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A terrain-dependent reference atmosphere determination method for available potential energy calculations

By THOMAS L. KOEHLER, *Department of Meteorology, University of Wisconsin, Madison, Wisconsin 53706, USA*

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ABSTRACT

An iterative technique that determines the reference atmosphere which incorporates the effects of uneven surface topography is presented. This method has been successfully applied in several available potential energy studies. An alternative method due to Taylor is also evaluated. While Taylor presented excellent continuous formulations of the available potential energy that include topography, his method for determining the reference atmosphere distributions failed to provide the accuracy needed to produce reliable available potential energy estimates. Since topography has a significant influence on the general circulation, it is important to employ techniques that incorporate its effects in the determination of available potential energy.

1. Introduction

The uneven topography of the earth's surface is a feature often ignored in the determination of available potential energy and its time rate of change. An excellent formulation of the continuous available potential energy (APE) expressions that incorporate the effects of uneven topography has been presented by Taylor (1979). Lorenz (1955) defined APE as the difference in the total potential energies of two atmospheric states: an original atmospheric state, and a reference atmospheric state derived from the original atmospheric state by an adiabatic redistribution of mass to a horizontal, stably stratified configuration. Therefore, a key factor in applying the Taylor (1979) expressions to real data sets has been the development of methods that incorporate the effects of surface terrain in determining the distribution of atmospheric variables such as pressure, potential temperature and geopotential height in the reference atmosphere.

In a study evaluating the loss of thermal gradient information in satellite temperature retrievals, Koehler (1979) developed a terrain-dependent computational method for determining the reference atmosphere over a limited North American region. This method has since been modified for hemispheric and global applications, and has been employed in studies such as Min and Horn (1982), and Koehler and Min (1984) in evaluating the zonal and eddy APE distributions during the FGGE year. This paper describes the modified computational method, and briefly evaluates a method suggested in an appendix to Taylor (1979) that employs finite difference approximations of his continuous expressions. No new definitions of the reference atmosphere or reformulations of the APE expressions are presented.

2. The reference state determination algorithm

The original definition of the reference state by Lorenz (1955) provides the basis for the algorithm. The redistribution of mass from the original state to the reference state is adiabatic, making potential temperature (θ) the natural choice for the reference determination. Data required by the method are gridded terrain elevations (Z_s), surface pressures (p_s) and potential temperatures (θ_s) at those elevations, and the gridded pressure (p) analyses on each of the isentropic surfaces in the domain.

Most meteorological analyses are not made in isentropic coordinates; thus, vertical interpolations from the original coordinate system must usually be performed. In many cases, interpolations of p and θ to the terrain height (Z_s) are also necessary. The horizontal resolution in the topography should be commensurate with that in the atmospheric data being used. Since this method is computational in nature, discrete forms of the expressions in θ coordinates are presented.

In an adiabatic redistribution, the atmospheric mass between two isentropic surfaces must be conserved. Since the mass above a specified θ level is proportional to the average pressure on that surface (in a hydrostatic atmosphere), mass conservation will occur when,

$$\bar{p}_k = \hat{\bar{p}}_k, \qquad k = 1, \ldots, K, \tag{1}$$

where the overbar denotes an areal average over the domain being considered, ˆ designates the reference atmosphere, and k, the isentropic level extending from the minimum surface θ value in the sample ($k = 1$) to the top level in the domain ($k = K$). Following Lorenz (1955), $\bar{p}$ on a surface intersecting the ground is determined using the observed p values when the surface is above ground, and p_s, the pressure at the ground, when the surface is at or below ground. Over a region with uniform surface topography, $\hat{\bar{p}}_k = \bar{p}_k$ for all k. With uneven surface topography, $\hat{\bar{p}}_k > \bar{p}_k$ for surfaces intersecting the ground in the reference atmosphere, while $\hat{\bar{p}}_k = \bar{p}_k$ for reference surfaces entirely in the free atmosphere.

The fact that the reference atmosphere is horizontal is the next important feature in the reference algorithm. It implies that p and θ are functions only of the elevation above sea level (Z) in the reference atmosphere, irrespective of a point's geographical location. Also, since the reference atmosphere is stably stratified, the potential temperature at the lowest elevation in the domain must be the minimum potential temperature in the sample. Assuming hydrostatic balance, and a linear relationship of θ versus p^κ between two consecutive θ surfaces ($\kappa = R/c_p$), the following hydrostatic expression can be used to relate Z, θ and p in the reference atmosphere.

$$\hat{Z}_{k+1} = \hat{Z}_k + \frac{c_p}{2gp_{00}^k}(\hat{p}_k^\kappa - \hat{p}_{k+1}^\kappa)(\theta_k + \theta_{k+1}), \qquad k = 1, \ldots, K-1 \tag{2}$$

where g is the gravitational acceleration and $p_{00} = 1000$ mb. Furthermore, since the reference atmosphere is horizontal, the reference pressure and potential temperature at a given point on the earth's surface anywhere in the domain can be determined simply by specifying its elevation (Z_s) and using the following expressions.

$$\hat{\theta}_s = \left\{\frac{2gp_{00}^\kappa}{c_p(\hat{p}_k^\kappa - \hat{p}_{k+1}^\kappa)}[\theta_{k+1}(Z_s - \hat{Z}_k) + \theta_k(\hat{Z}_{k+1} - Z_s)] - \theta_k\theta_{k+1}\right\}^{1/2} \tag{3}$$

$$\hat{p}_s = \left[\hat{p}_{k+1}^\kappa + \frac{2gp_{00}^\kappa(\hat{Z}_{k+1} - Z_s)}{c_p(\theta_{k+1} + \hat{\theta}_s)}\right]^{1/\kappa}, \tag{4}$$

where $\hat{Z} \leqslant Z_s \leqslant \hat{Z}_{k+1}$ and the s subscript denotes the surface. Eqs. (3) and (4) are consistent with the linear θ versus p^κ distribution assumed between levels k and $k + 1$ in eq. (2), and are derived from two expressions similar to eq. (2), only one is for the layer from level k to the earth's surface (Z_s), and the other is for the layer from the surface to level $k + 1$.

The reference determination begins by computing $\bar{p}_k$ on each θ level for a given original atmospheric state, proceeding from θ_1, the minimum observed surface θ, to the top of the domain, θ_K. The $\bar{p}_k$ values are then compared to the minimum observed surface pressure to determine θ_{K^*}, the highest isentropic surface intersecting the ground in the reference state. Levels $K^* + 1$ to K have average pressures less than this minimum surface pressure and are assumed to be entirely above ground in the reference atmosphere where $\hat{\bar{p}}_k = \bar{p}_k$. The remaining θ levels probably intersect the ground, and the reference pressures on these levels must be determined in an iterative fashion.

The iteration can be summarized as a multistep process outlines as follows: (1) using the principle of mass conservation, estimate a $\hat{p}_k$ profile for the levels intersecting the earth's surface; (2) compute the $\hat{Z}_k$ profile from these $\hat{p}_k$ estimates; (3) determine $\hat{p}_s$ using eqs. (3) and (4) at each point at the earth's surface; (4) compute $\hat{\bar{p}}_k$ at each level, comparing the results to the $\bar{p}_k$ values; and (5) readjust the $\hat{p}_k$ profile to try to conserve mass, returning then to step (2). The iteration stops when the mass in each θ layer is conserved.

Any realistic p versus θ distribution could be used as an initial estimate of $\hat{p}_k$. Since $\hat{p}_s$ must be determined at every grid point of the domain during each iteration, reducing the number of iterations by using a more accurate initial estimate of $\hat{p}_k$ can produce considerable savings. The most straightforward option is to set $\hat{p}_k$ equal to $\bar{p}_k$. However, $\bar{p}$, can underestimate $\hat{p}$, by as much as 150 mb in complex terrain distributions. A simple and fast iterative approach shown in Fig. 1 employs only the original $\bar{p}_k$ and p_s distribution to provide better initial estimates of $\hat{p}_k$ on the levels intersecting the ground in the reference atmosphere.

Keeping the surface pressures fixed to observed values, $\hat{p}_{K^*}$ is set equal to $\bar{p}_{K^*}$ as shown in Fig. 1b, with some of the mass in the layer between θ_{K^*} and θ_{K^*+1} appearing below ground. This below ground mass is then evenly distributed over the area on the θ_{K^*} surface that was above ground, increasing $\hat{p}_{K^*}$ as shown in Fig. 1c. Continuing this procedure until all the mass in the layer is above ground yields the final estimate of $\hat{p}_{K^*}$ (Fig. 1d). A similar process of estimating $\hat{p}_k$ from $\bar{p}_k$, then redistributing the mass in the layer appearing below ground would be repeated for each subsequent layer. The initial estimate of $\hat{p}_1$ from this method will always equal the maximum surface pressure in the original atmosphere, which will normally be closer to the true value of $\hat{p}_1$ than will $\bar{p}_1$.

The $\hat{Z}_k$ distribution is then computed using eq. (2) from this initial estimate of $\hat{p}_k$ (step 2 of the iterative method described earlier), and $\hat{p}_s$ is determined from the $\hat{Z}_k$ distribution (step 3). The resulting $\bar{\hat{p}}_k$ values will more than likely depart from the observed $\bar{p}_k$ values. Subsequent adjustments to $\hat{p}_k$ are based on these departures.

$$\hat{p}_k^{n+1} = \hat{p}_{k+1}^{n+1} + \frac{(\bar{p}_k - \bar{p}_{k+1})}{(\bar{\hat{p}}_k^n - \bar{\hat{p}}_{k+1}^n)} (\hat{p}_k^n - \hat{p}_{k+1}^n),$$

$$k = K^*, \dots, 1; n = 1, \dots, N, \qquad (5)$$

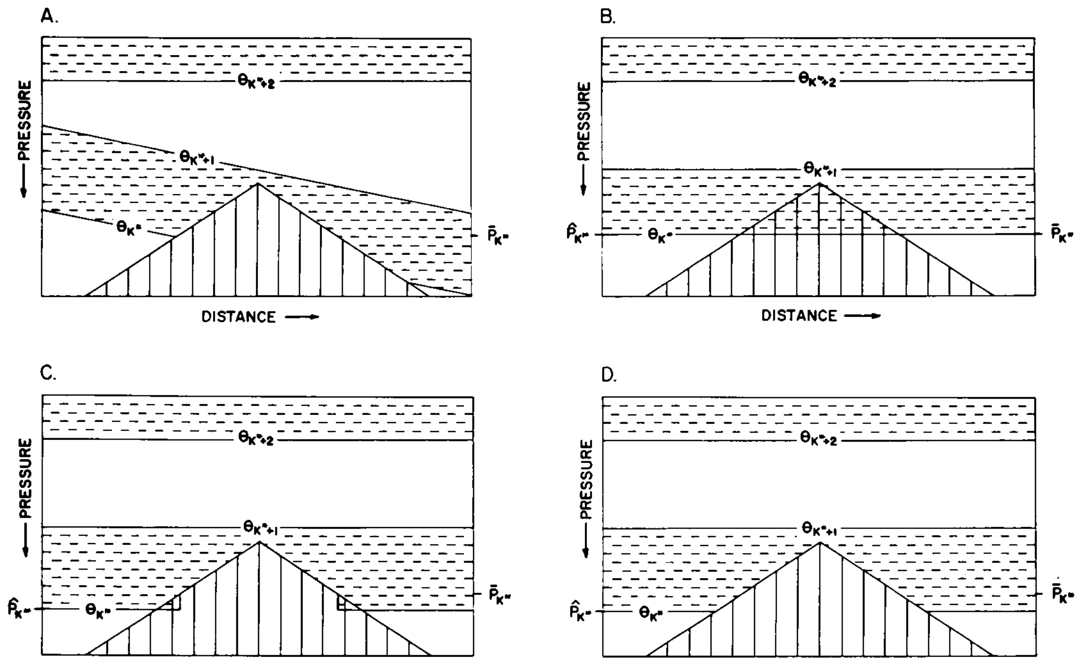

Fig. 1. A schematic of the method for determining the initial reference pressure distribution for the iterative algorithm. The original atmospheric distribution is shown in panel (a), where the vertically hatched triangular region represents the surface pressure distribution on the sloping terrain, and alternate isentropic layers are shaded. Panels (b) and (c) show successive steps in the determination of $\hat{p}_{K^*}$, the initial reference pressure estimate on the uppermost θ level intersecting the ground. Panel (d) illustrates the resulting $\hat{p}_{K^*}$ estimate.

where $\hat{p}^{n+1} = \hat{p}^n = \hat{p} = \bar{p}$ at level $K^* + 1$, the first level entirely above ground. While (5) is only an empirical relationship, it will act to increase (decrease) the reference pressure difference between two consecutive θ surfaces if the mass in that layer was underestimated (overestimated) in the previous iteration (n).

In practice, this technique has been applied to several data sets varying from limited regions to the entire globe, with a 10 K θ increment, and has proven to be quite stable. Convergence is based upon two criteria. First, $\hat{p}_k$ for each level must be within 10^{-3} mb of $\bar{p}_k$ for each θ level. Also, the change of $\hat{p}_s$ from one iteration to the next must be less than 10^{-2} mb. These criteria are normally met in less than 10 iterations. In some limited regions with complex terrain configurations, such as the zonal rings over Antarctica, convergence is much slower. In these cases, a separation of solutions for the odd and even iterations is often present. To handle this problem, an averaging of the consecutive $\hat{p}_k$ solutions is employed after the 20th iteration, which usually leads to convergence before the 40th iteration.

Two examples of the reference pressure distributions produced using the method described above are presented in Fig. 2. Both examples are for the

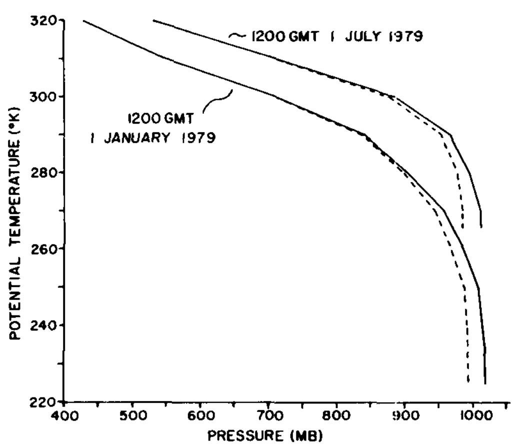

Fig. 2. Two examples of the reference pressures produced using the iterative technique with uneven topography. These results are for the northern hemisphere and were derived from the FGGE Level III-a analyses produced operationally at NMC. Solid lines depict the reference pressures ($\hat{p}_k$), while the dashed lines indicate average pressures ($\bar{p}_k$) on the various θ levels.

northern hemisphere, one winter example at 1200 GMT 1 January 1979, and the other a summer example 1200 GMT 1 July 1979. The original atmospheric distributions were derived from the Level IIIa FGGE data sets produced operationally at the National Meteorological Center (NMC). Note that the reference pressures near the minimum θ values are nearly 30 mb greater than the average pressures, and the lowest layers are very stable, as indicated by large θ changes accompanied by relatively small $\hat{p}$ changes.

3. Evaluation of Taylor's method

An alternative for determining the reference pressure distribution in the presence of uneven topography was presented by Taylor (1979) in an appendix to his APE formulations. Only a brief description of this method (hereafter called Taylor's method) is provided here, with some minor changes in notation. A more detailed description is provided in his article.

One of the biggest drawbacks to the technique described in Section 3 is the repeated computation of the surface reference pressure distribution using eqs. (3) and (4) over the entire domain for each iteration. The Taylor method circumvents this problem by introducing another variable, $\hat{S}_k$, which in this paper is defined as the fraction of the total area that is above ground on level θ_k in the reference atmosphere. For example, $\hat{S}_k = 0.75$ would indicate that 75% of the θ_k surface is above ground in the reference atmosphere. Since the reference atmosphere is horizontal, this variable can be related to the fraction of the area above ground at a given elevation (σ_i), which can be determined from the terrain distribution. The addition of these variables makes the determination of the reference atmosphere a one-dimensional problem in three variables $\hat{p}_k$, $\hat{Z}_k$, and $\hat{S}_k$, instead of a three-dimensional problem in the variables $\hat{p}_k$, $\hat{Z}_k$ and $\hat{p}_s$.

Four equations are used in the Taylor method to relate the three unknowns and two knowns, $\bar{p}$ and σ_i. The first two equations represent the mass conservation in the adiabatic redistribution, relating $\hat{p}$ and $\hat{S}$ to $\bar{p}$.

$$\hat{p}_{k+1} = \hat{p}_k - 2(\bar{p}_k - \bar{p}_{k+1})/(\hat{S}_k + \hat{S}_{k+1}),$$
$$k = 1, \ldots, K^* - 1, \tag{6}$$

$$\hat{p}_1 = \left[2\bar{p}_1 - \sum_{k=2}^{K^*+1} \hat{p}_k (\hat{S}_{k+1} - \hat{S}_{k+1}) \right] \Big/ (\hat{S}_1 + \hat{S}_2). \quad (7)$$

Eq. (2), the hypsometric equation, is used to relate $\hat{Z}$ to $\hat{p}$. The final equation defines the relationship between the unknowns $\hat{Z}$ and $\hat{S}$, and the known distribution of σ_i:

$$\begin{aligned} \hat{S}_k = {} & [(Z_{i+1} - \hat{Z}_k)\sigma_i \\ & + (\hat{Z}_k - Z_i)\sigma_{i+1}]/(Z_{i+1} - Z_i), \\ & k = 1, \ldots, K^*, \end{aligned} \quad (8)$$

where $Z_i \leqslant \hat{Z}_k \leqslant Z_{i+1}$, and the σ_i values are defined at uniformly spaced intervals of elevation (Z_i). These four equations are all finite difference approximations to the original continuous expressions, and thus can only approximate the conservation of mass in the adiabatic redistribution, and hydrostatic balance. Thus, it is important to estimate how errors in finite differencing may affect APE computations from the resulting reference atmosphere.

A straightforward evaluation of Taylor's method is provided using the following idealized terrain and atmospheric configurations. The terrain is a square-shaped plateau at an elevation of 2.5 km, with sloping sides reaching sea level over a distance of 500 km (Fig. 3). The area of the domain where $Z_s > 0$ is 25% of the total area. For simplicity, the original atmosphere is horizontally distributed with θ varying linearly with respect to p^κ. The θ and p values at sea level are equal to 287 K and 1013 mb, respectively, and at 2.5 km, the top of the plateau, θ and p are 296 K and 746.85 mb. Eq. (2) becomes an exact expression for such an atmosphere, and eq. (8) can be replaced by a continuous expression relating $\hat{Z}$ and $\hat{S}$ for the terrain specified. Also, since the original and reference atmospheres are one and the same for this situation, the APE must equal zero. The experiment then becomes an assessment of the accuracy of the mass conservation equations (6) and (7).

An iterative solution of eqs. (6), (7) and (2) was produced using these p_k values in the free atmosphere as a first guess. An exact solution would have produced $\hat{p}_k$ values identical to these guess values. However, as shown in Table 1, the $\hat{p}_1$ values produced by the Taylor iteration consistently overestimated the correct solution for the range of θ increments tested (10 K to 0.5 K). Although not shown, the $\hat{p}$ values for every level intersecting the

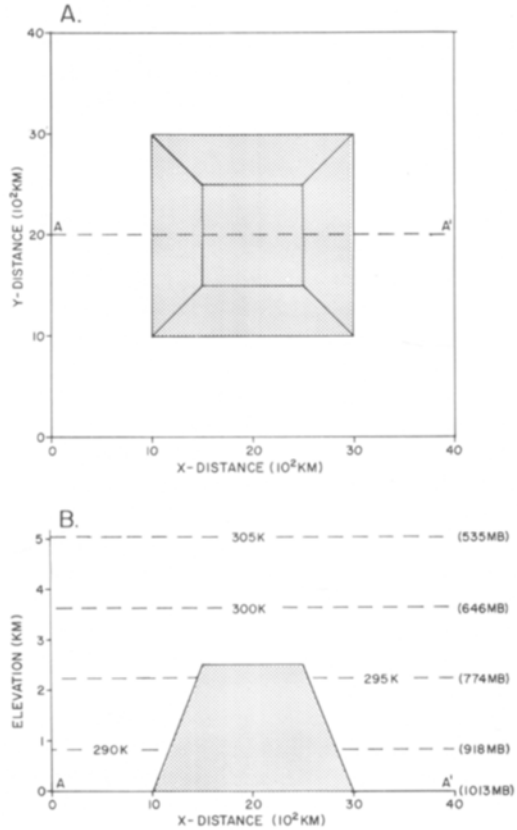

Fig. 3. A depiction of the idealized state used in testing the Taylor method. The upper panel (a) shows a horizontal view, with the region of surface heights above sea level shaded. Panel (b) illustrates a vertical cross section through the center of the terrain feature (along line AA′). The numbers in parentheses to the right are the pressures in the free atmosphere on the isentropes shown.

Table 1. *A comparison of reference state values for Taylor's method solutions at various θ increments for the idealized example*

$\Delta\theta$ (K)	$\hat{p}_1$ (mb)	$\hat{\pi}$ (10^5 J m^{-2})	$\pi - \hat{\pi}$ (10^5 J m^{-2})
10	1017.44	8184.11	−117.53
5	1016.63	8157.04	−90.46
2	1014.78	8110.91	−44.33
1	1013.88	8088.49	−21.91
0.5	1013.44	8077.45	−10.87
exact solution	1013.00	8066.58	0.00

ground were overestimated. The reference total potential energy per unit area ($\hat{\pi}$) at any of the points with a surface elevation at sea level (75% of the total area) were computed using the Taylor method solutions for the 287 K to 300 K layer and the following expression for $\hat{\pi}$.

$$\hat{\pi} = \frac{c_p}{g p_{00}^{\kappa}(\kappa + 1)} \left[\int_{\theta_1}^{\theta_{K'}} \hat{p}^{\kappa+1}\, \mathrm{d}\theta + \theta_1 \hat{p}^{\kappa-1} - \theta_{K'} \hat{p}_K^{\kappa+1} \right] + \hat{p}_1 \hat{Z}_1 - \hat{p}_{K'} \hat{Z}_{K'}, \tag{9}$$

where K' refers to the 300 K level, and the $\hat{p}_1 \hat{Z}_1$ term will vanish because $\hat{Z}_1 = 0$. In approximating the integral in eq. (9), a linear variation of θ versus p^{κ} between consecutive θ levels was assumed. Since θ also varies linearly with p^{κ} in the idealized atmosphere, $\hat{\pi}$ values computed from the exact solution at different θ increments would be identical to the continuous solution for the idealized atmosphere. As shown in Table 1, $\hat{\pi}$ values from the Taylor solutions all exceed the exact solution, yielding negative APE ($\pi - \hat{\pi}$) values. Considering that hemispheric APE values are typically between 20×10^5 and 70×10^5 J m^{-2}, these numerical errors are quite large. Increasing the vertical resolution beyond 0.5 K could decrease this error, but the determination of the $\bar{p}_k$ values for the method could become expensive. Also, many of the global data sets have rather coarse vertical resolution, such as mandatory pressure levels, and the interpolation to θ levels spaced less than 2 K would probably be unjustified.

In summary, this idealized example demonstrates that the Taylor method can produce reference atmosphere distributions, but since the method fails to conserve mass, the reference values produced are not accurate enough for APE calculations. In contrast, the iterative method presented in Section 2 would have recognized the initial guess as the solution for each of the θ increments. Also, the method described for estimating the initial reference pressures for the iterative method would have reproduced the exact solution in this idealized case.

As a final test, Taylor's method was applied to a global data set at 10 K resolution. While the method described in Section 2 was able to produce reference distributions that yielded realistic APE values, Taylor's method produced some highly unrealistic reference pressure distributions and negative APE values.

4. Concluding remarks

A method that incorporates the effects of uneven surface topography into the determination of the reference atmosphere used in APE calculations has been presented. This method has been applied successfully to several data sets over both global and limited domains, yielding physically reasonable solutions. An evaluation of one of the few alternatives, the method suggested by Taylor (1979) has also been presented for an idealized situation. The inaccuracies introduced by the numerical approximations used in applying Taylor's method produced physically unrealistic APE values, both in the idealized case and when applied to a real global data set. Thus, while the method described in this paper requires considerable more computations than Taylor's method, more accurate and meaningful solutions are produced.

Heating and cooling at high elevations provides the predominate forcing for many large-scale atmospheric circulation features. For example, the development of the tropical easterly jet and Asian monsoon during the summer is directly influenced by heating over the Tibetan plateau. Also, southern hemisphere circulation patterns are influenced dramatically by the elevated Antarctic continent. Therefore, with the methods currently available, it seems unreasonable to continue the practice of ignoring surface terrain features in the determination of available potential energy and its time rate of change.

As a final note, the methods presented here are based on a dry adiabatic redistribution of mass to determine the reference state. Lorenz (1978) has presented the concept of moist available potential energy. However, considerable effort is needed before this concept can be transformed into a workable reference atmosphere determination method, with or without terrain.

5. Acknowledgements

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