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Statistical mechanical equilibria of the shallow water equations

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ABSTRACT

Statistical mechanical equilibria of the shallow water equations, valid in the limit of weak flow, are found to imply that quasi-geostrophic flow is unstable in the statistical mechanical limit. In equilibrium, most of the energy ends up in short-scale gravity-inertial waves implying an energy transfer to short scales during relaxation, regardless of the nature of the initial state. The situation is therefore quite unlike that described by the quasi-geostrophic version of the equations in which energy flow to short scales is prohibited. The transfer occurs via the divergent part of the flow which, in turn, appears to draw upon the energy of the large scale rotational modes. Following Sadourny, it is conjectured that this mechanism persists in forced-dissipative flows at large Reynolds numbers, leading to a weak direct energy cascade.

1. Introduction

Equilibrium statistical mechanics can be a powerful tool in the study of nonlinear fluid flows and leads to what appears to be some of the few exact results for systems with a large number of interacting degrees of freedom. These methods were first applied to a continuous two-dimensional fluid by Kraichnan (1967, 1975) to investigate the directions of the various cascades in two-dimensional flows and have since been employed in a number of investigations of geophysical flows—e.g., barotropic motion on a sphere and simple baroclinic flows (Frederiksen and Sawford, 1981; Salmon et al., 1976; see also Salmon, 1982)—to test analytical theories of turbulence as well as to investigate various finite difference formulations of the equations of motion (Bennett and Middleton, 1983). Most of the studies to date have focused on purely rotational flows since quasi-geostrophic arguments suggest that these should be relevant for large-scale geophysical flows.

The purpose of the present paper is to extend the statistical mechanical arguments to include the effects of compressibility. While of interest in itself, this problem may also be relevant to the study of quasi-geostrophic flows in general, and to quasi-geostrophic turbulence in particular, since it offers an opportunity to study the evolution of flows with small but finite Rossby number within the context of more complete equations. In two-dimensional, quasi-geostrophic flows, conservation of energy and enstrophy provide strong constraints which, among other things, severely limit the flow of energy to short scales. In compressible and/or three-dimensional flows, the enstrophy constraint is formally broken, and it would be of interest to determine to what extent the energy flow to shorter scales is modified, since even a weak direct energy cascade could lead to a significant modification of the energy spectrum at small scales. A related question concerns the stability of quasi-geostrophic flow to neglected degrees of freedom, or equivalently, the stability of the "slow manifold" (Leith, 1980). Numerical solutions of the shallow water equations (Sadourny, 1975), and the three-dimensional Boussinesq system (Errico, 1984) indicate that significant excitation of the ageostrophic modes may be possible in the inviscid limit, even when the initial state is close to

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geostrophic. The equipartition spectra for the shallow water equations found here are consistent with this conclusion, although different from those of the Boussinesq system.

In what follows we restrict our attention to the shallow water system and are consequently forced to deal with the fact that none of the constraints are quadratic. Quadratic constraints play a central rôle in the two-dimensional, incompressible case since they appear to be the only isolating constraints which survive the truncation process. As argued in Section 2, this difficulty can be avoided by considering the limit of weak flow for which energy and (potential) enstrophy are again quadratic. These constraints, which are slightly more general than those of the quasi-geostrophic counterpart, are used in Section 3 to obtain the equilibrium spectra. These spectra should be valid for any discretized version of the governing equations which respect the constraints (at least approximately). They differ markedly from the geostrophic case in that significant energy can be transferred to short scales and that typically most of the energy ends up in the divergent component of the flow. Some aspects of the relaxation process are discussed in Section 4. Some possible implications for forced-dissipative flows are discussed in Section 5.

2. Conservation laws

Consider the shallow water equations, with or without rotation, and restrict the flow to be doubly periodic over a square with sides of length D. The governing equations may be written

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \times f\boldsymbol{k} \times \boldsymbol{v} = -\nabla\eta,$$
$$\frac{\mathrm{d}\eta}{\mathrm{d}t} + (1+\eta)\nabla\cdot\boldsymbol{v} = 0, \tag{1}$$

where η, $\boldsymbol{v} = (u, v)$ and f are the dimensionless free surface departure, horizontal velocity and Coriolis parameter respectively, scaled by the mean depth H_0, c $(=(gH_0)^{1/2})$, and c/L. L is a horizontal scale taken to be $L_R = c/f$, the Rossby deformation radius, when there is rotation. With this choice, $f = 1$ with rotation; otherwise it vanishes. Denote the dimensionless domain size by d.

If a horizontal average is denoted by a bar, i.e. if

$$\overline{(\quad)} = \frac{1}{d^2}\int_A (\quad)\,\mathrm{d}s,$$

then the system (1) has the following invariants;

$$M = \bar{\eta}, \tag{2}$$

$$\hat{E} = \overline{\tfrac{1}{2}(1+\eta)\boldsymbol{v}\cdot\boldsymbol{v} + \tfrac{1}{2}(1+\eta)^2}, \tag{3}$$

$$V_F = \overline{(1+\eta)F\left(\frac{\zeta+f}{1+\eta}\right)}, \tag{4}$$

corresponding to the conservation of mass, energy, and potential vorticity respectively. Here $\zeta = \boldsymbol{k}\cdot\nabla \times \boldsymbol{v}$ and F is an arbitrary function. Further if $\mathbf{P}$ is the mean momentum defined as

$$\mathbf{P} = \overline{(1+\eta)\boldsymbol{v}},$$

then

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} + f\boldsymbol{k} \times \boldsymbol{P} = 0. \tag{5}$$

In contrast to the incompressible case, an average velocity $\boldsymbol{u}$ can be generated, even if it vanishes initially, due to the interaction between the rotational and divergent parts of the velocity. This can be seen from

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{k} \times \boldsymbol{u} = -\boldsymbol{k} \times \overline{\nabla^2\psi\nabla\chi}, \tag{6}$$

where χ and ψ are related to the velocity through

$$\boldsymbol{v} = \boldsymbol{u} + \nabla\chi + \boldsymbol{k} \times \nabla\psi.$$

The right-hand side of (6) vanishes for pure rotational flow; otherwise it vanishes only exceptionally.

The invariants (3) and (4) can be compared to those for two-dimensional incompressible flow, i.e.,

$$E_2 = \tfrac{1}{2}\overline{\boldsymbol{v}\cdot\boldsymbol{v}},$$
$$Z_{2F} = \overline{F(\zeta)}.$$

Two of these, the energy and the enstrophy, are quadratic. These play a distinguished rôle in the theory of statistical mechanical equilibria, apparently because they are either the only constraints to survive the truncation process or because they are the only isolating first integrals.

The existence of quadratic constraints (assuming the satisfaction of the Liouville property) is essential if the equipartition arguments are to remain tractable.

In contrast to the purely rotational case, none of the invariants of (1) are quadratic, and it is unlikely that any survive the truncation process (see also Lorenz (1980)). It would seem that equilibrium statistical mechanics is not applicable, at least not in any strict sense. At best, one must be content with approximations. In spite of these difficulties, some progress should be possible in limiting cases where some of the non-quadratic constraints become quadratic. This occurs in the quasi-geostrophic limit, which involves the dual assumptions of small Rossby number and slow timescale, where (1) reduces to

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \tag{7}$$

with

$$q = \nabla^2 \psi - f^2 \psi,$$

and $\eta = f\psi$. Eqs. (3) and (4) become

$$E_g = \tfrac{1}{2}\overline{(\nabla\psi)^2 + \tfrac{1}{2} f^2 \psi^2}, \tag{8}$$

$$Z_g = \tfrac{1}{2}\overline{(\nabla^2 \psi - f^2 \psi)^2}. \tag{9}$$

While eqs. (8) and (9) are exact invariants of (7), they are only approximate under (1) and even then only if the flow remains close to geostrophic. It turns out that quadratic invariants also arise when the motion is weak, i.e., in the limit $\boldsymbol{v}, \eta \to 0$*. This can be regarded as the limit of small Rossby number *without* the additional assumption of slow time scales. This is weaker than the quasi-geostrophic constraint and consequently should include that case. For weak flow, eqs. (3) and (4) become

$$\hat{E} = \tfrac{1}{2} + M + E + \cdots,$$

$$\hat{V}_F = F(f) + \{F(f) - fF'(f)\} M + F''(f) V + \cdots,$$

assuming F is sufficiently regular. The dots denote cubic and higher order terms, while

$$E = \tfrac{1}{2}\overline{(\boldsymbol{v}\cdot\boldsymbol{v}) + \tfrac{1}{2}\eta^2}, \tag{10}$$

$$V = \tfrac{1}{2}\overline{(\zeta - f\eta)^2}. \tag{11}$$

* The importance of this limit was impressed on the author by Dr. R. Sadourny.

These will be referred to as the energy and the potential enstrophy respectively. In view of (2) it is clear that they are approximately conserved provided only that the motion remains weak; it is *not* required that it remain quasi-geostrophic. It is not clear *a priori* whether spectral truncations of the equations respect these "quasi-constraints", although it will be assumed that they do. Since finite difference versions of the equations can be constructed which do conserve total energy and potential enstrophy (Arakawa and Lamb, 1981), eqs. (10) and (11) will necessarily be approximately conserved in the limit of weak flow. E and V cannot "drift" even over long time scales in such circumstances. The considerations below should be valid in this case. Note also that if M is taken to be zero and $\boldsymbol{P}$ vanishes initially, then $\boldsymbol{u}$ vanishes to first order and will not contribute to (10) or (11). It is hereafter assumed that this is the case.

3. Equilibrium spectra

It is important to verify that the Liouville property holds in an appropriately defined phase space. Observe that

$$W = \begin{bmatrix} u \\ v \\ \eta \end{bmatrix},$$

has the representation

$$W = \sum_{k,\alpha} A_k^\alpha(t)\, W_k^\alpha e^{ik\cdot r}, \tag{12}$$

where W_k^α is an eigenvector of the linearized version of (1) and $\boldsymbol{k} = (k_1, k_2)$ is a wavevector consistent with the cyclic boundary conditions. α assumes the values ± 1 or 0 depending on whether the mode is gravity-inertial or rotational (see Appendix A for details). The frequencies of the rotational modes vanish in the linear limit while the gravity-inertial wave frequency is $\pm\omega_k$, where $\omega_k = (f^2 + k^2)^{1/2}$. The coefficients satisfy the reality condition

$$A_k^{\alpha *} = A_{-k}^{-\alpha},$$

and evolve according to

$$\frac{dA_k^\alpha}{dt} + i\omega_k^\alpha A_k^\alpha = i \sum_{\substack{p,q \\ \beta,\gamma}} \mu_{kpq}^{\alpha\beta\gamma}\; A_p^{\beta *} A_q^{\gamma *}. \tag{13}$$

Explicit forms for some of the interaction coefficients are given in Appendix A. As usual, they vanish unless the interaction condition

$$\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0,$$

is satisfied; however unlike the purely rotational case, triads involving parallel wave vectors may interact. In particular, self-interactions are possible. Since $A_{\boldsymbol{0}}^{0}$ is constant, a consequence of mass conservation, it can be taken to be zero. The energy and enstrophy are given by

$$E = \tfrac{1}{2} \sum_{\alpha, \boldsymbol{k}} |A_{\boldsymbol{k}}^{\alpha}|^2, \tag{14}$$

$$V = \tfrac{1}{2} \sum_{\boldsymbol{k}} \omega_k^2 |A_{\boldsymbol{k}}^{0}|^2. \tag{15}$$

Note that the energy involves all the modes, while the enstrophy involves only the rotational part. The exact representation for the energy involves an additional "interaction" energy

$$I = \tfrac{1}{2}\overline{\eta \boldsymbol{v} \cdot \boldsymbol{v}},$$

$$= \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} I_{\boldsymbol{kpq}}^{\alpha\beta\gamma},$$

where

$$I_{\boldsymbol{kpq}}^{\alpha\beta\gamma} = \sigma_{\boldsymbol{kpq}}^{\alpha\beta\gamma} A_{\boldsymbol{k}}^{\alpha} A_{\boldsymbol{p}}^{\beta} A_{\boldsymbol{q}}^{\gamma},$$

which depends on the entire configuration and cannot be written as the sum of energies of individual modes. If the flow is not weak, this contribution can be significant; it then becomes difficult to define the energy spectrum. Similar remarks apply to the enstrophy.

The form of the energy implies that a truncated version of (13) may not be conservative, since it can be seen that the rate of change of the cubic interaction energy does not necessarily vanish when the amplitude of one of the modes is zero. Setting all but a finite set of modes to zero implies zero interaction energy for triads involving a member outside the set, but a nonzero flow of interaction energy into or out of the triad. The resulting inconsistency can lead to a net loss or gain in the total energy of the retained modes.

The presence of self-interactions makes the Liouville theorem less evident than in the incompressible case. If the real and imaginary parts of the A's are used to define a set of generalized phase-space coordinates, then since

$$\frac{\partial \dot{a}_{\boldsymbol{k}}^{\alpha}}{\partial a_{\boldsymbol{k}}^{\alpha}} + \frac{\partial \dot{b}_{\boldsymbol{k}}^{\alpha}}{\partial b_{\boldsymbol{k}}^{\alpha}} = 2\mathrm{Re}\left[\frac{\partial \dot{A}_{\boldsymbol{k}}^{\alpha}}{\partial A_{\boldsymbol{k}}^{\alpha}}\right],$$

$$= -4\mathrm{Im}\left[\sum_{\gamma} \mu_{\boldsymbol{k}-\boldsymbol{k}\boldsymbol{0}}^{\alpha-\alpha\gamma} A_{\boldsymbol{0}}^{\gamma *}\right],$$

$$= 0,$$

where $a_{\boldsymbol{k}}^{\alpha}$ and $b_{\boldsymbol{k}}^{\alpha}$ are the real and imaginary parts of $A_{\boldsymbol{k}}^{\alpha}$, it follows that phase-space volumes are conserved and the Liouville theorem holds. The last step uses (A5) and (A10) of Appendix A. The macrocanonical distribution

$$p = C\mathrm{e}^{-(\alpha E + \beta V)/2},$$

would therefore be invariant if E and V were exact first integrals (Tolman, 1938; Kinchin, 1949; Kraichnan, 1967, 1975). α and β are Lagrange multipliers, determined once the energy and enstrophy levels are specified. C is a normalization constant. It is hereafter assumed that this is correct to leading order. The real and imaginary parts of $A_{\boldsymbol{k}}^{\alpha}$ are consequently independent and gaussian with

$$\langle |A_{\boldsymbol{k}}^{s}|^2 \rangle = \frac{1}{\alpha}, \tag{16}$$

$$\langle |A_{\boldsymbol{k}}^{0}|^2 \rangle = \frac{1}{\alpha + \beta\omega_k^2}, \tag{17}$$

$s = \mp 1$ while the angle brackets denote expected values. The inertial-gravity waves equipartition energy amongst themselves while the rotational modes are described by Kraichnan's energy-enstrophy equipartition spectrum (Kraichnan, 1967, 1975; Fox and Orszag, 1973; Basdevant and Sadourny, 1975). When d, the domain size, approaches infinity,

$$\left(\frac{2\pi}{d}\right)^2 \sum_{\boldsymbol{p}} \rightarrow \int \mathrm{d}\boldsymbol{p},$$

the one-dimensional divergent and rotational spectra become

$$E_{\mathrm{G}} = \frac{2k}{a}, \tag{18}$$

$$E_{\mathrm{R}} = \frac{k}{a + b\omega_k^2}, \tag{19}$$

where a and b are constants. In the quasi-geostrophic limit, the gravity-wave contributions can be neglected. The full energy spectrum is then given by (19) alone. In that case, it is readily shown that if k_T is the truncation wavenumber, then

$$a + bf^2 \sim \frac{k_T^4}{2V} e^{-k_T^2/k_{1R}^2} \quad \text{and} \quad b \sim \frac{k_T^2}{2V}, \tag{20}$$

in the limit $k_T \to \infty$, $k_{1R}^2 = V/E^R$ fixed, where $E^R = \int E_R \, dk$ (Kraichnan, 1967, 1975). The peak in the energy spectrum is located at k_{Rm}, where

$$k_{Rm} \sim k_T e^{-k_T^2/2k_{1R}^2},$$

while the enstrophy has a peak at k_T, reflecting the tendency for energy to be transferred upscale and enstrophy downscale. When the gravity waves are present it follows that

$$a \sim \frac{k_T^2}{E} \quad \text{and} \quad b \sim \frac{k_T^2}{2V}, \tag{21}$$

and the spectral peaks for both energy and enstrophy are located at k_T. The rotational spectrum has a peak at

$$k_m \sim (f^2 + 2k_1^2)^{1/2},$$

where $k_1^2 = V/E$. The additional degrees of freedom can lead to a suppression of the reverse energy cascade, with most of the energy ending up in the inertial-gravity waves at equilibrium. At this stage, the total energy in the rotational modes is

$$E^R \sim \frac{V}{k_T^2} \ln (k_T^2/k_m^2).$$

If this is used in (20), (21) is recovered.

As noted by Kraichnan (1975), the macro-canonical distribution is stable for arbitrary couplings which preserve the energy and (potential) enstrophy. This is *not* the case for couplings involving *new* degrees of freedom due to a leakage of energy into those modes. Eqs. (16) and (17) suggest that an initially quasi-geostrophic flow will ultimately become non-geostrophic due to a transfer of energy into the high frequency inertial-gravity waves, and consequently that quasi-geostrophic flow is unstable in the statistical mechanical limit. Partial support for this can also be found from numerical solutions of the shallow water equations (Sadourny, 1975). The rotational and divergent energy spectra were observed to exhibit enstrophy and energy equipartition, respectively, at large wave numbers (see Fig. 7 of that paper). Disagreement at lower wavenumbers may be due to the fact that full equilibrium has not been achieved. Since the turnover time of the inertial-gravity modes is much greater than their linear period, the relaxation time is expected to be long due to the inhibition of energy flow to those modes by wave processes (Rhines, 1985; see also Section 4). Recent numerical experiments with a three-dimensional, inviscid, rotating, Boussinesq system (Errico, 1984) also imply that significant energy transfer into the divergent component of the flow is possible. In that system, there is apparently only a single quadratic invariant, the energy, and complete equipartition between the rotational and divergent parts was observed even though the initial states were nearly geostrophic. It was argued that dissipation is essential if geostrophic balance is to be maintained.

4. Relaxation to equilibrium and quasi-geostrophic breakdown

The final equilibrium of the inviscid truncated system will be decidedly non-geostrophic after sufficient time has elapsed, regardless of the initial state. If the initial flow is nearly geostrophic, the quasi-geostrophic constraint must ultimately be violated, and it is of interest to examine the nature of the breakdown. It can occur in at least two ways.

(a) The solution is nonuniform in k in the sense that the local Rossby number r_k becomes of order unity for some values of k, where r_k is defined as

$$r_k = \frac{(E_s k)^{1/2} k}{f},$$

and $E_s = E_R + E_G$. The local Rossby number will increase with k whenever the spectral slope exceeds -3, as it does for the quasi-geostrophic equipartition spectrum at large k. As our study of statistical mechanical equilibria is motivated by the insights they can provide into the forced-dissipative problem, a breakdown of this type might be regarded as somewhat artificial within the context of the full continuum equations, since it is an artifact of the truncation rather than a reflection of a fundamental underlying mechanism. It turns out that the increase in Rossby number with decreasing scale is counteracted by a reduction in the energy intensity due to the spreading of energy amongst the available modes. The breakdown does not appear to be due to a failure of the local Rossby number to remain small (see below).

(b) The quasi-geostrophic expansion may not be uniformly valid on some long timescale, even when the local Rossby number is small, due to a transfer of energy into the inertial-gravity waves by neglected higher order terms.

Unfortunately, equilibrium statistical mechanical arguments provide no information concerning the nature of the relaxation process. Arguments given below and the numerical experiments of Errico (1984) suggest that the relaxation occurs in two stages. On time scales of order εt, where ε is a measure of the strength of the flow, resonant interaction theory implies that the rotational modes decouple from the inertial-gravity modes to leading order and relax according to quasi-geostrophic dynamics, even if the overall flow is not quasi-geostrophic. This is because the gravity modes are weakly nonlinear in the present limit with the strongest exchanges occurring between resonantly interacting triads. For initially balanced flow, the overall motion is quasi-geostrophic with the (weak) gravity modes "slaved" to the rotational modes through the spectral equivalent of the ω and balance equations. During this stage, the trajectories are confined to the neighborhood of a surface or manifold embedded in the full phase space. This has been termed the slow manifold (Leith, 1980). Its dimension is equal to the number of rotational degrees of freedom (see below). Trajectories in the vicinity of the manifold are quasi-geostrophic and largely free of high-frequency gravity waves whereas elsewhere in phase space, the solution involves fast oscillations. In forced-dissipative systems, the manifold is thought to be attracting (Lorenz, 1980), reflecting the tendency for initially unbalanced states to become quasi-geostrophic. In conservative systems, attractors are precluded by Liouville's theorem. Finally, on longer time scales, weak coupling is expected to drive the system towards the final state described by (16) and (17); the motion on the slow manifold is consequently expected to be weakly unstable.

4.1. *Quasi-geostrophic stage; statistics on the manifold*

To investigate the first stage of the evolution, (13) is written as the coupled pair

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \varepsilon N_1(R,R) + \varepsilon N_2(R,G), \tag{22}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} + i\Lambda G = \varepsilon M_1(R,R) + \varepsilon M_2(R,G) + \varepsilon M_3(G,G), \tag{23}$$

where εR and εG are vectors whose components comprise the rotational and gravity wave amplitudes, respectively, Λ is a diagonal matrix with elements ω_k^s while N_1 etc., are vectors with components which are quadratic in the indicated variables. It can be seen that an inertial-gravity mode pair cannot interact to modify a rotational mode since this would lead to a violation of the enstrophy constraint (see A8). When ε is small, the method of multiple scales can be employed with timescales t and $\tau = \varepsilon t$ to reduce the system to

$$\frac{\mathrm{d}R}{\mathrm{d}\tau} = N_1(R,R), \tag{24}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\tau} = \tilde{M}_2(R,g), \tag{25}$$

to leading order. Here $G = \mathrm{e}^{i\Lambda t}g$, while $\tilde{M}_2$ includes only the resonant contributions from M_2, i.e., only those terms associated with triads satisfying the interaction condition and

$$\omega_k^\alpha + \omega_p^\beta + \omega_q^\gamma = 0.$$

Eqs. (24) and (25) follow on noting that all triads involving only rotational modes are "resonant" while there are none involving only inertial-gravity modes. The only mixed resonant interactions occur between a single rotational mode and two inertial-gravity modes with the same scale. These are catalytic in the sense that the rotational mode decouples, a consequence of the fact that the relevant interaction co-efficient vanishes identically (see A8 of Appendix A). As (24) is equivalent to (7), the rotational dynamics are quasi-geostrophic, even though the overall flow may not be, and the rotational modes will relax to (19) on a τ time scale, regardless of the initial state of the inertial gravity modes. a and b are determined from the enstrophy and the initial rotational energy.

Resonant interaction theory can be expected to hold in the limit $\varepsilon \to 0$ with N, the number of degrees of freedom, fixed. When N is large, and the resonant interactions sparse, ε may have to be severely restricted to ensure that the cumulative effect of non-resonant triads remain relatively small. Since the resonant interactions in the rotational equation occur over a set of full measure in wavenumber space, (24) may be valid over a wider range of ε. In this sense, quasi-geostrophic motion may be quite robust on time scales of order τ.

If the inertial-gravity modes are sufficiently small initially, then according to the usual arguments, all modes are expected to evolve on the slow time scale εt in which case the motion is quasi-geostrophic. In this limit, (25) is replaced by

$$G = -i\varepsilon \Lambda^{-1} M_1(R,R),$$

to $O(\varepsilon^2)$, or

$$\omega_k^s G_k^s = \varepsilon \sum_{p,q} \mu_{kpq}^{s00} R_p^* R_q^* \tag{26}$$

which is just the leading order approximation to the slow manifold (Leith, 1980). Eq. (26) is simply a reflection of the fact that the divergent part of the motion is determined diagnostically from the rotational part in quasi-geostrophic flow. It implies that the inertial-gravity mode statistics are determined from the statistics of the rotational modes. When the latter relax according to (24), they become independent and gaussian. Since

$$\mu_{kpq}^{s00} = -\frac{2\omega_k^s (\boldsymbol{p}\times\boldsymbol{q})^2 + if(\boldsymbol{p}\times\boldsymbol{q})(p^2-q^2)}{2\sqrt{2}k\omega_k\omega_p\omega_q} \tag{27}$$

from Appendix A, it follows that

$$\langle G_k^s \rangle = 0,$$

$$\langle G_{k_1}^s R_{k_2} \rangle = 0,$$

$$\langle |G_k^s|^2 \rangle = \frac{2\varepsilon^2}{\omega_k^2} \sum_{p,q} |\mu_{kpq}^{s00}|^2 \langle |R_p|^2 \rangle \langle |R_q|^2 \rangle, \tag{28}$$

during the first stage of the relaxation process. The last expression results on using (27) and

$$\langle cdef \rangle = \langle cd \rangle \langle ef \rangle + \langle ce \rangle \langle df \rangle + \langle cf \rangle \langle de \rangle,$$

for c, d, e, and f gaussian. Eq. (28) then implies

$$E_G = \frac{4k}{\pi\omega_k^2} \int_0^{k_T} dp \int_0^{2\pi} d\theta |\mu_{kpq}^{s00}|^2 E_R(p) E_R(q)/q. \tag{29}$$

It is shown in Appendix B that on using (19), this becomes

$$E_G \sim \frac{c_1}{b^2 k} \quad \text{when} \quad f \ll k \ll k_T, \tag{30}$$

and

$$E_G \sim \frac{k^3}{2\pi f^4 b^2} \quad \text{as} \quad k \to 0. \tag{31}$$

c_1 is a constant. Furthermore

$$r_k \sim \frac{\sqrt{2}V^{1/2}}{f} \frac{k}{k_T} \quad \text{for} \quad f \ll k \ll k_T.$$

Although r_k increases linearly over a large range of k, it remains uniformly small for weak flow suggesting that the breakdown is not a result of alternative (a) above but rather due to a slow transfer of energy into the gravity-inertial modes by terms which are neglected in quasi-geostrophic theory. This idea is pursued below.

4.2. *Final stage—a Langevin equation*

The above suggests that the last stage of the relaxation will be slow with the rotational models passing through a succession of quasi-equilibrium states, each described by (19), but with the parameter a determined from (20) and the instantaneous value of E^R. The parameter b remains constant because of enstrophy conservation, so the main changes in the rotational spectrum appear at low wavenumber (see Fig. 1). The gravity-inertial modes also undergo substantial changes. If the flow is initially geostrophic, there must be an evolution from a quasi-equilibrium state on the manifold to a full equipartition spectrum involving most of the initial energy. The relaxation necessitates a con-

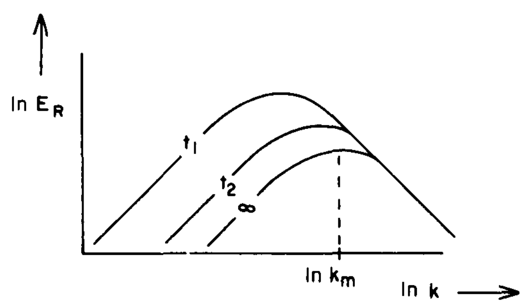

Fig. 1. Long-time relaxation of rotational energy spectrum. Note that the main transfer comes from wavenumbers $k \leqslant k_m$.

tinuous extraction of energy from the rotational modes. While the precise nature of this transfer is unclear and complicated by the fact that the truncated system is not exactly conservative, it may be possible to capture some aspects of the transfer using a simple model system consisting of (23), with the nonlinearities involving gravity-inertial modes excluded, and (24). While the system is admittedly rather crude since it ignores feedback processes on the rotational modes, it may be useful in describing the transfers during the early stages. More importantly, the model provides an example of how the multiple scale arguments underlying the quasi-geostrophic expansion may fail.

Since the rotational modes decouple, the model gravity modes evolve according to a Langevin equation

$$\frac{\mathrm{d}G_k^s}{\mathrm{d}t} + i\sigma_k^s G_k^s = f_k^s, \tag{32}$$

with

$$f_k^s = i \sum \mu_{kpq}^{s00} R_p^* R_q^*,$$

providing a random gaussian forcing once the rotational modes have relaxed. It will be convenient for the discussion that follows to include a small damping term on the gravity inertial modes ($\sigma_k^s = \omega_k^s - i\eta_k^s$, where $\eta_k^s \geqslant 0$). If $G_k^s = B_k^s$ initially, with $\langle B_k^s f_k^s \rangle = 0$ and f_k^s stationary, then

$$\langle G_k^{s^2} \rangle = \langle |B_k^s|^2 \rangle \mathrm{e}^{-2\eta_k^s t} + \frac{2\mathrm{e}^{-\eta_k^s t}}{\eta_k^s} \mathrm{Re}\left[\int_0^t \sinh \eta_k^s (t-\tau) \mathrm{e}^{i\omega_k^s \tau} R_k^s(\tau) \mathrm{d}\tau\right], \tag{33}$$

where

$$R_k^s(\tau) = \langle f_k^s(t+\tau) f_k^{s*}(t) \rangle.$$

It turns out that the limiting behaviour for large ω_k^s, t and small η_k^s is nonuniform. Consider the following special cases.

(i) $\eta_k^s \to 0$, $t \to \infty$, in that order.

When $\eta_k^s \to 0$, (33) reduces to

$$\langle |G_k^s|^2 \rangle \sim 2t \int_0^\infty \mathrm{e}^{i\omega_k^s \tau} R_k^s(\tau) \mathrm{d}\tau,$$

for large t. Evidently there is a continuous transfer of energy into the gravity-inertial modes. The unlimited growth is due to the neglected feedback into the rotational modes.

(ii) $\omega_k^s, t \to \infty$, η_k^s fixed and nonvanishing.

When the wave and nonlinear time scales are separated, an asymptotic representation of (33) in inverse powers of ω_k^s can be obtained by integration by parts. For large t,

$$\langle |G_k^s|^2 \rangle \sim \frac{R_k^s(0)}{\omega_k^2}.$$

which is just the leading order statistic on the slow manifold. On comparing with (i), it is clear that viscosity is necessary if the motion is to remain near the slow manifold. To examine this in more detail, we consider:

(iii) $R_k^s(\tau) = R_k^s(0)\mathrm{e}^{-\alpha\tau}$, $t \to \infty$.

Direct evaluation gives

$$\langle |G_k^s|^2 \rangle \sim \frac{R_k^s(0)(\eta_k^s + \alpha)}{\eta_k^s |\alpha + i\sigma_k^s|^2},$$

for large t. If $\omega_k^s \gg \eta_k^s$, then there are two subcases;

$$\langle |G_k^s|^2 \rangle \sim \frac{R_k^s(0)}{\omega_k^2}, \qquad \text{for} \qquad \alpha \ll \eta_k^s,$$

as in (ii) and

$$\langle |G_k^s|^2 \rangle \sim \frac{\alpha R_k^s(0)}{\eta_k^s \omega_k^2}, \qquad \text{for} \qquad \eta_k^s \ll \alpha.$$

It follows that when the flow is sufficiently viscous, (32) relaxes to the vicinity of the model slow manifold but otherwise it does not. Apparently when the viscous dissipation timescale is much longer than the decorrelation time, α^{-1}, then the gravity-inertial modes are much more energetic than they would be on the manifold. The motion cannot be expected to remain quasi-geostrophic in this case.

(iv) $t \to \infty$, $\eta_k^s \to 0$, in that order.

Similar conclusions hold for more general R_k^s. In fact it is not difficult to show from (33) that for small enough dissipation,

$$\langle |G_k^s|^2 \rangle \sim \frac{S(\omega_k^s)}{2\eta_k^s}, \qquad \text{as} \qquad t \to \infty,$$

where S is the power spectrum of f_k^s. The gravity-inertial waves are again found to be substantially larger than would be implied by quasi-geostrophic theory. The model system remains in the vicinity of its slow manifold only when the damping is sufficiently large.

5. Conclusions

Statistical mechanical equilibrium solutions of the inviscid shallow water equations, in the limit of weak flow, suggest that a nearly geostrophic state degrades with time until the bulk of the energy resides in short-scale, high-frequency, inertial-gravity waves. This implies a direct energy cascade. made possible by the fact that the divergent components do not participate in the enstrophy constraint. It remains an open question as to whether this transfer mechanism plays a rôle in forced-dissipative flows. According to the model equation introduced in Section 4, it may, for large-enough Reynolds numbers, provided the decorrelation time for the nonlinear transfer term is less than the dissipative time scale. In this case, one might expect a Rossby number dependent damping of the rotational energy due to the action of the high frequency modes and a subsequent cascade to short scales. For small enough Rossby numbers, the loss would probably be small while for moderate values, such as those sometimes observed in the atmosphere, it could be significant.

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7. Appendix A

The normal modes of (1) are solutions of

$$\frac{\partial W}{\partial t} + iMW = 0,$$

where $W = (u, v, \eta)$ and M is the self-adjoint operator

$$M = \begin{bmatrix} 0 & if & -i\partial/\partial x \\ -if & 0 & -i\partial/\partial y \\ -i\partial/\partial x & -i\partial/\partial y & 0 \end{bmatrix}.$$

For doubly periodic flow, these are of the form

$$W_k^\alpha e^{i(k\cdot r - \omega t)},$$

where $\alpha = 0, \pm 1$ and $k = (2\pi l/d^2, 2\pi m/d^2)$ where l and m are integers. α denotes one of the wave types which fall into two classes.

(a) $\alpha = 0$; rotational modes

$$\omega_k^0 = 0 \quad \text{and} \quad W_k^0 = \omega_k^{-1} \begin{bmatrix} -ik_2 \\ ik_1 \\ f \end{bmatrix}, \tag{A1}$$

where $\omega_k = (k^2 + f^2)^{1/2}$. These are characterized by exact geostrophic balance. The component with $k = 0$ is associated with a spatially uniform free surface departure and can be assumed to vanish.

(b) $\alpha = s$ where $s = \pm 1$; inertial-gravity modes

$$\omega_k^s = s\omega_k \quad \text{and}$$

$$W_k^s = (\sqrt{2}\omega_k k)^{-1} \begin{bmatrix} s\omega_k k_1 + ifk_2 \\ s\omega_k k_2 - ifk_1 \\ k^2 \end{bmatrix}, \tag{A2}$$

except when $k = 0$. In that case

$$\omega_0^s = sf \quad \text{and} \quad W_0^s = \sqrt{\tfrac{1}{2}} \begin{bmatrix} is \\ 1 \\ 0 \end{bmatrix}. \tag{A3}$$

These correspond to high-frequency oscillations under normal atmospheric scaling. It turns out that the $k = 0$ inertial-gravity modes can participate in the interactions and are associated with the mean flow u discussed in Section 2.

The eigenvectors satisfy

$$W_k^{\alpha *} = W_{-k}^{-\alpha},$$

$$(W_k^\alpha, W_k^\beta) = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and the brackets indicate the scalar product

$$(u, v) = \sum u_i^* v_i.$$

Combining these with the trigonometric functions, gives the representation (9) which leads to (11) when introduced into (1). The interaction coefficients are given by

$$\mu_{kpq}^{\alpha\beta\gamma} = \tfrac{1}{2}(c_{kpq}^{\alpha\beta\gamma} + c_{kqp}^{\alpha\gamma\beta}), \tag{A4}$$

where

$$c_{kpq}^{\alpha\beta\gamma} = (W_p^\beta, q)(W_k^\alpha, W_q^{\gamma *}) + (W_k^\alpha, \hat{z})(W_p^\beta, \hat{z})(W_q^\gamma, q),$$

provided $k + p + q = 0$, otherwise they vanish.

Wave vectors in these expressions are to be regarded as three tuples where necessary and $\hat{z} = (0, 0, 1)$. Eqs. (11) and (12) in Section 3 follow from

$$E = \tfrac{1}{2}(W, W),$$

$$V = \tfrac{1}{2}(MW, MW),$$

where

$$M = \begin{bmatrix} -\dfrac{\partial}{\partial y} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & -f \end{bmatrix}$$

Certain relations used in the main body of the text are listed below. Assuming $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$ then

$$\mu^{\alpha\beta\gamma}_{\boldsymbol{kpq}} = \mu^{\alpha\gamma\beta}_{\boldsymbol{kqp}}, \tag{A5}$$

$$\mu^{000}_{\boldsymbol{kpq}} = \frac{i(\boldsymbol{p} \times \boldsymbol{q})(p^2 - q^2)}{2\omega_k \omega_p \omega_q}, \tag{A6}$$

$$\mu^{00s}_{\boldsymbol{kpq}} = -\frac{\omega_p^2((\boldsymbol{k} \cdot \boldsymbol{q})\omega_q^s + if(\boldsymbol{p} \times \boldsymbol{q}))}{2\sqrt{2}q\omega_k\omega_p\omega_q}, \tag{A7}$$

$$\mu^{0s_1s_2}_{\boldsymbol{kpq}} = 0, \tag{A8}$$

$$\mu^{s00}_{\boldsymbol{kpq}} = -\frac{2\omega_k^s(\boldsymbol{p} \times \boldsymbol{q})^2 + if(\boldsymbol{p} \times \boldsymbol{q})(p^2 - q^2)}{2\sqrt{2}k\omega_k\omega_p\omega_q}, \tag{A9}$$

$$\mu^{s\,-s\,\gamma}_{\boldsymbol{k}\,-\boldsymbol{k}\,0} = \frac{1}{2\sqrt{2}}(-k_2 + isk_1). \tag{A10}$$

These follow on inserting the eigenvectors into A4. The expressions for the remaining two interaction coefficients are found in a similar fashion. A8 implies that there are no quadratic inertial gravity wave interactions in the rotational equation. A9 appears in the leading order approximation to the slow manifold while A10 is used in the demonstration of the Liouville property.

8. Appendix B

(*a*) *Long-wave limit*

Since

$$|\mu^{s00}_{\boldsymbol{kpq}}|^2 \sim \frac{k^2 p^4 \sin^2\theta}{2\omega_p^2\omega_q^2} \quad \text{as} \quad k \to 0,$$

(29) becomes

$$E_G \sim \frac{2k^3}{\pi f^2} \int_0^{k_T} \mathrm{d}p p^4 E_R(p) I_1(k, p)/\omega_p^2, \tag{B1}$$

where

$$I_1(k, p) = \int_0^{2\pi} \mathrm{d}\theta \frac{\sin^2\theta}{\omega_q^2(a + b\omega_q^2)}.$$

Since

$$I_1(k, p) \sim \frac{1}{2b\omega_p^2 p^2},$$

on using (20), (B1) gives (31).

(*b*) *Short-wave limit*

Eq. (27) can be written

$$E_G = E_G^1 + E_G^2 \tag{B2}$$

where

$$E_G^1 \sim \frac{2k}{\pi} \int_0^{k_T} \mathrm{d}p p^4 E_R(p) I_2(k, p)/\omega_p^2, \tag{B3}$$

$$E_G^2 \sim \frac{f^2}{2\pi k} \int_0^{k_T} \mathrm{d}p p^2 E_R(p) I_3(k, p)/\omega_p^2, \tag{B4}$$

when $k \gg f$. Here

$$I_2(k, p) = \int_0^{2\pi} \mathrm{d}\theta \frac{\sin^4\theta}{\omega_q^2(a + b\omega_q^2)}, \tag{B5}$$

and

$$I_3(k, p) = \int_0^{2\pi} \mathrm{d}\theta \frac{\sin^2\theta(k + 2p\cos\theta)^2}{\omega_q^2(a + b\omega_q^2)}.$$

When $k \gg f$

$$I_2(k, p) \sim -\frac{1}{4bk^2p^2} H'\left(\frac{k^2 + p^2}{2kp}\right), \tag{B6}$$

where

$$H(x) = \int_0^{2\pi} \mathrm{d}\theta \frac{\sin^4\theta}{x + \cos\theta}.$$

Using (B6) in (B3) gives (29). In a similar fashion, the contribution E_G^2 can be shown to be small.

REFERENCES

Arakawa, A. and Lamb, V. 1981. A potential enstrophy and energy conserving scheme for the shallow water equations. *Mon. Wea. Rev. 109*, 18–36.

Basdevant, C. and Sadourny, R. 1975. Ergodic properties of inviscid truncated models of two-dimensional incompressible flows. *J. Fluid Mech. 69*, 673–688.

Bennett, A. and Middleton, J. 1983. Statistical mechanics of finite difference approximations to the barotropic vorticity equation. *Q. J. R. Meteorol. Soc. 109*, 795–808.

Errico, R. 1984. The statistical equilibrium solution of a primitive equations model. *Tellus 36A*, 42--51.

Fox, D. G. and Orszag, S. 1973. Inviscid dynamics of two-dimensional turbulence. *Phys. Fluids 16*, 167–171.

Frederiksen, J. and Sawford, B. 1981. Topographic waves in nonlinear and linear spherical barotropic models. *J. Atmos. Sci. 38*, 69–86.

Kinchin, A. I. 1949. *The mathematical foundations of statistical mechanics*. Dover, 179pp.

Kraichnan, R. 1967. Inertial ranges in two-dimensional turbulence. *Phys. Fluids 10*, 1417.

Kraichnan, R. 1975. Statistical mechanics of two-dimensional flow. *J. Fluid Mech. 67*, 155–175.

Leith, C. 1980. Nonlinear normal mode initialization and quasi-geostrophic theory. *J. Atmos. Sci. 37*, 958–968.

Lorenz, E. N. 1980. Attractor sets and quasi-geostrophic equilibrium. *J. Atmos. Sci. 37*, 1685–1689.

Rhines, P. 1975. Waves and turbulence on a beta-plane. *J. Fluid. Mech. 69*, 305–319.

Sadourny, R. 1975. The dynamics of finite difference models of the shallow water equations. *J. Atmos. Sci. 32*, 680–689.

Salmon, R., Holloway, G. and Hendershott, M. C. 1976. The equilibrium statistical mechanics of simple quasi-geostrophic models. *J. Fluid Mech 75*, 691–703.

Salmon, R. 1982. Geostrophic turbulence Topics in Ocean Physics. Proc. Int. School of Physics. Enrico Fermi.

Tolman, R. C. 1938. *The principles of statistical mechanics*. Oxford University Press, 660pp.