Elementary Current, and defines the transport when the stratification is included (identical with the modern "mass transport"). He proceeds to consider the effects of precipitation and evaporation, and estimates the magnitude of the currents which these can provide, which he correctly finds to be negligibly small. In Section IV, he gives the complete set of field equations for the problem, in a form analogous to the ones found in his 1923 paper, but now with stratification included. Firstly, he deals with the problem in an f-plane, but later the β-effect is introduced. One can track down expressions that essentially reproduce the Sverdrup relation (see, for example, the formula just before eq. (33) in the following translation) for a case with no net divergence, stating that the meridional transport is proportional to curl T.

Ekman finishes with a comment on the path of the Gulf Stream, noting that it ought to turn into deep water as it pushes north. He states that he intends to return to this problem to complete his theory, but, as we know, this was not to happen.

The paper seems to me to be of enough interest historically as well as scientifically to warrant publication, in a translated English version. The translation, which follows, is intentionally made with preservation of the somewhat rigid, Swedish style which Ekman uses. His notations are kept unchanged: the reader should note that some variables (V, γ, S, G, T, etc.) are vectors, their components are written γ_x, γ_y, etc. Derivatives are, on the other hand, written out in full.

Seattle, October 17, 1984
PIERRE WELANDER

Outline of a Unified Ocean Current Theory

By V. WALFRID EKMAN,

(From an invited paper for the doctoral promotion in Lund, Sweden 31 May 1924)

I

The theoretical studies of the dynamics of ocean currents, to date, have mainly been confined to one or the other of two main types of currents: wind currents and convection currents[1]. The former are currents which are generated by the wind in an ocean basin with homogeneous water, while the latter run without direct influence of the wind, through the disturbance in hydrostatic balance between water masses having different densities.

As the theories for these two types of currents have followed widely different lines, these furthermore appear isolated, and attempts to unify them to a general theory for ocean currents have unfortunately been unsuccessful. This is regrettable since some of the most spectacular and important motions manifested in the sea obviously represent effects of both the wind and the disturbed hydrostatic balance, which accordingly have not been subjected to a logical theoretical study. For the same reason, one has had little success in the discussion of the question whether the wind or the internal equilibrium changes represent the most important cause of the ocean currents, and this old controversy has not yet been removed from the agenda.

The attempt which will be made here to build a foundation on which a unified theory for ocean currents of different origins can be built, is most closely connected to the theory for wind currents, which I have developed in some earlier publications. Among these publications, one finds in particular

[1] One exception is the peculiar theory which was forwarded by Mohn in 1887, but this exhibits such fundamental shortcomings that it nowadays is only of historic interest.

three[1] to which reference will be made later, and which, therefore, will be denoted simply by year of printing: (1905), (1906), and (1923), respectively.

As basis for this wind current theory lie, among others, the following three assumptions, I, II, and III, which to equal degrees remain valid also for convection currents. With regard to the motivations which form the basis for these assumptions, reference is given to the above-mentioned publications.

I. The vertical velocity component of the water masses play no role from the point of view of the dynamics, and accordingly one needs only to consider the velocity components u, v along two horizontal axes x, y. The y-axis is assumed to be positioned *contra solem* (against the sun's motion) from the x-axis, the x-axis then lies *cum sole* (with the sun's motion) from the y-axis. If, for example, the x-axis is drawn straight eastward, the y-axis is directed from the equator to the pole.

II. The accelerations of the water masses are not large enough to play a role in the dynamical equations, but the motion (relative to the earth) takes place according to the formally static equations in which, together with the real forces, enter certain "deviation forces" depending on the earth's rotation and the velocity of the water. These basic dynamic equations for the ocean currents state that:

$$u = cY, v = -cX \tag{1}$$

$$c = \frac{1}{2\omega \sin \phi} \tag{2}$$

where ω and ϕ denote the earth's angular velocity and latitude, and X, Y denote the components of the real forces acting on a mass unit of water: the pressure gradient and the friction.

III. The friction from water masses moving relatively to each other *sidewise* is unimportant; only the friction between water layers moving *over* each other requires consideration. The subject of the study is only the water motion at large, that is, the "current" in its real sense, not the associated turbulent eddies. The increased interaction between the water layers which the turbulence produces can formally be expressed through an increased "virtual" friction coefficient; the latter is normally not a constant but has varying values in different parts of the ocean, due to different wind conditions, etc.

The assumption II, and the equations (1) which follow from it, are not valid under all circumstances. Exceptions must be made for two types of oceanic regions, in which the theory developed here certainly is unapplicable. One exception occurs in a narrow—possibly a very narrow—belt at the equator. The other exception occurs in local current conditions in small regions, outside sharp points, etc., where the proximity to the coast or other special circumstances produce a strong curvature of the current paths.

Eqs. (1) assume strictly that the xy-plane, parallel to which the motion will take place, is completely horizontal. They are, however, approximately valid if this plane has a small slope angle, and actually it is sufficient that one can neglect the square of this, i.e., be able to set its cosine equal to one. The only difference is that one must include the components of the gravity force in X, Y.

The theory for wind currents which has been developed on the basis of the assumptions I–III firstly leads to a calculation of the current direction and speed at points on a vertical line from the bottom to the sea surface, when one knows the slope of the latter, and the wind force acting on the sea surface. This picture of the current, at points on a vertical line, I have denoted the "Elementary Current" (Elementarströmmen in Swedish, Elementarstrohm in German, translator's comment). It comprises a *bottom current* and a *surface current*, both having a variable speed and direction along the vertical line, and, in the main part of the sea between the bottom current and surface current, a *deep current* at all levels.

Under the assumption of *stationary motion* (stationary winds and a stationary current field in the ocean), the theory leads further, at least in principle, to the calculation of the shape of the sea surface, and thereby also to the elementary current in every part of the sea, when the shape of this surface and the winds are known. For different

[1] On the Influence of the Earth's Rotation on Ocean Currents, Arkiv för Mat., Astr. och Fysik, Bd 2, N:o 2 1905.

Beiträge zur Theorie des Meeresströmungen, Annalen der Hydr. und Maritim. Meteorologie. S. 423 ff. 1906.

Über Horizontalzirkulation bei winderzeugten Meeresströmungen, Arkiv. för Mat., Astr. och Fysik, Bd 17 N:o 26, Sthlm 1923.

reasons, and not the least because of the mathematical difficulties, this calculation cannot in general be carried out; results in this direction, which have been of use geographically, are at present confined to certain general theorems regarding the dependence of the current on the wind, the bottom topography, and on the horizontal direction and latitude (see 1923).

A particular difficulty appears when one tries to apply the theoretical results practically in one or the other of the two directions mentioned. One must then consider the empirical laws for the intensity of the turbulence, and the "virtual" friction coefficient which depends on these laws, about which one has so far had only meagre knowledge.

The modern theory for convection currents has its origin in the circulation theory by Bjerknes[1]. The latter assumes absence of friction. It can be made complete by adding a term which gives the mathematical expression for the influence of friction, but in a form which does not allow its practical use. Otherwise, the theorem is of a remarkably general kind, and independent of the assumptions I and II. If one now introduces these assumptions, the circulation theorem leads to an equation, by which the geometric differences between velocities at any two levels can be calculated, when the density field, that is, the distribution of the density in the sea, is known. In oceanography this special equation has largely replaced the original circulation theorem, in particular in quantitative calculations, and it is also the one to be used in the following. For the sake of uniformity of this study I find it justifiable to derive this equation directly, without reference to the circulation theorem. By this the deduction gains simplicity and shortness, although it may lose elegance.

We start from the first of the eqs. (1), and assume a perfectly horizontal xy-plane and a vertical z-axis, directed downward. In the absence of friction we have then

$$u = \frac{-c}{q}\frac{\partial p}{\partial y}$$

where p denotes the pressure and q the density of the water. We imagine two points A and B, in the sea, one above the other on the same vertical line, and denote the velocity component in the two points u_a and u_b. Through each of these points runs an isobaric surface, on which accordingly p is constant. Since the differences in the density of the water are small, each such isobaric surface lies on an approximately constant depth below the sea surface. A tangent to the isobaric surface, drawn perpendicular to the x-axis, and in the same direction as the positive y-axis, forms a small angle γ_a with the latter, and we agree to denote this as positive if the tangent is slanted downward, and negative if it is slanted upward. Then, as γ_a is a very small angle,

$$\gamma_a = tg\gamma_a = \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{-\dfrac{\partial p}{\partial y}}{\dfrac{\partial p}{\partial z}} = \frac{-\dfrac{\partial p}{\partial y}}{qg}$$

If one inserts the value of $\partial p/\partial y$ obtained from this formula, and replaces u by u_a, one finds

$$u_a = cg\gamma_a$$

In a corresponding way one finds

$$u_b = cg\gamma_b$$

where γ_b denotes the angle between the y-axis and the tangent to the isobaric surface at B, calculated in a similar way.

Since the slope of the sea surface itself cannot be observed, one cannot from the use of the hydrographic data calculate the angles γ_a and γ_b, and the velocity components u_a and u_b. However, $\gamma_b - \gamma_a$ can be calculated from the observational material, and one finds

$$u_a - u_b = cg(\gamma_a - \gamma_b) \tag{3}$$

This is the equation mentioned, deduced from the original circulation equation, representing the end result of the present theory of convection currents.

The fact that eq. (3) can be derived without use of the circulation theorem by Bjerknes[2], has of course, not made the latter indispensible for oceanography. It seems unnecessary to mention this for persons who know of the different forms into which the theorem in question has been developed and applied, and of the excellent graphic

[1] V. Bjerknes, Zirkulation Relativ zu der Erde, Transl. by K. Vet. Akad. Förh. 1901, No. 10.

[2] V. Bjerknes: Dynamic Meteorology and Hydrography, Part I, Statics, Carnegie Inst. of Wash. No. 88, 1910.

overview of the dynamic state which it can offer by use of the "solenoid field". The working methods founded on this theorem, however useful they may be, do not fall within the context of the development of a unified theory, which is the present aim.

Armed with a sufficient observational material of salinities and temperatures at different depths, one can, by use of specially constructed tables, calculate at which depths below the sea surface a given pressure value (excluding the atmospheric value) exists. This depth is calculated in a unit which varies with the gravity force g: $10/g$ meter, whereby the effect of the varying intensity of the gravity force is considered already in the first stage of the work. The corresponding unit for vertical level differences, the so-called "dynamic meter", is about 102 cm, at the surface and at 45° latitude.

If one has, in this way, constructed a topographic chart of a particular isobaric surface, one has everything that is necessary to directly read of the geometric difference $V_a - V_b$ between the water velocity V_a at the sea surface and V_b at the level of the selected isobaric surface. According to eq. (3) this difference $V_a - V_b$ is everywhere directed along the level lines mentioned, and *cum sole* from the direction in which the isobaric depth decreases; its magnitude $|V_a - V_b|$ is inversely proportional to the distance between the level lines. A numerical calculation shows that

$$|V_a - V_b| = \frac{0.62N}{\sin\phi} \tag{4}$$

where N denotes the number of level lines which are found within a perpendicular line segment of a distance corresponding to one latitudinal degree.

This is the method, worked out by J. W. Sandström (on the basis of the theorem by Bjerknes), which we intend to use as part of the theory to be outlined below. However, it is not in itself sufficient. Firstly, as already mentioned, no consideration is given to the friction, secondly, it gives no information on the *absolute* velocity at a given depth. One may imagine that the latter problem could be taken care of by placing the level of the lower isobaric surface at the sea bottom, where the velocity, or course, is zero. However, this method fails because the water layer next to the sea bottom is precisely the one for which it is *not* permissible to neglect the friction (compare 1906, p. 568).

In general it has been assumed that the velocities at large depths—that is, in the nearly homogeneous deep water—are very small. If this assumption is correct, eq. (4) could give information about the absolute velocity of the current in the upper water layers (of course, only to the extent the friction is negligible). We will later find that the theory, in some situations, give certain support for the correctness of this assumption.

II

We start again from eqs. (1), placing the xy-plane horizontally, and including in the force components the friction terms which are well known from the wind current theory. The equations are then

$$u = -\frac{c}{q}\frac{\partial p}{\partial y} + \frac{\mu c}{q}\frac{\partial^2 v}{\partial z^2}, \quad v = \frac{c}{q}\frac{\partial p}{\partial x} - \frac{\mu c}{q}\frac{\partial^2 u}{\partial z^2} \tag{5}$$

The velocity components, which the water would have in the absence of friction but under the assumption of unchanged density distribution, are denoted by u_0, v_0; thus

$$u_0 = -\frac{c}{q}\frac{\partial p}{\partial y}, \quad v_0 = \frac{c}{q}\frac{\partial p}{\partial x}$$

Introducing these quantities, and using instead of c another quantity

$$D = \frac{\mu}{\sqrt{q\omega\sin\phi}} = \pi\sqrt{\frac{2\mu c}{q}} \tag{6}$$

the equations (5) transform to

$$\frac{D^2}{2\pi^2}\frac{\partial^2 u}{\partial z^2} + v = v_0; \quad \frac{D^2}{2\pi^2}\frac{\partial^2 v}{\partial z^2} - u = -u_0 \tag{7}$$

Here D is the "friction depth", introduced earlier in the wind current theory.

To eliminate v from the two eqs. (7) one must take a derivative of the first equation twice with respect to z, and then D should correctly be considered as variable (because of the different strength of the turbulence at different depths). This has been pointed out earlier, during the development of the wind current theory. Now, with the water no longer homogeneous but stratified, and with a varying degree of stability between the layers, one expects that the variation of the friction depth with the depth below the sea surface will be even more important (see 1906, pp. 428–430). However, to avoid large complications it seems justified, in this preliminary study, to assume D constant, i.e., independent of z. We will only consider the variation in D to the extent that we distinguish between one (mean) value D' for the uppermost water layer, the so-called "upper frictional depth", and another value D'', the "lower frictional depth", for the bottom layer. With regard to the magnitude of these quantities see (1923) §7 and 17. In the theory for windcurrents in homogeneous water it has been found (1905, pp. 43–51; 1923, §12) that such a simplification does not essentially change the results when the assumption is introduced in the convection current, it must, however, be stressed that it is, at this time, introduced only provisionally.

Under the assumption thus made, $\partial D/\partial z = 0$, we obtain from eqs. (7), if we disregard independent variables other than z, and replace symbols of partial derivatives by ordinary ones

$$\frac{D^4}{4\pi^4}\frac{D^4u}{dz^4} + u = u_0 + \frac{D^2}{2\pi^2}\frac{d^2v_0}{dz^2},$$

$$v = v_0 - \frac{D^2}{2\pi^2}\frac{d^2u}{dz^2} \tag{8}$$

If, instead of u, the variable $u' = u - u_o$ is introduced, the first of these equations can be written in the form

$$\frac{D^4}{4\pi^4}\frac{d^4u'}{dz^4} + u' = \frac{D^2}{2\pi^2}\frac{d^2v_0}{dz^2} - \frac{D^4}{4\pi^4}\frac{d^4u_0}{dz^4} \tag{9}$$

We will come back, in Section III, to the question how to calculate u_0 and v_0. At this stage we consider u_0 and v_0 as known, and the mathematical task is then to integrate a linear differential equation with constant coefficients, and with a "right hand side" (forcing term); this is a problem that always can be solved. It becomes particularly simple in case the "right hand side" is absent, or takes on a suitable, simple form.

If u_0 and v_0 are *constants*, which means that no solenoid field exists (that is, the water is either homogeneous, or stratified in horizontal, homogeneous layers), one recovers the windcurrent problem treated earlier (1905). On the other hand, if the solenoid field is limited to an upper layer, below which the water can be considered homogeneous, with sufficient accuracy, the elementary current in the homogeneous layer is precisely similar to the elementary current in a sea which is everywhere homogeneous. That is, it must comprise a *bottom current* and a *uniform deep current*, plus a "surface current" just below the solenoid field, which we conveniently can call an *internal surface current*.

But even in the region of the solenoid field the integration of eqs. (8) or (9) can be carried out explicitly without difficulty, as soon as u_0 and v_0 can be written in the forms of a polynomial or a trigonometric function of z, and accordingly it is not difficult to calculate the elementary current with sufficient accuracy for almost all solenoid fields, which may occur in the sea, should have a mentioned, it is, however, required that one firstly is able to determine the water velocity at *one* level, for example, in the uniform deep current, which was required to make the calculation of u_0 and v_0 by use of the solenoid field. Further, it is assumed that the friction depth D can be considered as constant.)

Professor Björn Helland-Hansen and the author of this invited paper have shown that the solenoid fields, which may occur in the sea, should have a relatively simple form, under certain assumptions.

All isothermal and isohaline surfaces, which intersect the same vertical line, must have such a slope that their horizontal tangents are *parallel*. (A solenoidal field of such type may be denoted a "*parallel solenoidal field*".) Further, the velocity vector of the elementary current, at all levels—inside as well as below the solenoid layer—must lie in the same vertical plane, that is, it must have either the same single direction, or two opposite directions.

As has already been mentioned, the above theorem is valid only under certain assumptions (see 1923, §67). Thus, it is valid only in a *stationary* hydrographic situation, and not in cases of periodic or instantaneous surges of water of anomalous salinity and temperature, such as have been seen in Skagerack-Kattegatt, and elsewhere. Also, it may not apply near sharp points, as for example at Skagen. Further, it must be stated that, strictly, the theorem assumes absence of friction. If, in the following, we will use it as starting point for the calculation of the effect of friction on convection currents, we must consider it as an *approximate* calculation in cases where this effect is relatively small, that is, in solenoid fields with a depth larger than the frictional depth.

Independently of the theoretical foundation this theorem has been subjected to a beautiful verification in the Gulf Stream region between latitudes 40° N and 60° N (1923, §71); and, although the empirical study of the validity of the theorem is not completed by this verification, it is nevertheless obvious that nearly parallel solenoid fields play a prominent role in the dynamics of ocean currents. It seems probable to me that these fields, in stationary hydrographic situations, represent the rule; and thus it seems justified, in examples of eqs. (8) and (9), to consider in particular the simpler case of parallel solenoid fields.

If the solenoid field is parallel and the elementary current lies in a plane, one may place the x-axis in a direction that makes $v_0 = 0$. Instead of eqs. (8) and (9), one then obtains

$$\frac{d^4u'}{dz^4} + \frac{4\pi^4}{D^4}u' = -\frac{d^4u_0}{dz^4},$$

$$v = -\frac{D^2}{2\pi^2}\frac{d^2(u_0 + u')}{dz^2} \tag{10}$$

The simplest case is

$$u_0 = \text{constant } (= U) \tag{11}$$

in the entire solenoid field, i.e., one has two homogeneous water layers, separated by a sloping interface.

Applying eq. (10) to only the upper water layer, u_0 falls out and one obtains an integral of the form

$$\begin{aligned} u' &= C_1' e^{\pi z/D} \cos(\pi z/D + c_1') + C_2' e^{-\pi z/D} \cos(\pi z/D + c_2') \\ v &= C_1 e^{\pi z/D} \sin(\pi z/D + c_1) + C_2 e^{-\pi z/D} \sin(\pi z/D + c_2) \end{aligned} \tag{12}$$

That says, the velocity components u', v in one part form a common "pure wind current", corresponding to the tangential stress acting at the sea surface, and in another part form an inverted "pure wind current", corresponding to a tangential stress acting on the internal interface from below. If the thickness of the water layer is approximately as large as D (or larger), the current mentioned firstly is practically determined by the tangential stress at the interface.

The integral valid for the bottom layer takes on a similar form. The eight integration constants which enter in total are determined by four pairs of conditions:

(1) the velocity is zero at the bottom of the sea;
(2) the velocity varies continuously from the lower to the upper layer;
(3) the same condition applied to the tangential stress, thus also for the derivatives du/dz and dv/dz;
(4) the tangential stress at the sea surface is determined by the wind.

The components u' and v have the same form when $d^2u_0/dz^2 = 0$, i.e., when u_0 is a linear function of z (a homogeneous solenoid field). Only the calculation of the integration constants, carried out in a way similar to the one described above, gives a change in the result. For example, let d_s be the depth of the solenoid layer, and U be the velocity at the surface, and assume that the pressure gradient in the bottom layer vanishes. Then, under the assumption made,

$$u_0 = \mathrm{U}(1 - z/d_s) \tag{13}$$

in the region of the solenoid layer.

I have earlier (using a more time-consuming method) calculated some cases of convection currents according to assumptions corresponding to eqs. (11) and (13), in all cases in the absence of wind (1906, pp. 574–576, Figs. 33 and 34).

As has already been mentioned, the integration of eq. (10) can be carried out with practically the same ease if u_0 is a polynomial after z. only the calculation of the integration constants becomes somewhat more complicated. Thus, it would not be difficult to write down a systematic list of solenoid fields corresponding to the most common types existing in the sea, and to show graphically the currents generated by these fields. Such a systematic list may possibly be published in another form, suitable for this purpose.

Of special interest are solenoid fields for which u_0 varies as a cosine function after z, according to the equation

$$u_0 = \tfrac{1}{2}U(1 + \cos \pi z/d_s) \tag{14}$$

where U and d_s have the same meaning as before. Such a solenoid field corresponds to some extent to a "discontinuity layer" (see the curved line in the figure below). From eq. (10) the following particular solution is obtained

$$u = u_0 + u' = \tfrac{1}{2}U\left(1 + \frac{4n^4}{1+4n^4}\cos\frac{\pi z}{d_s}\right)$$
$$v = \tfrac{1}{4}U\frac{n^2}{1+4n^4}\cos\frac{\pi z}{d_s} = \frac{1}{2n^2}(u - \tfrac{1}{2}U) \tag{15}$$

where $n = d_s/D$.

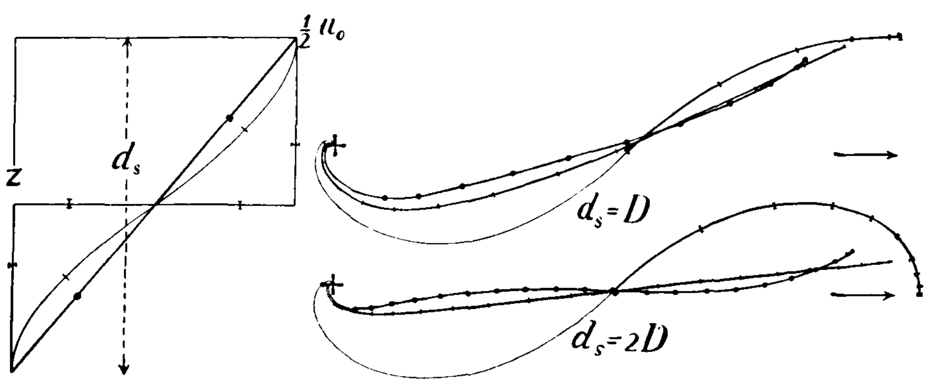
According to this solution, which satisfies the surface boundary condition in a windfree situation, all the geometric velocity differences between different levels will be *parallel*, as in the case of no friction. However, the magnitude of the velocity difference is reduced in a certain proportion μ, while at the same time all directions are turned *contra solem* a certain angle ν. The following short table gives an impression about the relation between μ, ν, and n, that is, the effect of the friction on the speed and direction of the current, in a solenoid field of the assumed kind, with different thicknesses:

n	μ	ν
2	0.985	7°
1	0.8	27°
0.75	0.56	42°
0.5	0.2	63°
0.25	0.015	83°

Equation (15) does not, however, satisfy the condition that the velocity should vary continuously from the solenoid layer to the underlying homogeneous water. Therefore, it becomes necessary to add a term of the form (12), and the corresponding "internal surface current" in the bottom water, in which case the current to some extent is modified, in particular in the lower part of the solenoid layer.

The figure shows a few examples of currents of the kind described above. The diagram to the left shows, in the form of a relation between u_0 and z, three types of solenoid layer: a completely discontinuous layer (the broken line), a homogenous solenoid field (the straight, slanted line), and the solenoid field given by eq. (14) (the curved line). The diagram to the right shows, for each of these solenoid fields, the elementary current it causes, under the assumption $d_s = D$ (above) and for a layer of double this thickness (below), in both cases under the assumption that there is no deep current. The velocity at each level is represented by an arrow drawn from the centre of the "Maltese cross" to a point on the corresponding curve. The fine part of the curves represents the "internal surface current" in the bottom layer, the heavy part represents the solenoidal layer, and in the latter case marks are made for the points of the arrows at equidistant levels, at distances of $0.1D$. The corresponding marks —, ●, I are furthermore inserted on the curves of the left diagram, and thus show the solenoidal field to which a particular velocity diagram belongs. The arrow point indicated, calculated with starting point in the "Malteser cross", shows the velocity u_0 which the same solenoidal field would produce in the absence of friction.

From the form of eqs. (8) or (10) it is obvious that the same diagrams are valid also in the case where there is a deep current, in the bottom water; the only difference being that the starting



point for the arrows must be moved, such that an arrow drawn *toward* the "Malteser cross" represents the deep current and further, that to the diagram must be added a logarithmic spiral, corresponding to the deep current (compare 1906, Figs. 13–15).

No fundamentally new effect is caused by the wind, in the calculation of the elementary current. Since eq. (10) is linear in its form, one must only add a pure wind current corresponding to the direction and strength of the wind, such that the velocity at every level is formed by the geometric sum of the two current components. In other words, the pure wind current penetrates the solenoidal layer in the same way as in homogeneous water, the only difference being the addition of the convection current in the former case. One should notice that the wind can also cause an increase in the frictional depth, and in this way affect also the convection current. However, we deal then with parts of the theory for which the necessary empiric laws still are lacking, and which parts we must omit here.

III

As has already been mentioned, one cannot calculate u_0 or v_0 completely from the hydrographic data, but an unknown constant remains. Thus it remains the task of determining this constant, that is, the speed and direction of the current at a specified depth, or equivalently, to calculate the shape of the sea surface, when one knows the density at every point, and the winds. This part of the theory is the most difficult, from a mathematical point of view. I have already (1923) considered this problem under the assumption of a sea of homogeneous water, that is, for wind currents alone. That study was limited to the mathematical formulation of the problem; a solution of the problem in a few particularly simple cases, which can hardly be applied under real, geographic conditions, and finally, a derivation of certain general laws, by help of which one may, to some extent, form a picture of the direction into which the theory would lead, in case it could be completely carried out.

A similar attempt, though in some respects even more limited, will be made here for the case in which the water is *not* homogeneous, in total or in every particular horizontal layer; in other words, under the assumption that a (general) solenoidal layer exists.

One concept which played a fundamental role in the corresponding study of wind currents, was the so-called "current transport" S, the components S_x, S_y of which are defined by the equations

$$S_x = \int_0^d u\mathrm{d}z, \quad S_y = \int_0^d v\mathrm{d}z$$

This "current transport" gives the water volume which flows, per unit time, between two vertical lines, drawn from the bottom to the surface. Two properties of the "current transport" formed the basis for the previous study, in particular: first, that the current mass in the "pure wind current" is directed perpendicular *cum sole* from the wind direction; second, that $\operatorname{div} S = 0$, the latter being a condition for the stationarity of the sea surface. No consideration was given in this case to changes in water volume due to precipitation or evaporation. Furthermore, it was assumed that the water had uniform density, and thus was incompressible. If one considers the compressibility, and the differences in the density q, one finds that the two properties just mentioned belong not to the current mass defined above, but to another vector S, with components defined by the equations

$$S_x = \int_0^d qu\mathrm{d}z, \quad S_y = \int_0^d qv\mathrm{d}z \tag{16}$$

One may call this for the current transport based on momentum instead of velocity.

As long as one considered currents in homogenous water, it was in reality of little importance whether one or the other definition of the current mass was used. In the extension of the problem made here, it turns out however, to be advantageous—in order that the theory should not lose its formal simplicity—to introduce the current transport defined through eq. (16), which, anyway, would have been the correct one from the start. In connection with this change, it has been found convenient, in accordance with eq. (2), to introduce a new quantity c instead for the quantity denoted in the same way in the paper of 1923, which has q in the denominator.

In close correspondence with the real conditions in the sea, we assume that the solenoidal field is limited to an upper layer, below the lower boundary of which—at a few hundred or possibly a few thousand meters depth—the water accordingly is homogeneous.

Analyzing the state of the motion, we can distinguish between the concepts defined below, which will appear in the following investigation:

G = the velocity of the uniform "*deep current*", which exists in the homogeneous bottom water, from the bottom current up the "internal surface current"; G_x, G_y denoting its components.

S'' = the current transport in the "*gradient current*". By this we mean the current resulting if all interfaces between water layers of different densities were placed parallel to the sea surface, but the slope of the latter then was adjusted such that the pressure gradient components ($\partial p/\partial x$ and $\partial p/\partial y$) in the homogenous bottom water remained unchanged. Under assumption of homogeneous, incompressible water, this would mean that the pressure gradient would be the same from the bottom to the surface; in incompressible water, on the other hand, it would be everywhere of the same direction and proportional to the density. In both cases, the current velocity everywhere above the bottom current would be equal to G.

γ = the slope of the sea surface, under the same assumptions. The components are

$$\gamma_x = (-1/qg)(\partial p/\partial x), \quad \gamma_y = (-1/qg)(\partial p/\partial y)$$

where p and q denote pressure and density in the region of the deep current, and g is the constant gravity.

$S'' + S'''$ = the current transport which would occur if the wind ceased, but the form and level of the sea surface, as well as the density of the water, remained the same, i.e., if the pressure gradient in the *entire* sea were equal to the existing one. S''' will be called the transport of the *convection current*.

S = $S' + S'' + S'''$ = the existing, total current transport. S' is the transport of the "*pure wind current*".

One may possibly object to the use of the name "convection current", as it does not cover the concept which usually is associated with the word convection current, i.e., a current due to inhomogeneities in the water. This may cause a gradient current also in the homogeneous deep water, while the current transport S'' only arises from currents in the region of the solenoidal field, and in the *uppermost* layer of the bottom water (the internal surface current). As an argument for the terminology introduced here, one may mention the inconvenience of introducing a new definition without necessity, and also the result—as will come out from the later investigation—that the *gradient currents* which, according to standard terminology, should be included in the "convection currents", in most cases can be assumed to be quite small.

The two-dimensional divergence and curl operators are denoted by div and curl in the following, thus, as example:

$$\operatorname{div} G = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}, \quad \operatorname{curl} G = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}$$

As a self-evident condition that the sea level should stay stationary, I have earlier (1923, p. 18) assumed that the entire current transport is divergence-free:

$$\operatorname{div} S = 0 \tag{17}$$

which says that the watermasses flowing into and out of a given region of the sea are equally large, and, as already mentioned, S should then correctly be the current transport defined by eq. (16).

When precipitation and evaporation are considered, it is, however, necessary to make a further study of the problem. Where the precipitation is larger than the evaporation, div S must be positive, to allow the sea surface to remain stationary, and where the evaporation dominates, div S must be negative. If, through the combined effect of precipitation and evaporation, the sea surface rises by n units of distance per second, it follows

$$\operatorname{div} S = n \tag{18}$$

An overview of the magnitude of the precipitation and evaporation in different parts of the sea can be obtained from an excellent study by G. Wüst[1] According to this study, the precipitation is in excess at latitudes above 40° N, and between 0° N and 10° N. In other zones the evaporation is in excess.

[1] G. Wüst, Die Verdunstung auf dem Meere—Veröffentl. der Inst. für Meereskunde, Neue Folge, Reihe A, Heft 6, 1920.

The largest transport of water within a connected Atlantic region caused by this effect comes from the excess of evaporation between 10° N and 40° N. This excess of evaporation would, according to Wüst, correspond to a yearly lowering of the sea surface of 94 cm at 10°–20° N, 95 cm at 20°–30° N, and 43 cm at 30°–40° N. Including the Central American and Mediterranean seas, a water volume of 620,000 m^3 will be removed to the air per second under this assumption, and it must be replaced from elsewhere. The water transports which are added from land through the rivers, are relatively small. Using data of the water transports in some larger rivers, and a comparison of the precipitation amounts corresponding to the regions of these and other river systems, I have approximately estimated the total water transport coming from the land regions (mainly from America) to be 30,000 m^3 per second, which, however, is found sufficient to replace the water which is reaching the Mediterranean through the Gibraltar Strait. The transport of 620,000 m^3 must thus flow from north and south across the latitudes 10° N and 40° N, along a line of total length 9400 km, that is, on an average 0.066 m^3 per meter length of this line. Thus, the current transport toward the region, between 10° N and 40° N, caused by the evaporation, is on an average 0.066 m^2/s.

The most obvious assumption is that the slope γ under consideration is directed perpendicularly to the boundary line of the region considered, and in any case, this must be the condition on an average, since curl $\gamma = 0$.

If, in accordance with this, we assume that the required current transport is due to the bottom current alone, one may calculate the velocity G of the bottom current which may be set up by the evaporation, by use of (1923) eqs. (15), (10) and (14). One finds

$$S'' = B\gamma = (B/k)G = (D''/2\pi)G \tag{19}$$

For the quantity D'', the "lower friction depth", appearing in the right hand side, I have (1923, eq. 25) formulated the hypothetical value

$$D'' = 600G \tag{20}$$

If the evaporation were the *only* reason for deep currents in this region of the sea, one obtains, by elimination of D'' between eqs. (19) and (20)

$$S'' = (600/2\pi)G^2 = 96G^2$$

thus

$$G = \sqrt{0.066/96} = 0.026 \text{ m/s} = 2.6 \text{ cm/s}$$

The lower friction depth corresponding to this value is 16 m.

The velocity G calculated here is, however, certainly too large, for several reasons:

(1) The coefficient 600 in eq. (20), obtained in analogy with the surface currents, is probably too small. While the surface water generally is stably stratified, to some degree, the bottom water is indifferent (neutral) and possibly in an unstable equilibrium[1], which must strengthen the turbulence and thus increase the virtual friction coefficient and the magnitude of the friction depth. By empirical arguments (1923, p. 74) I have also reasons to believe that eq. (20) will give too small values for D''. Too small a value for D'' will yield too large a value for G, according to eq. (19).

(2) In addition, the winds produce deep currents as well, generally increasing the total value for G. For this reason as well, D'' becomes larger, and accordingly the contribution to G from the evaporation becomes smaller.

(3) The depth of the sea is in reality not uniform, and the water transport into the region of evaporation will in such a case to a large part be carried by the deep current, thereby reducing the velocity of the current strongly (see 1923, §63).

As the calculation made above is, in its essential, applicable to other oceans as well, we can conclude that the deep currents arising from evaporation and precipitation, with near certainty, do not reach values of 3 cm/s as a maximum, but probably only reach a small fraction of one cm/s. In the latter case, they can be considered as unimportant, at least in the present stage of oceanographic research. In accordance, we disregard this cause of the currents, and instead of eq. (18) keep the simpler eq. (17).

IV

If one multiplies both sides of eqs. (1) with a mass element dm and integrates over the

[1] G. Schott. Adaibatische Temperaturänderung in grossen Meerestiefen. Ann. d. Hydr. u. Mar. Met. 1914, p. 321.

total mass of a vertical water cylinder, with unit cross-section area from the surface to the bottom, the left hand sides become the current transport components S_x and S_y. For a pure wind current (in homogeneous water and in the absence of a pressure gradient) one accordingly obtains, since bottom friction is absent,

$$S'_x = cT_y, S'_y = -cT_x \tag{21}$$

where T_x and T_y denote the components of the tangential stress by the wind. The equations are the same as (9) in (1923), the only difference being that the compressibility of the water is now taken into account. Since eqs. (1) are linear, they are valid as well for separate component forces X_1, Y_1, etc., into which we divide X, Y; and the component velocities u_1, v_1, etc. which these cause. Eqs. (21) are thus valid for the current transport of the pure wind current, even when solenoidal fields and pressure gradients simultaneously appear.

In the region of the uniform deep current there is no friction, and eqs. (1) thus give

$$G_x = gc\gamma_y, G_y = -gc\gamma_x \tag{22}$$

In the calculation of the transport of the gradient current (1923, eqs. 14 and 15), it was assumed that the speed of the bottom current varied after the distance from the bottom in accordance with the known logarithmic spiral diagram. Under the assumption of homogeneous, incompressible water, this requires a constant friction coefficient μ. With present assumptions, the condition for the logarithmic spiral diagram is that the *kinematic* friction coefficient μ/q is constant. Considering that the density variation within the region of the bottom current probably does not vary by more than some parts per thousand, while the uncertainty in μ certainly is many times larger, it seems obvious that the two conditions are practically satisfied. Thus, we can retain eqs. (15), 1923, mentioned above, that is, write

$$S''_x = B\gamma_x + b\gamma_y, \quad S''_y = B\gamma_y - b\gamma_x \tag{23}$$

Using the present notation the above coefficients B and b are

$$B = \frac{cgq''D''}{2\pi}, \quad b = cg\bar{q}d - B \tag{24}$$

where q'' denotes the water density in the region of the bottom current (considered constant), and $\bar{q}$ denotes the mean density in the sea, from the bottom to the surface.

In the following we make, in the beginning, the simplifying assumption that ϕ as well as d and D'' can be considered constant. With regard to q and g, they could immediately be considered as constant at each separate level. Then c, B, and b are also independent of x, y, and one finds by taking derivatives of eqs. (21), (22) and (23), considering that curl $\gamma = 0$:

$$\operatorname{div} S' = c \operatorname{curl} T \qquad \operatorname{curl} S' = -c \operatorname{div} T \tag{25a,b}$$

$$\operatorname{div} S'' = B \operatorname{div} \gamma \qquad \operatorname{curl} S'' = -b \operatorname{div} \gamma \tag{26a,b}$$

$$\operatorname{div} G = 0 \qquad \operatorname{curl} G = -gc \operatorname{div} \gamma \tag{27a,b}$$

The last two equations are valid at each separate level, in the region of the deep current. In the same way, one obtains from eq. (5) with V denoting the vector formed by the components u and v,

$$q \operatorname{div} V = \mu c \frac{\mathrm{d}^2 (\operatorname{curl} V)}{\mathrm{d}z^2}$$

$$q \operatorname{curl} V = c\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) - \frac{\mathrm{d}^2 (\operatorname{div} V)}{\mathrm{d}z^2}$$

Let us apply these equations solely on the gradient and convection currents, that is, making the assumption that no tangential stress acts at the sea surface. If one multiplies by dz and integrates along a vertical line from the sea surface ($z = 0$) to the lower level of the internal surface current current ($z = d_i$), the term furthest to the right disappears, since not only du/dz but also the derivatives of these with respect to x and y are zero at the two integration limits. The left hand sides of the equations will comprise the parts of div ($S'' + S'''$) and curl ($S'' + S'''$), respectively, which refer to the water layers between the sea surface and $z = d_i$. If one forms the equation mentioned, for the gradient current and the convection current, and also for the gradient current alone, and then subtracts the latter equations from the former ones, one finds in the left hand sides accordingly div S''' and curl S'''. Thus

$$\operatorname{div} S''' = 0, \operatorname{curl} S''' = c\int_0^{d_i} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) \mathrm{d}z + d_i cgq' \operatorname{div} \gamma \tag{28a,b}$$

where q' denotes the average value of q in the region of the convection current.

If one then subtracts eqs. (25a) and (28a) from eq. (17), one obtains

$$\operatorname{div} S''' = -c \operatorname{curl} T$$

or, considering equation (26a)

$$\operatorname{div} \gamma = -\frac{c}{B} \operatorname{curl} T$$

as in homogeneous water. Further, one finds from this, considering eq. (27b)

$$\operatorname{curl} G = \frac{gc^2}{B} \operatorname{curl} T$$

independent of the existence of a solenoidal field. And, finally, one obtains from eq. (28b), since curl T and consequently also div γ are zero for the pure convection current,

$$\operatorname{curl} S''' = c \int_0^{d_i} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) \mathrm{d}z$$

We may summarize some of the results obtained as follows:

1. The transport of the convection current (the internal surface current included) is divergence-free, but in general it is not irrotational; rather, it has a curl depending on the solenoidal field.

2. The uniform deep current is divergence-free, but has a curl which only depends on the wind field. The deep current, which is solely caused by the solenoidal field, is thus divergence-free and irrotational (it could be found by solving Laplace's equation at the level of the sea surface), and must accordingly, in general, be relatively small. If, at some place in the ocean, by some reason, light surface water is produced and retained, as in the Sargasso Sea, a convection current will encompass this water *cum sole* in the surface region of the solenoidal layer; but this circulation will not, apart from an internal surface current, reach into the homogeneous deep water, unless the winds prevailing in this region also form an anticyclonic air vortex.

What has been stated here with regard to the divergence and curl of the convection currents, relates, as has already been mentioned, to the momentum of the currents, that is, mass times velocity. It is to be noticed, however, that the horizontal differences in water density in a uniform current region hardly can exceed 1 or 2 parts per thousand. The differences in the current speed of this percentage magnitude lie far below the level of accuracy which can be reached at present, either in direct observation or in calculation of the current. As long as this accuracy is not essentially raised, one may thus apply the previous theorems to the velocity of the water.

The problem becomes more complex when we consider the differences in the depth of the sea d, and the friction depth D, and the latitudinal difference, that is, the spherical shape of the earth. In these respects I can only give some general indications. As in the wind current theory (1923, §21), one may conveniently direct the x-axis eastward, and the y-axis along the meridian, in the direction of the north pole. If the earth's radius is denoted by R, and b/B by $\tan \beta$, and if one considers that, for an increase or decrease of the depth of the sea d, the added or subtracted water column has a density q'', one finds, as in the earlier study, but in the new notations:

$$\begin{aligned}
&\frac{\partial c}{\partial x} = 0 \\
&\frac{\partial c}{\partial y} = -\frac{c}{R \tan c\phi} \\
&\frac{\partial B}{\partial x} = \frac{B}{D''} \frac{\partial D''}{\partial x} \\
&\frac{\partial B}{\partial y} = \frac{B}{D''} \frac{\partial D''}{\partial y} - \frac{B}{R\ tan\ \phi} \\
&\frac{\partial b}{\partial x} = cgq'' \frac{\partial d}{\partial x} - \frac{B}{D''} \frac{\partial D''}{\partial x} \\
&\frac{\partial b}{\partial y} = cgq'' \frac{\partial d}{\partial y} - \frac{B}{D''} \frac{\partial D''}{\partial y} - \tan \beta \frac{B}{R \tan \phi}
\end{aligned} \qquad (29)$$

The fact that q and g as well are functions of x and y is not considered. The terms, which accordingly are neglected, are, however, so small in relation to the the derivatives of c, d, and D'', that they hardly can affect the results noticeably.

Considering the above equations, one finds

$$\operatorname{div} S' = c \operatorname{curl} T + \frac{cT_x}{\mathrm{R} \tan \phi} \tag{30}$$

$$\begin{aligned}\operatorname{div} S'' &= B \operatorname{div} \gamma \\ &+ \frac{B}{D''}\left[(\gamma_x - \gamma_y)\frac{\partial D''}{\partial x} + (\gamma_x + \gamma_y)\frac{\partial D''}{\partial y}\right] \\ &+ cgq''\left(\gamma_y \frac{\partial d}{\partial x} - \gamma_x \frac{\partial d}{\partial y}\right) \\ &+ \frac{B}{R \tan \phi}(\gamma_x \tan \beta - \gamma_y)\end{aligned} \tag{31}$$

$$\operatorname{div} G = \frac{gc\gamma_x}{R \tan \phi},$$

$$\operatorname{curl} G = -gc \operatorname{div} \gamma + \frac{gc\gamma_y}{R \tan \phi} \tag{32a,b}$$

The result, to this extent, is completely independent of the solenoidal field. To calculate div S''' we start from eqs. (5), or better, from the corresponding equations which do not assume that μ is independent of z. If we denote the tangential stress, which is exerted at an arbitrary level z of an overlying water layer on an underlying layer, by τ (components: τ_x, τ_y), eqs. (5) could be replaced by

$$qu = -c\frac{\partial p}{\partial y} - c\frac{\partial \tau y}{\partial z}, \quad qv = c\frac{\partial p}{\partial x} + c\frac{\partial \tau_x}{\partial z}$$

Taking derivatives of these with respect to x and y, respectively, and adding, one obtains, considering eq. (29)

$$\begin{aligned}q \operatorname{div} V &= -\frac{c}{R \tan \phi}\frac{\partial p}{\partial x} \\ &- \frac{c}{R \tan \phi}\frac{\partial \tau_x}{\partial z} - c\frac{\partial(\operatorname{curl} \tau)}{\partial z} \\ &= -\frac{qv}{R \tan \phi} - c\frac{\partial(\operatorname{curl} \tau)}{\partial z}\end{aligned}$$

If, finally, one multiplies with dz and integrates from $z = 0$ to a level $z = d_i$ at the lower boundary of the internal surface current, one obtains, since $\tau = 0$ at $z = d_i$ and $\tau =$ the wind stress T at $z = 0$,

$$\int_0^{d_i} q \operatorname{div} V \,\mathrm{d}z = -\frac{1}{R \tan \phi}\int_0^{d_i} qv \mathrm{d}z + c \operatorname{curl} T$$

If one forms this equation, firstly under existing conditions, and secondly under assumed changed conditions, with the solenoid field removed, but with the deep current as well as the wind unchanged, one obtains an equation, the left side of which is div S‴. The right hand side is $S_y'''/R \tan \phi$, since the last two terms cancel. Thus

$$\operatorname{div} S''' = -\frac{S_y'''}{R \tan \phi} \tag{33}$$

which equation corresponds completely to eq. (43) in (1923).

Considering eq. (17), one obtains, by addition of eqs. (30), (31) and (33), division by B and rearrangement of terms:

$$\begin{aligned}\operatorname{div} \gamma &= -\frac{c}{B}\operatorname{curl} T - \frac{1}{D''} \\ &\times \left[(\gamma_x - \gamma_y)\frac{\partial D''}{\partial x} + (\gamma_x + \gamma_y)\frac{\partial D''}{\partial y}\right] \\ &- \frac{cgq''}{B}\left(\gamma_y \frac{\partial d}{\partial x} - \gamma_x \frac{\partial d}{\partial y}\right) - \frac{1}{R \tan \phi} \\ &\times \left(\gamma_x \tan \beta - \gamma_y + \frac{c}{B}T_x - \frac{S_y'''}{B}\right)\end{aligned}$$

If, finally, one multiplies this equation by $-gc$, and moves two of the terms in the right hand side over to the left hand side, one obtains, considering eq. (32b)

$$\operatorname{curl} G - \frac{gc}{D''}\left[(\gamma_x - \gamma_y)\frac{\partial D''}{\partial x} + (\gamma_x + \gamma_y)\frac{\partial D''}{\partial y}\right] = \frac{gc^2}{B}\operatorname{curl} T +$$

$$+\frac{c^2g^2q''}{B}\left(\gamma_y\frac{\partial d}{\partial x}-\gamma_x\frac{\partial d}{\partial y}\right) \qquad (34)$$

$$+\frac{cg}{R\tan\phi}\left(\gamma_x\tan\beta+\frac{c}{B}T_x-\frac{S_y'''}{B}\right)$$

This equation, which corresponds to eq. (42) in (1923), contains a new term representing the effect of the convection current on the curl of the deep current. It is found that as a rule the solenoidal field in the upper water layer must to some extent affect the underlying, homogeneous water, and generate a deep current. For if no wind nor any deep current existed (that is, curl $T = T_x = \gamma_x = \gamma_y = 0$) the equation would be reduced to

$$\text{curl } G = -\frac{cg}{BR\tan\phi}S_y'''$$

from which it is seen that the deep current can be absent only at single places (single points), in cases where no winds act at the sea surface and S_y''' is different from zero. The velocity of the deep currents caused by the solenoidal field are, however, likely to be small. The east-west transport component of the convection current, which must be dominating at low latitudes, has, as can be seen, no effect on the deep current; and at higher latitudes, where $\tan\phi$ is not a small number, the coefficient for S_y''' in eq. (34) becomes small. In the case of the Gulf Stream region, between the Florida Strait and 40° N, a closer examination of the different terms of the equations shows, however, that a deep current must occur, and that it—and thus the total current—must run in a direction toward increasing depths, that is, on its way northward gradually move away from the coast. This is, however, only mentioned in passing. The use of eqs. (32a) and (34) on such detailed oceanographic problems requires—apart from necessary improved studies at sea—a continued formal development of the theory, to which I hope to return later on.