examined using analytic solutions for an isothermal, uniformly stratified, vertically semi-infinite atmosphere on an f-plane. A nondimensional measure of the depth of flow and hence of the importance of the vertical density variations of the basic state is the ratio of the Rossby height of the flow, H_R, to the density scale height H. Here $H_R = fa/N$ where a is the mountain halfwidth and N is the buoyancy frequency. The flow is deep for $\lambda = H_R/H \geqslant O(1)$ and shallow for $\lambda \simeq 0$.

It is found that the mountain-induced surface anticyclone is stronger for a deep flow than for a shallow one. Unlike the shallow case, no region of cyclonic vorticity is generated. The enhanced response for deep flow decreases the critical mountain height necessary for the formation of a stagnation point (e.g., Taylor cone). This critical height decreases with decreasing scale height and with increasing stratification. In addition, the mountain's influence decays less rapidly with height for deep flow.

The far-field response for the deep case exhibits a circulation consistent with the lift force acting on the mountain. Since it is shown that the volume of vertically displaced fluid at any level equals the mountain volume, the transmission of the lift to the fluid arises from the decreasing density field, while volume expansion generates the far-field anticyclonic vorticity.

These results hold for both the standard and the modified quasi-geostrophic equations. In the modified theory, however, the mountain surface is not isentropic.

1. Introduction

Smith (1979) articulated the importance of vertical density variations on quasi-geostrophic flow over mountains. He argued that in an atmosphere with a finite density scale height, H, a far-field circulation is generated that is consistent with the lift force acting on the mountain. In a semi-infinite Boussinesq ($H = \infty$) atmosphere, no such circulation arises. Though the identical lift force is present at the surface in both models, a nonvanishing response only arises for the atmosphere of finite mass per unit surface area.

The purpose of the present study is to extend Smith's f-plane analysis and present complete solutions of the flow and not just the far-field. The present results confirm Smith's for the far-field. They also agree qualitatively with Janowitz's (1977) β-plane, rigid lid model with respect to the strength and vertical decay of the near-field.

Section 2 describes the model physics and presents the mathematical solution. Here the standard form of the quasi-geostrophic equation is employed. Section 3 presents the results for this set while Section 4 examines the consequences of using the modified quasi-geostrophic set of White (1977). Unlike the standard equations, this modified set does not ignore the vertical variation of the basic state potential temperature over a density scale height. It is found that only the potential temperature field is altered by this modification. In particular the mountain no longer corresponds to an isentropic surface. Section 5 applies some of the findings to a recent study of frontal propagation over mountains and concludes with a discussion of the limitations of the theory. An appendix addresses the effects of a rigid lid, a situation relevant to both the ocean and the laboratory.

2. The model

The adiabatic, inviscid, quasi-geostrophic dynamics of a stably stratified, compressible fluid with uniform potential vorticity satisfies, e.g. Pedlosky (1979)

$$\nabla^2 \psi + (f^2/N^2)(\psi_{zz} - \psi_z/H) = 0, \tag{1}$$

where $\psi = p/\rho_s(z) f$ is the streamfunction, p the dynamic contribution to the pressure, and $\rho_s = \rho_0 \exp(-z/H)$ the static density field. Here $f > 0$ is the constant Coriolis parameter, N the constant buoyancy frequency, $\rho_0 = 1$ kg m^{-3} the surface density, and H the constant density scale height (i.e., the atmosphere is isothermal).

Far upstream, a uniform flow of amplitude U is incident on a circularly symmetric mountain of height $h(r)$ given by

$$h = h_0/(1 + r^2/a^2)^{3/2}, \tag{2}$$

where h_0 and a are the mountain height and halfwidth respectively. Cylindrical coordinates (r, θ, z) centered on the mountain are used with $\theta = 0$ denoting the downstream direction. Fig. 1 summarizes the model physics and geometry.

Solution to (1) is sought subject to the constraint that the far-field velocity is the incident flow U and to the kinematic boundary condition at the surface

$$\psi_z = -N^2 h/f \qquad \text{at } z = 0. \tag{3}$$

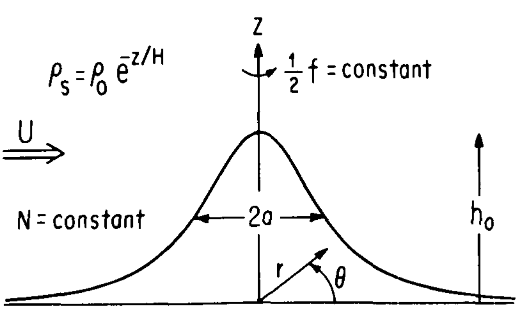

Fig. 1. A schematic illustration of the model. A uniform wind of amplitude U is incident on an isolated, circularly symmetric mountain of height h_0 and halfwidth a centered at the origin of an f-plane. The uniformly stratified model atmosphere is isothermal with constant density scale height H and constant buoyancy frequency N.

By symmetry the streamfunction may be written

$$\psi = -Ur \sin\theta + \phi(r, z), \tag{4}$$

where the first term represents the incident flow and the second represents the mountain contribution.

The method of solution could entail either a double Fourier transform in the horizontal Cartesian coordinates (x, y) or a Green function technique. In the latter, a solution for a point mountain, h_G,

$$h_G = h_0 a^2 \delta(r)/r, \tag{5}$$

is readily found by separation of variables. In (5) δ is the Dirac delta function and the mountain volume

$$MV = 2\pi a^2 h_0, \tag{6}$$

is the same as that for (2). The solution for (2) is obtained by convolution. The result for ϕ is

$$\phi = h_0 aN \int_0^\infty d\kappa\, \gamma^{-1} \kappa \exp(-\kappa) \exp(-\gamma \hat{z}) J_0(\kappa \hat{r}), \tag{7}$$

where $\gamma = \sqrt{\kappa^2 + \lambda^2/4} - \lambda/2$, κ is a nondimensional radial wavenumber, and J_k is the kth order Bessel function of the first kind. The integral in (7) is nondimensional, and hence the dimensional amplitude of ϕ is $h_0 aN$. Here the nondimensional coordinates $(\hat{r}, \hat{z}) = (r/a, z/H_R)$ have been introduced where $H_R = fa/N$ is the Rossby height.

The parameter $\lambda = H_R/H = fa/NH$ is a nondimensional measure of the depth of the flow. λ decreases with increasing scale height H and vanishes for a Boussinesq fluid ($H = \infty$). For typical atmospheric values ($N = 10^{-2}$ s^{-1}, $f = 10^{-4}$ s^{-1}, H = 10 km and $a = 10^3$ km), $\lambda = 1$. For simplicity we define the flow as being shallow for $\lambda \simeq 0$ and deep for $\lambda \geqslant 1$. In contrast, Smith refers to the shallow case as Boussinesq and the deep case as compressible.

The integral in (7) appears analytically tractable only for the shallow ($\lambda = 0$) case which is discussed by Buzzi and Tibaldi (1977). Here the quadrature is performed numerically with a 15-point Laguerre formulation (Abramowitz and Stegun, 1964, p. 923). Note that (7) reduces to the Green function solution for the mountain profile (5) if unity replaces the factor $\exp(-\kappa)$. This result emphasizes the generality of the results presented in Section 3.

3. Results

The general features of the flow have been discussed by several investigators, e.g. Buzzi and Tibaldi (1977). Salient characteristics include: a region of anticyclonic vorticity and high pressure centered over the mountain peak, a strengthening of the flow to the left (looking downstream) of the mountain and a weakening to the right, ascent upstream of the mountain and descent downstream, the absence of a mountain drag, and a reduction of the mountain induced flow with height. As emphasized by Merkine and Kálnay-Rivas (1976), the mountain response is independent of the amplitude U of the incident flow provided it is nonzero. In this discussion we assess the effects of a finite density scale height by comparison of deep ($\lambda = 1$) and shallow ($\lambda = 0$) cases.

3.1. Surface response

Fig. 2 displays the horizontal variation of the mountain contribution to the streamfunction ϕ, azimuthal velocity $v = \phi_r$, and relative vorticity $\zeta = (rv)_r/r$ at the surface. In the shallow case, ϕ decreases monotonically from a maximum of $h_0 aN$ at the mountain peak ($r = 0$) to zero at infinity. For typical atmospheric values, this variation corresponds to a 10 mb pressure change with $h_0 = 1$ km. In the deep case, ϕ has a logarithmic singularity (the integrand in (7) tends to λ/κ as $\kappa \to 0$) and its horizontal variation is infinite. This behaviour is consistent with the far-field circulation discussed below. In Fig. 2 the φ-curve in the deep case is adjusted so $\phi(r = 0) = h_0 aN$ for comparison with the shallow case.

The enhanced pressure differential for the deep case is reflected in a stronger clockwise azimuthal velocity. Despite the logarithmic behavior of ϕ, the v-field is everywhere bounded and no potential vortex exists.

Fig. 2 shows that the maximum anticyclonic vorticity is augmented $\approx 25\%$ by the finite scale height. This behavior increases with increasing λ (see Fig. 3). As noted by Buzzi and Tibaldi (1977),

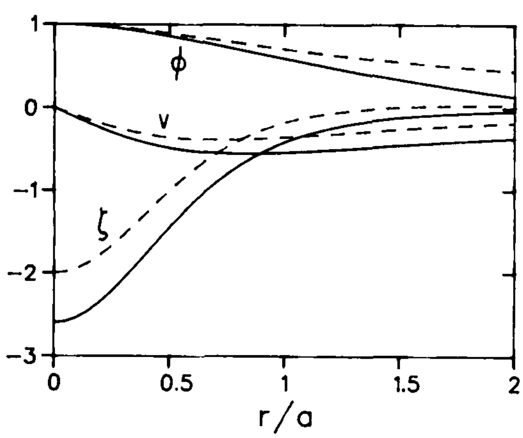

Fig. 2. Mountain streamfunction, φ, azimuthal velocity, v, and relative vorticity, ζ, at the surface as a function of radial distance from the mountain peak. The units of ϕ, v, and ζ are $h_0\,aN$, $h_0\,N$, and $h_0\,N/a$, respectively. The solid curves are for the deep ($\lambda = fa/NH = 1$) case; the dashed curves for the shallow ($\lambda = 0$) case.

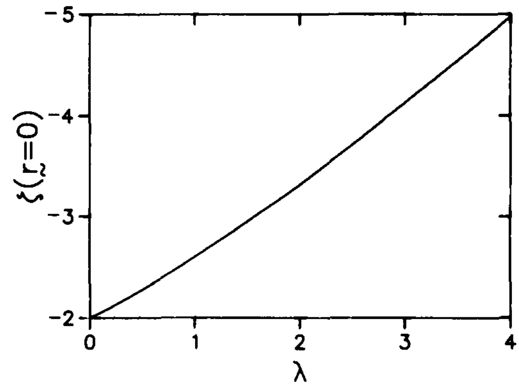

Fig. 3. The relative vorticity ζ (in units of $h_0\,N/a$) at the origin as a function of the parameter $\lambda = fa/NH$.

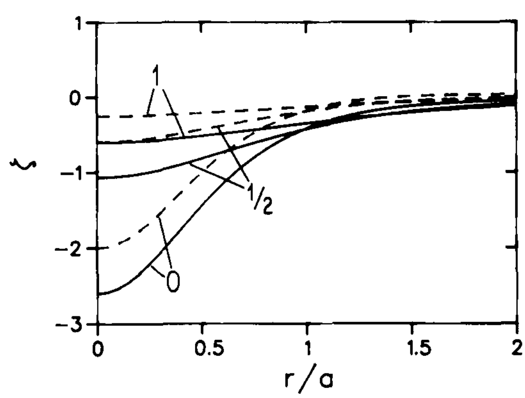

Fig. 4. The relative vorticity ζ (in units of $h_0\,N/a$) as a function of radius at various heights $\hat{z} \equiv z/H_R$ where $H_R = af/N$ is the Rossby height. The solid and dash curves are for $\lambda = 1$ and 0 respectively.

weak cyclonic vorticity exists beyond $r = a\sqrt{2}$ for the shallow case. This feature is absent for deep flow, as it is also for barotropic flow (Bannon, 1980).

3.2. *Vertical variation*

The factor $\exp(-\gamma\hat{z})$ in the integrand of (7) governs the vertical variation of the response. Here γ is a nondimensional measure of the decay rate with height. Noting that $\partial\gamma/\partial\kappa = \kappa/\sqrt{\kappa^2 + \lambda^2/4}$ and $\partial \ln \gamma/\partial\lambda = -1/(2\sqrt{\kappa^2 + \lambda^2/4})$, we can conclude that the small scale features (large wavenumber κ) decay more rapidly with height and that the decay rate decreases with increasing λ. These features are evident in Fig. 4.

3.3. *Vertical displacement*

The vertical displacement $\eta = -(f/N^2)\,\phi_z$ of fluid parcels is everywhere positive and is a measure of the strength of the potential temperature anomaly, $-\eta\partial\theta_s/\partial_z$. Here

$$\eta(r, z) = h_0 \int_0^\infty \mathrm{d}\kappa\, \kappa \exp(-\kappa) \exp(-\gamma\hat{z})\, J_0(\kappa\hat{r}). \tag{8}$$

Fig. 5 reveals that η corresponds to the mountain height $h(r)$ at the surface but decays in amplitude and broadens horizontally with height. Consistent with the preceding discussion, the decay is more rapid for the shallow case.

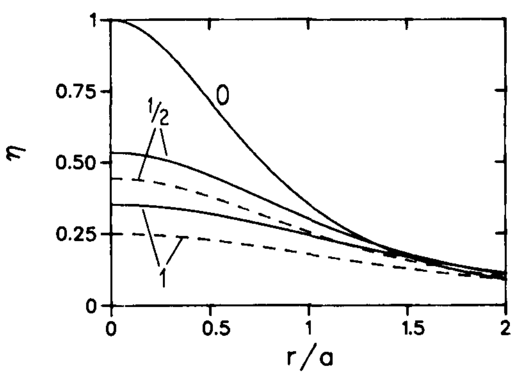

Fig. 5. Vertical displacement η (in units of h_0) as a function of radius at various heights $\hat{z}$. The solid and dash curves are for $\lambda = 1$ and 0 respectively. At $\hat{z} = 0$, all curves correspond to the mountain profile $h(r)$.

The volume V of displaced fluid at any level z is

$$V(z) \equiv \int_0^\infty \eta\, 2\pi r \mathrm{d}r. \tag{9}$$

Substituting (8) into (9) and using the fact that

$$\int_0^\infty \mathrm{d}\hat{r}\, \hat{r}\, J_0(\kappa\hat{r}) = \delta(\kappa)/\kappa,$$

one finds that $V(z)$ equals the mountain volume MV. This result, assumed by Smith, is relevant to the discussion of the far-field circulation and lift (see Subsection 3.5).

3.4. *Mechanisms of vorticity generation*

Changes in the relative vorticity of a fluid parcel arise due to vortex stretching, $f\partial w/\partial z$, and volume expansion, $-fw/H$, where

$$f\frac{\partial w}{\partial z} = \left(\frac{h_0\, UN \cos\theta}{a^2}\right)\int_0^\infty \mathrm{d}\kappa\, \gamma\kappa^2 \exp(-\kappa) \times \exp(-\gamma\hat{z}) J_1(\kappa\hat{r}), \tag{10}$$

$$\frac{-fw}{H} = \left(\frac{h_0\, UN \cos\theta}{a^2}\right)\lambda \int_0^\infty \mathrm{d}\kappa\, \kappa^2 \exp(-\kappa) \times \exp(-\gamma\hat{z})\, J_1(\kappa\hat{r}). \tag{11}$$

These fields are antisymmetric about the line $|\theta| = \frac{1}{2}\pi$ where they vanish. Fig. 6 plots the fields along the downstream radius ($\theta = 0$).

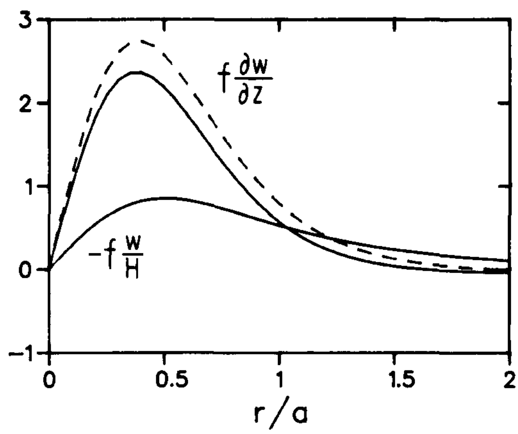

Fig. 6. Vortex stretching, $f\partial w/\partial z$, and the volume expansion, $-fw/H$, at the surface as a function of downstream radius. The units are $h_0\, UN/a^2$ and $\lambda h_0\, UN/a^2$ respectively. The fields are modulated by the factor $\cos\theta$. The solid and dash curves are for $\lambda = 1$ and 0 respectively.

In the shallow ($\lambda = 0$) case, no volume expansion is present and the near-field vortex stretching is negative (positive) on the upstream (downstream) side of the mountain. For $\hat{r} > 2(1 + \hat{z})$, there is weak vortex stretching on the upstream side of the mountain associated with the greater displacement of fluid parcels aloft than at the surface (Buzzi and Tibaldi, 1977). This reversal of the vortex stretching occurs closer to the mountain for the deep ($\lambda = 1$) case. Though the finite scale height reduces the magnitude of the vortex stretching, it augments the net vorticity generation by allowing for volume expansion. Close to the mountain peak, vortex stretching dominates while volume expansion dominates the far field. Thus the far-field response arises from the action of volume expansion.

3.5. *Far-field circulation and lift*

The leading term in an asymptotic expansion of the azimuthal velocity

$$v = -h_0 N \int_0^\infty \mathrm{d}\kappa\, \gamma^{-1} \kappa^2 \exp(-\kappa) \exp(-\gamma \hat{z}) J_1(\kappa \hat{r}), \tag{12}$$

is $v \approx -\lambda h_0 N/\hat{r}$, as can be shown by repeated integration by parts (e.g., Mathews and Walker (1970) p. 80). Clearly the far-field v vanishes for the shallow ($\lambda = 0$) case. This clockwise motion has a negative circulation

$$\Gamma = -f\mathrm{MV}/H, \tag{13}$$

in agreement with Smith and consistent with the lift force $L = \rho_0 U f \mathrm{MV}$ acting on the mountain at the surface.

Smith's analysis of the net impulse and lift force reaction hold here as well. But having shown here that the volume of displaced fluid at any level equals the mountain volume, we can conclude that the transmission of the lift force arises from the decreasing density field while volume expansion dominates the far-field (Fig. 6).

For a Boussinesq fluid ($H \to \infty$) the circulation tends to zero as the finite lift L generates a vanishing response by the atmosphere of infinite mass per unit horizontal area.

3.6. *Stagnation point and Taylor cone*

A stagnation point arises in the fluid where the incident flow U is balanced by the upstream azimuthal velocity v. Such a point would first occur at the surface where the azimuthal velocity is a maximum ($r = r_c$, $z = 0$), and antiparallel to $U(\theta = -\frac{1}{2}\pi)$. For a given flow situation, the critical height h_c for the formation of such a stagnation point is

$$h_c = H_R\, \varepsilon \Big/ \int_0^\infty \mathrm{d}\kappa\, \gamma^{-1} \kappa^2 \exp(-\kappa) J_1(\kappa r_c/a), \tag{14}$$

where $\varepsilon = U/fa$ is the Rossby number and H_R the Rossby height. Inspection of Fig. 7 indicates that h_c decreases and the stagnation point moves outward with increasing λ. These findings are consistent with the effect of a finite scale height (see Fig. 2) on the amplitude of the maximum azimuthal speed and its radial location. They indicate that the Boussinesq analysis of Huppert (1975) overestimates h_c and underestimates r_c. Huppert's result that stratification reduces h_c is qualitatively confirmed here in Fig. 7 if one notes that the scaling of h_c by H_R involves the buoyancy frequency N.

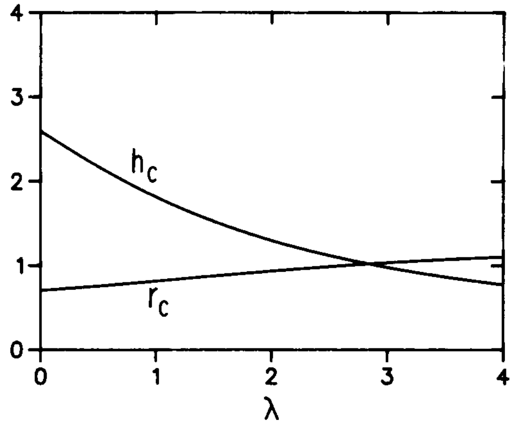

Fig. 7. Minimum mountain height, h_c, for the formation of a stagnation point and its radial distance, r_c (to the right of the mountain peak looking downstream) as a function of parameter λ. The units are εH_R and a respectively, where $\varepsilon = U/fa$ is the Rossby number.

4. Results for modified quasi-geostrophic theory

The preceding results are for the standard quasi-geostrophic theory. It is of some interest to ascertain the effects of using the modified theory.

White (1977) notes that the quasi-geostrophic potential vorticity equation is unaltered in this case. Hence (1) remains valid. The major difference is that the potential temperature deviation from the static case is

$$\theta/\theta_s = (f/g)\,[\psi_z - N^2\,\psi/g], \tag{15}$$

where neglect of the ψ-term is not made. Here θ and θ_s are the deviation and static potential temperature fields, respectively, and usage will clearly distinguish the former from the azimuthal angle. The steady form of the heat equation may be written as

$$J(\psi, \theta/\theta_s) + N^2 w = 0, \tag{16}$$

where J denotes the two-dimensional horizontal Jacobian. Substitution of (15) and the kinematic lower boundary condition

$$w = J(\psi, h) \qquad \text{at } z = 0 \tag{17}$$

into (16) yields

$$J[\psi, (f/g)\,(\psi_z - N^2\,\psi/g) + h] = 0 \qquad \text{at } z = 0. \tag{18}$$

Since $J(\psi, \psi) = 0$ identically, the ψ-term makes no contribution and the boundary condition (3) remains valid. Thus the mathematical problem for the streamfunction is the same for both the standard and the modified problem and the solution is again (7). Further it can be readily shown that the azimuthal velocity, relative vorticity, and vertical displacement are the same.

A convenient form of the vorticity equation for modified quasi-geostrophic flow is obtained from the steady form of White's equation (23)

$$J(\psi, \nabla^2\psi + f - f\theta/\theta_s) + f\nabla\cdot v_a = 0 \tag{19}$$

by eliminating the ageostrophic divergence $\nabla\cdot v_a$, using the steady continuity equation

$$J(\psi, \rho/\rho_s) + \nabla\cdot v_s + (1/\rho_s)\,\partial(\rho_s\,w)/\partial w = 0. \tag{20}$$

The result

$$J(\psi, \nabla^2\psi + f) = (f_0/\rho_s)\,\partial(\rho_s\,w)/\partial z \tag{21}$$

is identical to that for standard quasi-geostrophic theory. Thus all the results of the preceding section hold for the modified theory as well.

It should be noted that, by (15), the potential temperature field differs in the two cases. One finds

$$\theta/\theta_s = -(N^2/g)\,\eta \qquad \text{(standard)}, \tag{22}$$

$$\theta/\theta_s = -(N^2/g)\,[\eta + f/g(-Uy + \phi)] \qquad \text{(modified)}. \tag{23}$$

Interestingly the incident zonal wind in the modified theory is associated with a cross-stream meridional potential temperature gradient even in the absence of a vertical shear. This reflects the modified thermal wind relation. A second difference with standard theory is the intensification of the strength of the cold anomaly associated with the ϕ-term in (23). This feature arises from cold advection by the anticyclone of the meridional gradient. Thus for the modified theory the mountain surface is not isentropic.

5. Conclusion

The present study reveals the qualitative and quantitative effects of a finite density scale height on quasi-geostrophic flow over mountains in a vertically semi-infinite atmosphere. One possible application of these results is the problem of frontal interaction with orography. The far-field anticyclonic relative vorticity for deep flow implies an ageostrophic flow that is divergent rather than convergent for shallow flow. Thus the frontal strengthening predicted in Bannon (1984) upstream and downstream of the mountain may be a consequence of the Boussinesq assumption. Similarly the predicted frontal weakening near the mountain peak may be underestimated in that study. Verification of these ideas requires extension of the present analysis into the semi-geostrophic flow regime.

An important aspect of the solution for the deep case is that the mountain induced pressure field decreases logarithmically with radial distance from the mountain top. This behavior is required for the maintenance of the far-field circulation. However, quasi-geostrophic theory assumes that the dynamic variation of the thermodynamic fields are small compared to the static fields. Sufficiently far from the mountain, the dynamic pressure

$$p \to -\rho_s f h_0\, a N\lambda \ln \hat{r} \qquad \text{as } \hat{r} \to \infty$$

will equal in magnitude the static pressure

$$p_s = \rho_s\, gH.$$

This occurs at r:

$$\hat{r} = \hat{r}_F \equiv \exp\,(g/\lambda^2 h_0 N^2)$$

or $\hat{r} \simeq \exp(100)$ for the mountain height of 1 km and typical atmospheric values. Thus the logarithmic behavior of the pressure field is a weak one. Nonetheless description of the ultra-far field ($\hat{r} \geqslant \hat{r}_F$) would require application of type-2 quasi-geostrophic theory.

6. Acknowledgments

Critical comments by Professor Arnt Eliassen led to improvements in the manuscript. Financial support in part was provided jointly by the National Science Foundation and the National Oceanic and Atmospheric Administration under Grant ATM84-02249.

7. Appendix. Effects of a rigid lid

Inclusion of a rigid horizontal lid at $z = D$ changes the upper boundary condition to the kinematic one:

$$\psi_z = 0 \qquad \text{at } z = D. \tag{A1}$$

The solution in this case is

$$\phi(r, z) = h_0\, aN \int_0^\infty \mathrm{d}\kappa\, \kappa^{-1} \exp(-\kappa)\, J_0(\kappa\hat{r})\, F(\hat{z}), \tag{A2}$$

where

$F(\hat{z}) = \exp(\tfrac{1}{2}\lambda\hat{z})\, [a \cosh u - (\tfrac{1}{2}\lambda) \sinh u]/\sinh a\delta$,

and $a^2 = \kappa^2 + \tfrac{1}{4}\lambda^2$, $u = a(\hat{z} - \delta)$, and $\delta = D/H_R$. This result reduces to (7) in the limit $\delta \to \infty$.

Far from the mountain, the azimuthal velocity is

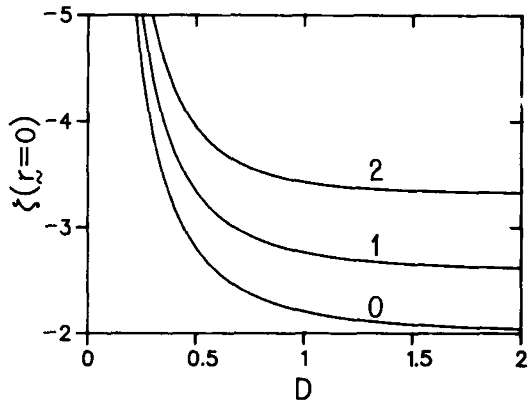

Fig. 8. The relative vorticity at the origin as a function of the fluid depth D for various values of the parameter λ. The units of ζ and D are $h_0 N/a$ and H_R respectively.

$$v \approx \frac{-\lambda h_0 N}{\hat{r}} \frac{1}{[1 - \exp(-\lambda\delta)]}, \tag{A3}$$

and is a factor $[1 - \exp(-\lambda\delta)]^{-1}$ greater than the semi-infinite result. As noted by Smith, the shallow case has $v \approx -h_0 N/\delta\hat{r}$ and the far-field circulation agrees with the semi-infinite deep case if $D = H$. A far-field consistent with the lift force L is obtained by adding to (A3) the homogeneous solution to (1). The appropriate cyclonic potential vortex,

$$\phi_H = \frac{\exp(-\lambda\delta)}{[1 - \exp(-\lambda\delta)]} \ln r, \tag{A4}$$

alters only the azimuthal velocity field but not the w, η, and ζ fields associated with (A3). As an example, Fig. 8 illustrates the amplification of the mountain anticyclone for both shallow and deep flow in the presence of a rigid lid.

REFERENCES

Abramowitz, M. and Stegun, I. A. (eds.) 1964. *Handbook of mathematical functions*, National Bureau of Standards, 1046 pp.

Bannon, P. R. 1980. Rotating barotropic flow over finite isolated topography. *J. Fluid Mech. 101*, 281–306.

Bannon, P. R. 1984. A semi-geostrophic model of frontogenesis over topography. *Contr. Atmos. Physics 57*, 393–408.

Buzzi, A. and Tibaldi, S. 1977. Inertial and frictional effects on rotating and stratified flow over topography. *Q. J. R. Meteorol. Soc. 103*, 135–150.

Huppert, H. E. 1975. Some remarks on the initiation of inertial Taylor Columns. *J. Fluid Mech. 67*, 397–412.

Janowitz, G. S. 1977. The effects of compressibility on the stably stratified flow over a shallow topography in the beta plane. *J. Atmos. Sci. 34*, 1707–1714.

Mathews, J. and Walker, R. L. 1970. *Mathematical methods of physics, 2nd edition*. Benjamin, 501 pp.

Merkine, L.-O. and Kálnay-Rivas, E. 1976. Rotating stratified flow over finite isolated topography. *J. Atmos. Sci. 33*, 908–922.

Pedlosky, J. 1979. *Geophysical fluid dynamics*. Springer-Verlag, 624 pp.

Smith, R. B. 1979. Some aspects of the quasi-geostrophic flow over mountains. *J. Atmos. Sci. 36*, 2385–2393.

White, A. A. 1977. Modified quasi-geostrophic equations using geometric height as vertical coordinates. *Q. J. R. Meteorol. Soc. 103*, 383–396.