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On the cut-off problem in linear CISK-models

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ABSTRACT

The shortwave cut-off properties of linear solutions to slightly different quasi-geostrophic layer models forced by diabatic heating due to latent heat release from convective clouds are discussed. The physics of the cut-off phenomenon is illustrated by means of an analytic and a layer model.

1. Introduction

Since the early formulation of the CISK-theory by Ooyama (1964) and Charney-Eliassen (1964), linear models including some kind of CISK mechanism have been used by a number of different authors to explain the formation and growth of tropical disturbances and more recently of polar lows (Rasmussen, 1979). A puzzling feature of many of these models, though not all, is the absence of a short wavelength cut-off, i.e., infinite damping of the smallest perturbations. As pointed out by Mak (1981), considerable effort has been made to account for or remedy the absence of a shortwave cut-off in the Charney-Eliassen model, as well as in other similar models. In some of the early studies, lateral diffusion was included in order to effect the cut-off. However, as mentioned by Chang and Williams (1974), "the inclusion of lateral diffusion is very unsatisfactory, since lateral diffusion can hardly be expected to have any significant effect before the storm matures". Chang and Williams instead attributed this shortcoming of the models to an erroneous use of the thermodynamic equation, and Rasmussen (1979) obtained a cut-off in an inviscid model similar to the one suggested by Chang and Williams. These works, however, seem to be little known and it is still widely believed that, for example, friction or non-linear effects must be included in order that a model produce a cut-off. Shukla (1978) ascribed the missing cut-off in his "mixed" CISK-barotropic-baroclinic model "either due to inadequate treatment of the sub-cloud layer or to the parameterization of cumulus heating by the quasi-equilibrium assumption". Ooyama (1969) in his study of the life cycle of tropical cyclones found rather "flat" growth rate curves for disturbances between 50 and 500 km and noted: "In the absence of a satisfactory explanation by linear models, it is reasonable to assume that the size of the convective area in observed tropical cyclones is controlled by non-linear mechanisms". Koss (1976) pointed out that the cut-off obtained by him in a linear stability analysis of CISK-induced disturbances in a primitive equation model was (relatively) independent of internal friction. He did not, however, discuss the type of mechanism responsible for the cut-off observed in his results. Emanuel (1983) with reference to Koss's work, mentions "that when the heating of the layers is equal as in Charney and Eliassen, no leveling off of the growth rate occurs at small wavelengths", and attributed this to the use of primitive rather than quasi-geostrophic equations. Further, Emanuel pointed out: "A more important result is that the growth rate distribution is very sensitive to the partitioning of the heating between the two layers, with shortwave cut-offs evident only when the heating is greater in the upper layer. This is a disturbing result since there exists little observational data regarding the distribution of cumulus heating; it also casts in doubt the viability of the use of crude vertical resolution in these models. When the vertical resolution is increased, the same sensitivity of the growth rates to η is indicated".

The present work was inspired by the growing interest in polar lows which according to some authors (see Økland (1977) or Rasmussen (1979)) bear some resemblance to tropical cyclones. It is the purpose of this paper, which should be seen as an extension of the work by Chang and Williams (1974), to demonstrate that the absence of a shortwave cut-off in a number of models is either due to the shortcoming first pointed out by Chang and Williams, or in some cases due to an inconsistent use of the vorticity equation on top of the Ekman layer. Further, we want to show that the sensitivity pointed out by Emanuel of the cut-off to the vertical distribution of diabatic heating is not a unique feature of the layer models but is also inherent in continuous models.

In Section 2, we will briefly discuss a number of simple and in many ways similar layer models which have been used since 1966 to study the effect of convection on the development of disturbances. Although the differences between the models may seem small, they are nevertheless important when we consider the cut-off phenomenon. In Section 3, we will present some results obtained by a simplified analytic model and discuss the physical explanation of the cut-off mechanism. In Section 4, we will consider especially the problems which arise in layer models when the vorticity equation is used at the level representing the top of the Ekman layer.

2. Some earlier model results

The models reviewed in this section have some general features in common. They are all linear and quasi-geostrophic and, at least partly, forced by latent heat released from convection. One of the models also includes baroclinic effects, but here only the pure "CISK-forced" solutions will be considered. The applicability of quasi-geostrophic models for the study of small-scale disturbances is not evident, but a linearised primitive equation model corresponding to the Rasmussen (1979) model gives results which are in agreement with those from his quasi-geostrophic model.

2.1. General equations

The atmosphere is considered hydrostatic, and an f-plane is assumed. We consider the equation of vorticity (1), the continuity equation (2), the hydrostatic equation (3) and the thermodynamic equation (4).

$$\partial \zeta/\partial t + V \cdot \nabla \zeta + f \nabla \cdot V = 0, \tag{1}$$

$$\nabla \cdot V + \partial \omega/\partial p = 0, \tag{2}$$

$$\partial \phi/\partial p = -\frac{R}{p}(p/p_0)^{R/c_p} \cdot \theta, \tag{3}$$

$$\partial \theta/\partial t + V \cdot \nabla \theta + \omega \partial \theta/\partial p = \frac{\theta Q}{T/c_p}, \tag{4}$$

where Q is the diabatic heating and the rest of the variables have their usual meaning.

Linearizing the equations for:

(i) a basic state of rest (denoted by an overbar) where the variables only depend on pressure;

(ii) assuming slab symmetrical perturbations of the form $(\hat{\ })_p\, e^{ikx}\, e^{\gamma t}$, k wavenumber and γ growth rate;

(iii) representing the horizontal perturbation velocity by a streamfunction $\psi = \phi/f$ and a velocity potential κ;

We find, when the hydrostatic equation is substituted into the thermodynamic equation:

$$k^2 \gamma \hat{\phi} + f k^2 \hat{\kappa} = 0, \tag{5}$$

$$-k^2 \hat{\kappa} + \frac{\partial \hat{\omega}}{\partial p} = 0, \tag{6}$$

$$\gamma \partial \hat{\phi}/\partial p - \frac{R}{p}(p/p_0)^{R/c_p}\, \hat{\omega}\, \partial \bar{\theta}/\partial p = \frac{\hat{Q} R}{p c_p}. \tag{7}$$

The total diabatic heating of an atmospheric column, Q_{ac}, having unity cross-section, is parameterised as

$$Q_{ac} = -\frac{L\, q_E}{g}\, \omega_E \tag{8}$$

where ω_E is the vertical p-velocity and q_E the mixing ratio, both at the top of the Ekman layer, and L the heat of condensation. All water vapour entering the bottom of an atmospheric column is assumed to be condensed within this and all the latent heat released is assumed to heat the column. The last assumption is not always realistic. Frank (1983) points out that the heat will only remain in the area where it is actually released if the horizontal scale of the disturbance is comparable to the local Rossby deformation radius. Otherwise it will be dispersed by internal gravity waves. Since we have assumed a basic state of rest, the Rossby radius of deformation for a stably stratified fluid is given by $R = N \cdot H/f$. Using $N = 10^{-2}\ \mathrm{s}^{-1}$ (corresponding to the conditions over the stable layer on the Bear Island sounding from 1200 GMT December 12, 1982 shown as Fig. 9 in Rasmussen, 1985), $H = 6000$ m and $f = 1.4\ 10^{-4}\ \mathrm{s}^{-1}$, we find $R \simeq 430$ km. This is roughly the scale of many polar lows, at least in the Norwegian and Barents Sea-region, so for those lows, the basic assumption seems to be reasonably well-founded.

The Ekman pumping formula is given by

$$\omega_E = -K \nabla^2 \phi_E = K k^2\, \phi_E, \tag{9}$$

where K is a positive constant and ϕ_E the perturbation geopotential at the top of the Ekman layer. In the following the '^' over the amplitude functions will be ignored. Introducing this into (7) gives

$$\gamma\, \partial\phi/\partial p + \sigma\omega - \eta K k^2\, \phi_E = 0, \tag{10}$$

where

$$\sigma = -\frac{R}{p}\,(p/p_0)^{R/c_p}\cdot \partial\bar{\theta}/\partial p$$

is the static stability, $\eta(p) = r\, L\, R\, q_E/g\, p\, C_p$ is a heating coefficient and r is a function describing the vertical distribution of the heating in a column of the atmosphere. r must fulfill the constraint $\int_{p_t}^{p_b} r\, \mathrm{d}p/g = 1$, where the integral is taken over the part of the atmosphere which is affected by the diabatic heating.

Usually when the quasi-geostrophic vorticity equation is used, a substitution of the horizontal divergence is made from the continuity equation giving

$$k^2 \gamma \phi + f^2\, \partial\omega/\partial p = 0. \tag{11}$$

Eqs. (10) and (11) are applied in a simple layer model as shown in Fig. 1. In the following we will discuss results obtained by using slightly different versions of this same basic model. For all models considered, level 5 corresponds to the top of the Ekman layer, which generally will be used as the lower boundary of the models.

2.2. *The Syono–Yamasaki model*

In Syono and Yamasaki (1966), (SY), a primitive equation as well as a quasi-geostrophic model (their Section 5 case 2) was presented, but as noted above, only the quasi-geostrophic model will be considered here. It should be noted, though, that SY find no qualitative difference between the two models when they consider the quasi-geostrophic mode of interest in the present context.

Syono and Yamasaki use the equation of horizontal motion, hydrostatic balance, mass continuity and the first law of thermodynamics explicitely in their model. In order to compare the different models, their equations have been transformed into the thermodynamic equation (10) and the vorticity equation (11) with ϕ and ω as independent variables. This, of course, does not change the solution.

MODEL	SY	CW$_1$	CW$_2$	R	SW
0 – – –				ω_0	ω_0
1 ———	Φ, ω_0 (11)	Φ, ω_0 (11)	Φ, ω_0 (11)	Φ (11)	Φ (11)
2 – – – ΔP	ω (10)	ω (10)	ω (10)	ω (10)	ω (10)
3 ———	Φ (11)	Φ (11)	Φ (11)	Φ (11)	Φ (11)
4 – – –	ω (10)		(10)	(10)	ω (10)
5 ———	Φ, ω_E (11)	ω_E	Φ, ω_E	Φ, ω_E	Φ (11)
Ekman Layer 6 ———					ω_E

Fig. 1. Schematic representation of layer models. Levels where variables are defined and equations applied are indicated. ω_0 and ω_E respectively indicate where ω is set equal to zero and the Ekman pumping-formula (9) is applied.

In the SY model, ϕ is defined at levels 1, 3 and 5 and ω at levels 1, 2 and 4, with $\omega = 0$ at level 1. The vorticity equation (11) is applied at levels 1, 3 and 5, while the thermodynamic equation (10) is applied at levels 2 and 4 (see Fig. 1). The partial derivatives with respect to p in (10) and (11) are approximated by centered finite differences except for $\partial\omega/\partial p$ at levels 1 and 5 in (11), where uncentered finite differences are used. $\omega_5 = \omega_E$ is given by (9).

The equations can be solved for the growth rate, γ_{SY}, giving

$$\gamma_{SY} = \gamma_{SY1} + \gamma_{SY2} + \gamma_{SY3},$$
$$\gamma_{SY1} = -4Kf^2/(S_1\,\Delta p),$$
$$\gamma_{SY2} = 2(\eta_2 - \sigma_2)\,Kk^2\,\Delta p/S_1,$$
$$\gamma_{SY3} = (\eta_4 - \sigma_4)\,Kk^2\,\Delta p\,(6 + 2\sigma_2\,(\Delta p)^2\,k^2/f^2)/S_1,$$
$$S_1 = 8 + 3\,(\Delta p)^2\,k^2\,(\sigma_2 + \sigma_4)/f^2 + \sigma_2\,\sigma_4\,(\Delta p)^4\,k^4/f^4. \tag{12a}$$

In Fig. 2, an example of γ_{SY} as a function of the wavelength is given.

The leading term of γ_{SY} for $k \to \infty$ is

$$2\,(\eta_4 - \sigma_4)\,Kf^2/(\sigma_4\,\Delta p). \tag{12b}$$

It follows that the SY model does *not* exhibit a shortwave cut-off since (12b) is a constant, independent of k.

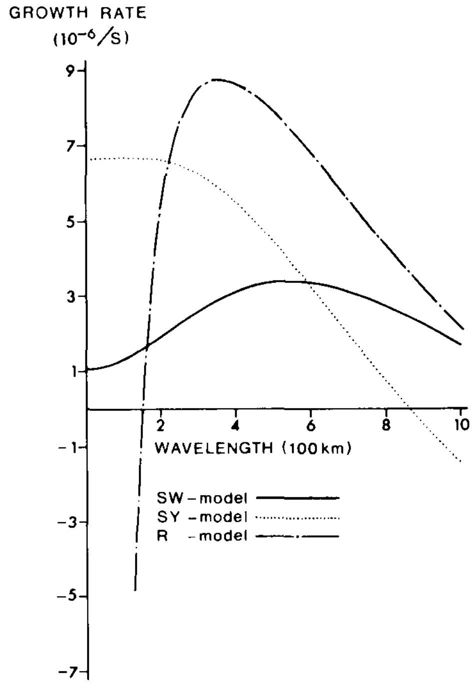

Fig. 2. Growth rate curves. Input parameters: $f = 1.26\cdot10^{-4}$, $(\partial\bar{\theta}/\partial p)_2 = -3.70\cdot10^{-4}$, $(\partial\bar{\theta}/\partial p)_4 = -2.25\cdot10^{-4}$, $K = 2.5\cdot10^7$, $q = 6\cdot10^{-3}$, 67% heating in upper layer (SI units). The top of the Ekman layer: 900 mb, $\Delta p = 200$ mb.

2.3. *The Chang–Williams model*

Chang and Williams (1974), (CW), pointed out, that a layer model with CISK mechanism *should* contain a shortwave cut-off as this feature is inherent in a continuous analytic model by Chang (1971) and in agreement with simple physical arguments (see subsection 3.2).

In their analysis of a model with continuous vertical structure, CW showed that the growth rate contribution due to diabatic heating had a maximum finite value when the horizontal scale was decreased to zero, whereas the decay rate due to the adiabatic cooling associated with the Ekman pumping becomes infinite when the horizontal scale goes to zero. Thus, for small scales, the unbounded Ekman damping rate will always dominate. In a corresponding analysis of a simple layer model, CW considered the Ekman damping for two different model formulations, ignoring diabatic heating. Using the vorticity equation at levels 1 and 3 and the thermodynamic equation at level 2 (see CW_1 in Fig. 1), *and* setting $\phi_3 = \phi_5$, they show that the damping rate due to adiabatic cooling in this case is nearly independent of the horizontal scale. Thus, the shortwave cut-off mechanism of their continuous model is *not* present in this formulation of the layer model.

The cut-off mechanism, however, can easily be included, also in such a relatively simple model by, as demonstrated by CW, applying the thermodynamic equation in the layer on top of the Ekman layer (see CW_2 in Fig. 1). In Section 4 we will comment on the lack of cut-off in the SY model, *in spite* of the application of (10), the thermodynamic equation, at level 4.

2.4. *The Rasmussen model*

Rasmussen (1979), (R), used a model resembling the CW_2 model with ϕ and ω defined at the levels indicated in Fig. 1. $\omega_0 \equiv 0$ and ω_4 is obtained by linear interpolation between ω_2 and ω_5, i.e. $\omega_4 = 2/3\,Kk^2\,\phi_5 + 1/3\,\omega_2$. As in CW_2, (10) is applied at levels 2 and 4, while (11) is applied at levels 1 and 3.

Solving for the growth rate gives

$$\gamma_R = \gamma_{R1} + \gamma_{R2} + \gamma_{R3},$$

$$\gamma_{R1} = (\tfrac{1}{3} Kk^2 \sigma_4 \Delta p - \tfrac{2}{3} Kf^2/\Delta p)/S_{II},$$

$$\gamma_{R2} = (\eta_2 - \sigma_2)\, \Delta p Kk^2 (\tfrac{2}{3} - \tfrac{1}{3}\sigma_4 (\Delta p)^2 k^2/f^2)/S_{II},$$

$$\gamma_{R3} = (\eta_4 - \sigma_4)\, \Delta p Kk^2 (\tfrac{4}{3} + \sigma_2 (\Delta p)^2 k^2/f^2)/S_{II},$$

$$S_{II} = \tfrac{4}{3} + \sigma_2 (\Delta p)^2 k^2/f^2. \tag{13a}$$

The leading term of γ_R as $k \to \infty$ is

$$k^2 K \Delta p \left(\eta_4 - \tfrac{2}{3}\sigma_4 - \tfrac{1}{3} \cdot \frac{\sigma_4}{\sigma_2} \cdot \eta_2 \right), \tag{13b}$$

which with *appropriate* values of η_2, η_4, δ_2 and δ_4, yields a shortwave cut-off. In Fig. 2, an example of γ_R as a function of the wavelength is given.

2.5. *The Sardie–Warner model*

The CISK model in a recent paper by Sardie and Warner (1983), (SW), bears some resemblence to the SY model, since (10) and (11) are applied and ϕ defined at the same levels. ω is defined at levels 2 and 4, but at the upper boundary ω is set equal to zero at level 0 instead of level 1 as in the SY model (see Fig. 1).

The SW parameterization of the CISK forcing, however, differs somewhat from that of the other models considered. Instead of letting the Ekman pumping-formula (9) define the relationship between ϕ and ω at the top of the Ekman layer, it is applied at the surface (1000 mb), i.e. $\omega_6 = -K\nabla^2 \phi_6$, introducing an "extra" level 6, in the model. Consequently, the CISK forcing in (10) is given by $-\eta Kk^2 \phi_6$, where ϕ_6 is obtained by linear extrapolation $\phi_6 = \tfrac{3}{2}\phi_5 - \tfrac{1}{2}\phi_3$. The extrapolation of ϕ makes it possible to approximate $\partial\omega/\partial p$ in (11) by a centered difference at level 5.

The extrapolation, however, corresponds to the introduction of a baroclinic Ekman layer and may yield erronous results. Assume for example that ϕ has a small negative value at the top of the Ekman layer at level 5, then the Ekman pumping yields a (small) upward motion due to the cyclonic flow. But, if ϕ has a more negative value at level 3 than at level 5, the extrapolation to level 6 may result in a *positive* value of ϕ_6 with a corresponding erroneous downward motion, cf. (9).

In the SW model, all p-derivatives are approximated by centered differences, but like the SY-model, there is no shortwave cut-off, in the sense that the decay rate goes to infinity for zero horizontal scale. This is seen from the expression for the growth rate:

$$\gamma_{SW} = \gamma_{SW1} + \gamma_{SW2} + \gamma_{SW3}$$

$$\gamma_{SW1} = (\tfrac{1}{2} Kk^2 \Delta p\, \sigma_4 - Kf^2/\Delta p)/S_{III},$$

$$\gamma_{SW2} = (\eta_2 - \sigma_2) Kk^2 \Delta p (1 - \tfrac{1}{2}\sigma_4 (\Delta p)^2 k^2/f^2)/S_{III},$$

$$\gamma_{SW2} = (\eta_4 - \sigma_4) Kk^2 \Delta p (\tfrac{7}{2} - 2\sigma_2 (\Delta p)^2 k^2/f^2)/S_{III},$$

$$S_{III} = 3 + 2\,(\Delta p)^2 k^2 (\sigma_2 + \sigma_4)/f^2 + \sigma_2 \sigma_4 (\Delta p)^4 k^4/f^4 \tag{14a}$$

where the leading term for $k \to \infty$ is

$$(2\,\eta_4 - \tfrac{3}{2}\sigma_4 - \tfrac{1}{2}\sigma_4/\sigma_2 \cdot \eta_2)\, Kf^2/(\sigma_4 \Delta p). \tag{14b}$$

Note, however, that even if there is no shortwave cut-off inherent in the model, a preferred wavelength may still be obtained as shown in Fig. 2, where the maximum growth rate is found near the same value as obtained by R using the same input parameters.

3. The cut-off in simplified models

3.1. *An analytic model*

The existence of a shortwave cut-off in the R-model discussed in the previous section still depends critically on the vertical heating distribution to ensure that the infinite damping of the shortest wavelengths does not turn into an infinite amplification. This will occur, if too much of the diabatic heating is released in the lower layer (see Fig. 3). In this section, we will look at the cut-off in the context of an analytic model.

The equations constituting the model are the same as in Section 2, i.e., eq. (10) and (11), but the static stability in this section is assumed to be a constant in order to obtain an analytic solution, and we only consider the conditions in the troposphere, say below $p_T = 300$ mb.

The heating coefficient η in (10) is assumed to be of the form

$$\eta = \begin{cases} 0 & , \quad p_* > p \geqslant p_T, \\ E_0 \sin\left(\dfrac{p_1 - p}{p_1 - p_*}\pi\right) e^{\frac{p_1 - p}{p_1 - p_*} a} & , \quad p_E \geqslant p \geqslant p_*, \end{cases} \tag{15}$$

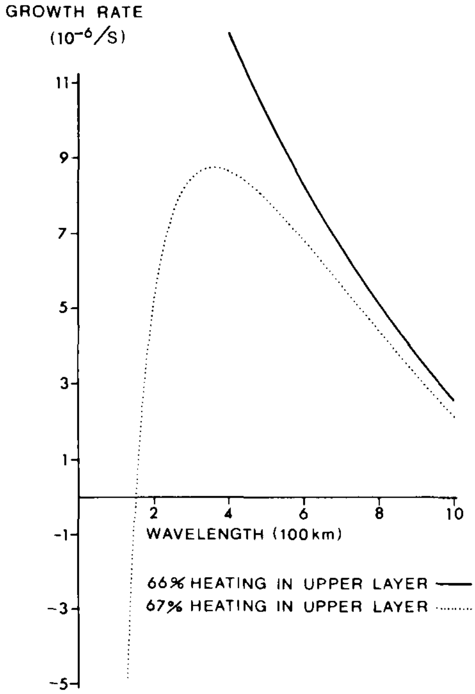

Fig. 3. Sensitivity of the growth rate in the R model to the heating distribution. Input parameters as given in Fig. 2 except for the heating distribution.

where p_1 is a constant pressure greater than or equal to p_E (the pressure at the top of the Ekman layer), and p_* shows how far the diabatic heating penetrates into the troposphere. In regions with ascent at the top of the Ekman layer, (15) gives a heating greater than zero between p_E and p_*. a is a constant giving the skewness of the heating profile. E_0 is a constant determined by the total amount of latent heat available for an atmospheric column.

Eliminating ω between (10) and (11) gives

$$\gamma\, \partial^2 \phi/\partial p^2 - \gamma\delta^2 \phi - \partial\eta/\partial p\, Kk^2 \phi_E = 0, \tag{16}$$

where $\delta = \sqrt{\sigma}\, k/f$.

The solution to (16) is given by

$$\phi = A \sinh (\delta(p_E - p)) + B \cosh (\delta(p_E - p)) + C \sin\left(\frac{p_1 - p}{p_1 - p_*}\pi\right) e^{\frac{p_1 - p}{p_1 - p_*}a} + D \cos\left(\frac{p_1 - p}{p_1 - p_*}\pi\right) e^{\frac{p_1 - p}{p_1 - p_*}a}, \quad p_E \geqslant p \geqslant p_*, \tag{17}$$

$$\phi = E \sinh (\delta p) + F \cosh (\delta p), \quad p_* > p \geqslant p_T, \tag{18}$$

where the constants C and D follow directly from substitution into (16), while the rest of the constants A, B, E and F follow from the external and internal boundary conditions:

(i) (17) must be valid at the lower boundary ($p = p_E$);
(ii) ϕ must be continuous at p_*;
(iii) $\partial\phi/\partial p$ must be continuous at p_* (corresponding to continuity of temperature);
(iv) ϕ, the perturbation geopotential must vanish at $p = p_T$.

Applying the solution (17) to (10) at the top of the Ekman layer yields a rather complicated expression for the growth rate which for $k \to \infty$ has the leading term

$$\frac{\mathrm{fKk}}{\sqrt{\sigma}} (\eta(p_E) - \sigma). \tag{19}$$

If we assume that the diabatic heating is zero at the top of the Ekman layer, i.e. $\eta(p_E) = 0$, (19) shows, that a cut-off will always be present.

Dropping the assumption of zero heating at the top of the Ekman layer, it follows from (19) that the shortwave cut-off will turn into a *shortwave blow-up* if $\eta(p_E) > \sigma$. This behaviour corresponds to the one exhibited by the layer models as will be elaborated in the following.

3.2. *The cut-off mechanism in a 2-layer model*

As shown above, an analytic model with a continuous vertical structure, but otherwise similar to the models discussed in Section 2, includes a cut-off mechanism provided the heating on top of the boundary layer is sufficiently small. In this section, we will show how a layer model must be formulated in order to simulate the cut-off mechanism inherent in the continuous model, and the physics of the cut-off mechanism will be discussed.

To achieve this, we will use a slightly simplified version of the R model discussed in Section 2. Instead of interpolating ω_4 by means of ω_E and ω_2, we simply set $\omega_4 \equiv \omega_E$. Numerical experiments have shown that the resulting growth rates will only be slightly changed by this approximation and qualitatively there will be no change.

The simplified R-model is given by the vorticity equation at levels 1 and 3:

$$\gamma k^2 \phi_1 + f^2 \omega_2/\Delta p = 0, \tag{20a}$$

$$\gamma k^2 \phi_3 + f^2 (Kk^2 \phi_5 - \omega_2)/\Delta p = 0, \tag{20b}$$

together with the thermodynamic equation at levels 2 and 4:

$$\gamma(\phi_3-\phi_1)/\Delta p+\sigma_2\,\omega_2-\eta_2 Kk^2\phi_5=0, \tag{21a}$$

$$\gamma(\phi_5-\phi_3)/\Delta p+(\sigma_4-\eta_4)Kk^2\phi_5=0. \tag{21b}$$

From (20a,b) and (21a,b) the following relations are easily obtained

$$\phi_1=-(1+f^2 K/[(\gamma_1+\gamma_2)\Delta p]\,\phi_3, \tag{22}$$

$$\phi_5=\gamma\phi_3/[\gamma+(\sigma_4-\eta_4)Kk^2\Delta p], \tag{23}$$

$$\gamma=\gamma_1+\gamma_2+\gamma_3,$$

with

$$\gamma_1=-K\,(f^2(\Delta p)/[1+\sigma_2(\Delta p)^2 k^2/f^2],$$
$$\gamma_2=Kk^2\,\Delta p\,(\eta_2-\sigma_2)/[1+\sigma_2\,(\Delta p)^2 k^2/f^2],$$
$$\gamma_3=Kk^2\,\Delta p\,(\sigma_4-\delta_4). \tag{24}$$

For $k\to\infty$ we find that:

$\gamma_1\to 0$ and is negative;

$$\gamma_2\to\frac{f^2 K}{\sigma_2\,\Delta p}(\eta_2-\sigma_2);$$

$$\gamma_3\to\infty \quad \text{for} \quad \eta_4>\sigma_4; \quad \gamma\to-\infty \quad \text{for} \quad \eta_4<\sigma_4. \tag{25}$$

It follows that a necessary condition for a shortwave cut-off is that $\eta_4<\sigma_4$, which on the other hand implies *that the lower layer of the model on top of the Ekman layer is being cooled in a surface low*. This is easily seen from the thermodynamic equation for layer (3–5) which may also be written

$$\partial(\phi_5-\phi_3)/\partial t=-\Delta p\,(\sigma_4-\eta_4)\,\omega_4, \tag{26}$$

where $(\sigma_4-\eta_4)$ is a modified static stability.

To obtain growth rate curves like that shown for the R model in Fig. 2, η_2 must be greater than σ_2. This assumption is not unrealistic when we consider the development of polar lows, provided all the heat remains in the area where it is actually released. Applying, for example, the data used by Rasmussen (1979) we find values for η_2/σ_2 around 3.

According to (23), ϕ_5 may be expressed as $\phi_5=F\phi_3$,

$$F\equiv\gamma/[\gamma+(\sigma_4-\eta_4)Kk^2\,\Delta p]$$
$$=(\gamma_1+\gamma_2+\gamma_3)/(\gamma_1+\gamma_2). \tag{27}$$

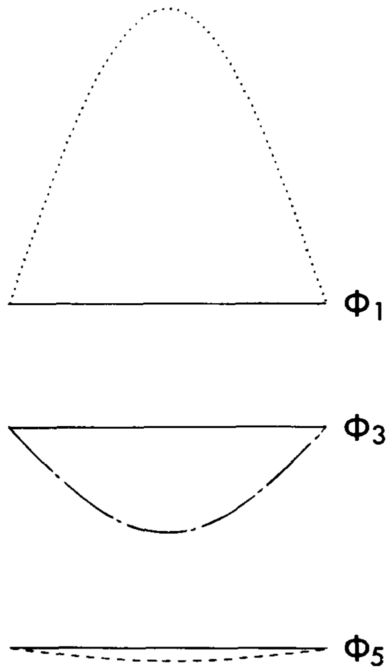

Fig. 4. Vertical distribution of the geopotential in the modified R model for a growing perturbation. Input parameters as given in Fig. 2 except for the heating distribution (74% applied in the upper layer).

For growth rate curves like that shown for the R-model in Fig. 2, it may be inferred from (24), (25), and (27), with L_0 being the wavelength corresponding to zero growth rate, that $0<F<1$ for $L>L_0$, and $F<0$ for $L<L_0$. The corresponding structure for the growing wave is shown schematically in Fig. 4. Since $(\gamma_1+\gamma_2)>0$ for all but the very long wavelengths, it follows from (22) that ϕ_1 and ϕ_3 are 180° out of phase for growing as well as decaying waves.

Fig. 4 and the relationships given above clearly demonstrate the physics of the shortwave cut-off mechanism. The contribution from γ_3 expresses the effect of the cooling of the layer on top of the Ekman layer and has the same sign for all wavelengths. Further, γ_3 will dominate for small wavelengths and, when negative, cause the shortwave cut-off. What physically happens for small, but still positive growth rates is that the cooling becomes so strong that the amplitude of ϕ_5 through the hydrostatic pressure effect becomes "very small" for a given amplitude of ϕ_3 as expressed through the relation $\phi_5=F\phi_3$ where $F\to 0$ for $L\to$

L_0. A small value of ϕ_5 means a small Ekman pumping, i.e., a small supply of latent heat and a resulting small growth rate. We can conclude *that in order to simulate the shortwave cut-off mechanism in a layer model, the model must be formulated in such a way that cooling, through the use of the thermodynamic equation, is allowed to take place in the layer immediately on top of the Ekman layer.* This is exactly the same effect as studied by CW by means of an analytic model, and our conclusion is similar to theirs.

4. A model formulation problem

As mentioned in Section 2, a cut-off should be expected in the SY model since the thermodynamic equation is applied at level 4 allowing for temperature variations in the layer on top of the Ekman layer. However, the use of the vorticity equation at level 5 imposes a constraint on the growth rate, preventing it from attaining an infinite limiting value for the perturbation wavelength approaching zero.

We may illustrate this in a simple way by comparing the expression for the growth rate, γ, in the simplified R-model and in the SY-model for $k \to \infty$. In this limiting case, the leading term for the growth rate, γ_∞, for the R-model can be found from the thermodynamic equation, (21b), alone. Writing (21b) as

$$\gamma(1 - \phi_3/\phi_5) + \Delta p\,(\sigma_4 - \eta_4)\,Kk^2 = 0, \qquad (28)$$

and noting that $\phi_3/\phi_5 = F^{-1} \to 0$ for $k \to \infty$, we obtain

$$\gamma_\infty = \Delta p K k^2\,(\eta_4 - \sigma_4),$$

i.e. an infinite limiting value of γ for zero wavelength.

In the SY model, the thermodynamic equation is applied at level 4 together with the vorticity equation at level 5

$$\gamma(\phi_5 - \phi_3)/\Delta p + \sigma_4\,\omega_4 - \eta_4\,Kk^2\,\phi_5 = 0,$$

$$k^2\,\gamma\phi_5 + 2f^2\,(\omega_5 - \omega_4)/\Delta p = 0. \qquad (29)$$

Introducing (9) into this yields

$$k^2\,\gamma + \frac{2f^2\,Kk^2}{\Delta p} - \frac{2f^2\,\eta_4\,Kk^2}{\Delta p\,\sigma_4} + \frac{2f^2\,\gamma}{\sigma_4\,(\Delta p)^2} - \frac{2f^2\,\gamma}{\sigma_4\,(\Delta p)^2}\frac{\phi_3}{\phi_5} = 0; \qquad (30)$$

for $k \to \infty$, the first three terms must balance each other giving

$$\gamma_\infty = \frac{2f^2\,K}{\sigma_4\,\Delta p}\,(\eta_4 - \sigma_4), \qquad (31)$$

identical to (12b). It follows that it is the extra constraint introduced by using (11) at level 5 which causes the lack of a cut-off in the SY-model.

It is still possible to obtain a shortwave cut-off applying the vorticity equation at level 5. This can be achieved by using the continuity equation explicitly instead of substituting it into the vorticity equation. The equations describing the model are then (5), (6) and (10) with the independent variables as ϕ, ω and κ. ϕ is defined at levels 1, 3 and 5, and ω at levels 2 and 0; ω_5 is as usual given by (9); κ is defined at levels 1, 3.5 (mid between levels 3 and 4) and 5. The vorticity equation is applied at levels 1, 3, and 5 and the continuity equation at levels 2 and 3.5. (κ_3 and ω_4 are obtained by linear interpolation.) Solving for the growth rate in this model gives

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3,$$

$$\gamma_1 = -\tfrac{8}{15}\,K f^4/[(\Delta p)^4\,S_{\mathrm{IV}}],$$

$$\gamma_2 = -K f^2\,k^2/(\Delta p)^2\,(\tfrac{16}{15}\,\sigma_4 - \tfrac{4}{3}\,\eta_4 + \tfrac{8}{15}\,\sigma_2 - \tfrac{1}{3}\,\eta_2)/S_{\mathrm{IV}},$$

$$\gamma_3 = -Kk^4\,(\tfrac{1}{3}\,\sigma_4\,\eta_2 + \tfrac{2}{3}\,\sigma_4\,\sigma_2 - \sigma_2\,\eta_4),$$

$$S_{\mathrm{IV}} = \sigma_2\,k^2/\Delta p + \tfrac{4}{3}f^2/(\Delta p)^3.$$

The leading term of γ for $k \to \infty$ is

$$\Delta p\,Kk^2\,(\eta_4 - \tfrac{2}{3}\,\sigma_4 - \tfrac{1}{3}\,\sigma_4/\sigma_2 \cdot \eta_2),$$

which is identical to (13b), the leading term in the R model, and a shortwave cut-off is possible for suitably chosen parameters.

5. Summary and concluding remarks

Mak (1981) pointed out, that since the original works on CISK from 1964, a large number of articles have attempted to identify and correct the shortcomings of the original CISK, and considerable efforts have especially been made to account for the absence of a short wavelength cut-off in the Charney-Eliassen model. Mak did not explicitly consider the cut-off problem in his paper but referred to the work by Chang and Williams (1974).

In Section 2, we have discussed 5 different layer models. Of these models, only the CW_2 and the R model produces a shortwave cut-off. The SY, the R and the SW models include a mechanism for release of latent heat through an Ekman pumping parameterization. These three models are basically identical, based on the same equations, but small differences exist, for example in the way that the vertical derivatives are defined.

In Section 3, we have shown that a model with a continuous vertical structure and with a constant static stability, but otherwise identical to the models considered in Section 2, produces a shortwave cut-off provided the heating at the top of the Ekman layer is zero or "small". Furthermore, in Section 3, we have showed by means of a slightly simplified version of the R model, how a layer model like those considered in Section 2 must be formulated in order to produce the cut-off inherent in the corresponding analytic model. Furthermore, we discussed the physical mechanism of the cut-off, i.e. the decreasing energy supply due to a decreased Ekman pumping. This in turn is caused by a diminished amplitude of the disturbance at the level corresponding to the top of the Ekman layer, because of cooling of the layer above.

cut-off mechanism may fail if the model is not suitable formulated. As an illustration of this, we discussed in Section 4 the SY model which fails to show a cut-off because the vorticity equation was applied on top of the Ekman layer. However, a cut-off can also be obtained in this case if the equations constituting the model are applied slightly differently.

The models which we have considered are basically alike and therefore easy to compare. We believe, however, that our conclusions are generally valid, especially since arguments for the importance of the low level cooling-effect can indirectly be found in several papers (e.g., Koss, 1976).

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