Tellus (1985), *37A*, 286–296

A laboratory study of baroclinic chaos on the f-plane

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(Manuscript received August 13; in final form November 26, 1984)

ABSTRACT

Laboratory experiments on the transition to chaos in a two-layer rotating fluid are described. The topography of the upper and lower bounding surfaces, and the rotation rate of the apparatus, are chosen so that the f-plane case of uniform basic potential vorticity in each layer is reproduced. Experiments are carried out in the near-critical single-wave region of parameter space where previous theory suggests a snap-through bifurcation from steady baroclinic waves to chaos. Instead, as the friction parameter is decreased past the curve of neutral stability of the axisymmetric basic state, we find several bifurcations before chaos is observed. The transition scenario includes successive limit cycles of doubled periods between the steady-wave and chaotic regions of parameter space. There is also evidence of ageostrophic behavior, even when the system Rossby number is less than 0.1.

1. Introduction

The baroclinic instability is a fundamental and significant process in many astronomical and geophysical flows, especially in the atmosphere and oceans. There has been much work on linear stability problems using both idealized and realistic basic flows. These have shown that baroclinic waves, growing by instability, should exist in a range of settings. Indeed, the wave structures predicted by linear theory are often similar to those seen in field observations. An important aspect of baroclinic instability is the fate of exponentially growing infinitesimal perturbations. As the amplitudes become finite, non-linear effects must become important and contribute to the time-asymptotic state of the system. This paper addresses the questions of non-linear equilibration and transition to chaos by making observations of a relatively simple baroclinic flow in the laboratory.

The flow that we study is a two-layer rotating fluid contained in a cylinder and driven by a differentially rotating upper lid. On the basis of a perturbation analysis for small differential rotation (Hart, 1972), it has been shown that this system should be described by the usual quasi-geostrophic two-layer vorticity equations. There have been a substantial number of theoretical and numerical studies of these equations in relation to non-linear baroclinic wave processes (see Hart (1979), for a review). Of particular interest here is the recent work of Pedlosky and Frenzen (1980). They investigated the nature of solutions of amplitude equations derived by a weakly non-linear expansion of the two-layer equations for baroclinic instability in an f-plane channel. That is, they studied the weakly non-linear dynamics of a two-layer zonal shear flow contained between two latitudinal walls. In the absence of a basic zonal current, the layer depth is constant, giving uniform basic potential vorticity or an "f-plane". If a simplification of the lateral boundary condition on the zonal component of the flow is employed, the equations for the slow-time behavior of a single wavenumber disturbance and mean flow correction reduce to the classical three-component convection equations of Lorenz (1963). Thus, the simplified problem describes a scenario for evolution of baroclinic waves equivalent to that for thermal convection of one mode between free boundaries. The Lorenz equations predict that as the system becomes supercritical, the instability first equilibrates to a constant amplitude. As the measure of the supercriticallity is raised, the amplitude becomes larger until at a second critical point there is an abrupt transition to chaotic behavior. This transition is sometimes called a snap-through bifurcation. Pedlosky and Frenzen showed that their model behaved similarly as the product of the linear growth rate and the viscous spin-down time was increased.

When the correct zonal flow boundary condition is used, the weakly non-linear evolution equations no longer reduce to a three-component set. Instead there is one component for the wave amplitude, one for the wave phase, and an infinite set of equations for the zonal flow. When this infinite set is truncated at a large finite value, Pedlosky and Frenzen (1980) showed by numerical integration that the scenario for transition to chaos on the f-plane was similar in structure to that for the three-component model. There are aditional complications, described further below, of windows of periodicity in the chaotic regime. As in the Lorenz model, there is an ultimate transition to periodic amplitude vacillation at small friction. It should be emphasized that the Pedlosky-Frenzen model is derived by formal asymptotic expansion, as opposed to the Lorenz equations which were obtained by an arbitrary truncation and whose interesting behavior occurs well outside their range of applicability (where other modes or three-dimensional effects enter in real flows). That is, the baroclinic model is expected to be valid near the curve of neutral stability. Since there are *two* governing dimensionless parameters in this problem, the viscous friction parameter and the inviscid Froude number (defined below in Section 2), it is possible to be both near the neutral curve *and* have almost inviscid behavior at the same time. This is not possible in the Lorenz equations since there is only one viscous parameter, the Rayleigh number. To make the convective flow relatively inviscid, one has to move well away from the neutral curve. For this reason the Pedlosky–Frenzen model is of substantial theoretical interest, since it is one of the few that self-consistently predict chaos in a hydrodynamic system.

One of the crucial questions that a laboratory study can address is that of convergence. Although the weakly non-linear model is asymptotically correct close to the neutral curve, and predicts chaos at small friction, is its region of validity large enough to be of practical significance? In other words, on the (Froude number–friction parameter) plane, does the transition go from steady waves directly to chaos over a reasonably substantial region, or is there another scenario that operates over all but a small area extremely close to the neutral curve? This is the issue addressed by the present study.

2. Statement of the problem

Fig. 1 shows a cross section of the experiment. A cylinder of radius $L = 22.8$ cm holds two immiscible fluids of nearly equal density $\rho_1 < \rho_2$. The upper layer fluid is 1.5 cs silicone oil and the lower is a mixture of about 75% ethyl alcohol and 25% water. The viscosities of each layer are adjusted to be equal within 15%. The system is rotated at rate Ω. The equilibrium interface with no differential driving ω is a parabola with height $z = \Omega^2 r^2/2g$, where r is the radius measured from the axis of rotation. The top and bottom surfaces are machined to within 0.003 cm of this parabola (for one value of Ω, 2.856 rad s^{-1}). Thus the equilibrium depths in each layer at this Ω are each a constant value of $D = 13$ cm. In this manner the equilbrium potential vorticity $2\Omega/D$ is constant, giving a simulation of f-plane dynamics. Motions are measured by two 0.0025 cm platinum wire probes stretched through the interface. One is on the axis, and one is at radius $r \simeq 0.75L$, as shown in Fig. 1. These miniature tide gauges measure the interface height which is proportional to electrical impedance to ground in the lower conducting fluid.

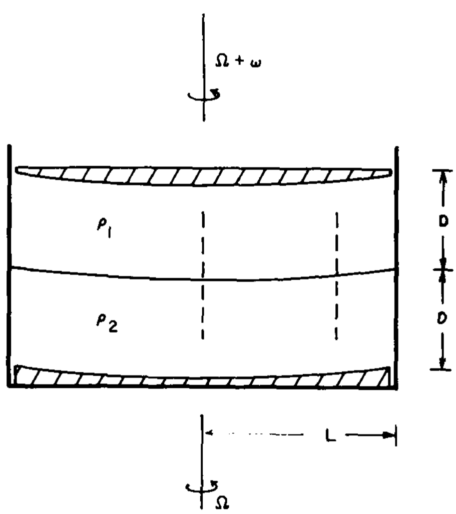

Fig. 1. Cross-section of the experiment. Vertical dashed lines indicate probe locations.

Further experimental details are given in Hart (1972). The only major differences are in the size of the apparatus, and the stability of the rotators, now being computer-controlled stepper motors with fluctuations of less than one part in 10^5.

Besides the constant aspect ratio D/L and nearly unit viscosity ratio, two parameters characterize the fluid motion outside narrow boundary layers at the sidewalls. These are the inviscid Froude number

$$F \equiv \frac{4\Omega^2 L^2}{g' D} \tag{1}$$

and the friction parameter

$$Q = \frac{\sqrt{2\Omega\nu}}{D\omega}, \tag{2}$$

where ν us the kinematic viscosity and g' the reduced gravity $(2g(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$.

When the Rossby is $\mathrm{Ro} = \Omega/2\omega$, asymptotically small, the dynamics should be decribed by the quasi-geostrophic equations derived in Hart (1972):

$$\left[\frac{\partial}{\partial t} + J(P_i, \quad)\right][\nabla^2 P_i + F(-1)^i (P_1 - P_2)]$$
$$= -|Q| \frac{\nabla^2(P_i - 2\delta_{1i})}{\sqrt{2}}$$
$$- |Q| (-1)^i \frac{\nabla^2(P_2 - P_1)}{\sqrt{8}} \tag{3}$$

Here P_i are the non-dimensional pressure fluctuations in each layer $i = 1, 2$, and $J(P_i, \quad)$ are the two-dimensional Jacobian operators representing advection in the ith layer.

In cylindrical coordinates, these equations should be solved using the boundary conditions that $P = 0$ at $r = 1$ ($r^* = L$) for the θ varying (wavelike) part of the flow, and $\partial^2 P/\partial r\, \partial t = 0$ at $r = 1$ for the zonal component of the flow. Regularity is imposed at $r = 0$. It is first noticed that there is a simple axisymmetric flow solution with solid body rotation in each layer at dimensional rates 0.75ω and 0.25ω in the upper and lower layers, respectively. This flow can become unstable to baroclinic waves with integer wavenumbers n. A solution for the amplitude of a single instable wave following the weakly non-linear expansion of Pedlosky (1971) indicates that there are no fundamental differences between the cylindrical version and the channel flow model. Eq. (3) can also be integrated numerically. We represent the pressure P as a sum of the orthogonal eigenfunctions of the linear stability theory. These are simply Bessel functions of the first kind. The partial differential equations are then converted into an infinite set of non-linear ordinary differential equations for the time-dependent amplitudes of the various spatial harmonics. If one retains the same spatial structure as is present in the weakly non-linear model, one reproduces almost exactly (for slow-time behavior) the qualitative behavior of the channel model.

Several numerical integrations of eqn. (3) without interfacial friction have been carried out for $F = 10$. Azimuthal wavenumber 1, radial mode number 1, is the only wave retained in the calculation as this is the single linearly unstable wave at this value of F (see the solid neutral curves in Fig. 3). Consistent with the Pedlosky–Frenzen model, a large number of zonal radial modes are also retained. Fig. 2 shows several solutions for decreasing values of Q along the line $F = 10$. The solutions are displayed as phase space orbits of the wave amplitude $X(t)$. In Fig. 2*a*, the axisymmetric flow is unstable, but the wave spirals in to a fixed point and ends up propagating around the cylinder with steady amplitude. At slightly lower "friction" Q, the orbits fill in on a sheet that looks remarkably like the chaotic attractor in Lorenz (1963). In Fig. 2b, only a few orbits are shown. They never reconnect. However, as Q decreases further, the chaos gives way to a reverse periodic cascade involving multiply looped periodic orbits (Fig. 2c), and eventually a period-one limit cycle that dominates for all very small Q values. In between the singly periodic limit cycle and the onset of chaos at about $Q = 0.025$, there are also islands of periodicity as found by Pedlosky and Frenzen (1980).

In summary, the single-wave and the weakly non-linear models both predict a transition directly from steady waves to chaos. This is followed ultimately by a reverse cascade involving successive period halvings until a period-one (single looped) limit cycle is found. Numerical experiments or calculations including interfacial friction are similar except that the values of Q should be divided by 2. The following sections describe experimental results that offer a different transition scenario.

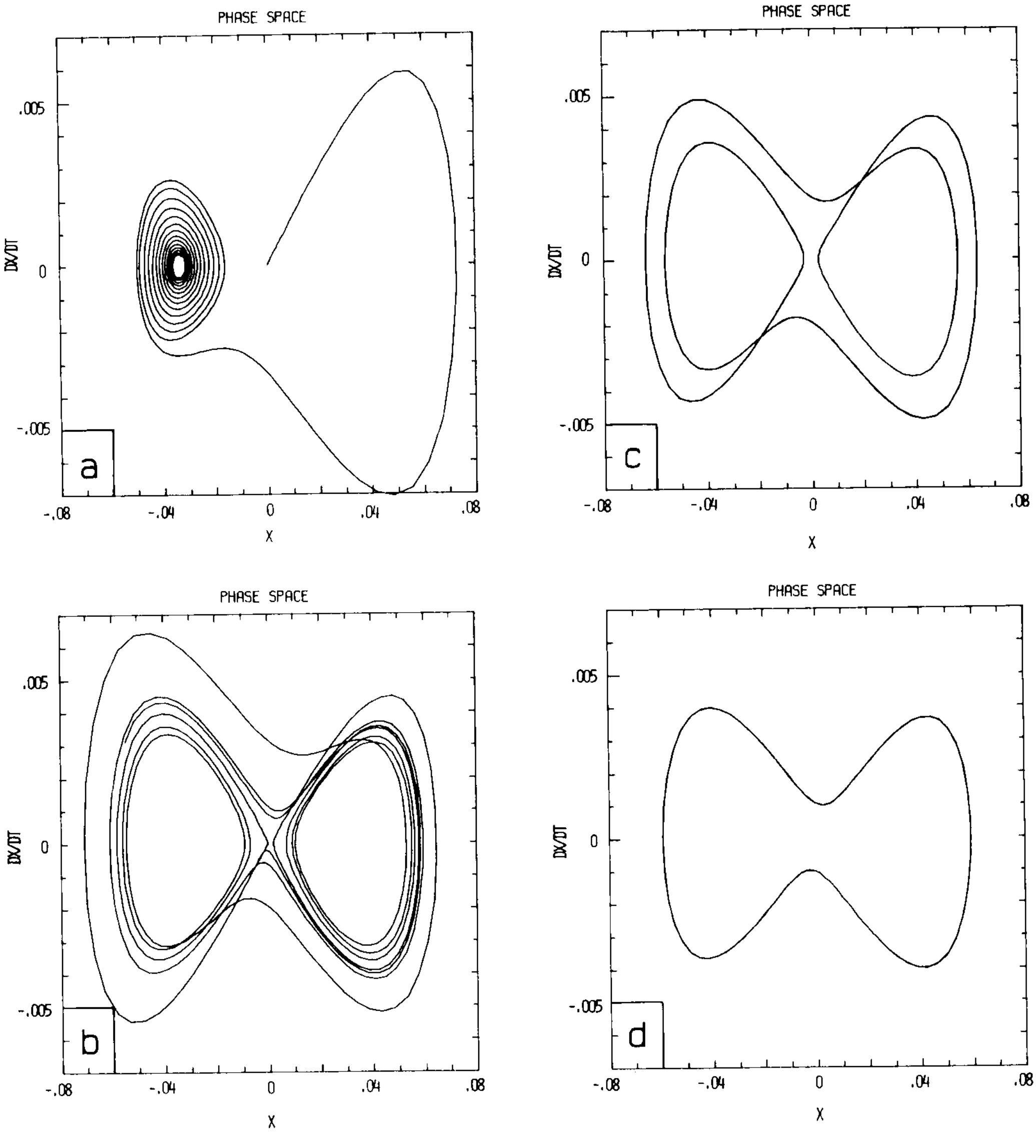

Fig. 2. Phase plots for the wave amplitude in a single-wave model. $F = 10$. (a) $Q = 0.03$, steady. (b) $Q = 0.023$ chaotic. (c) $Q = 0.017$ period 2 limit cycle. (d) $Q = 0.013$ period-1 limit cycle.

3. Experimental results

By monitoring the off-axis probe, one obtains a time series of "wave-height", where we specifically mean a signal that is dominated by the azimuthally varying travelling waves. In contrast, the on-axis probe measures an interface height whose only contribution comes from the azimuthally invariant or zonal component of the flow. This shall be called the "zonal-height".

Experiments were run by filling the tank to a specified density difference. This sets the Froude number, since the rotation rate is fixed to match the top and bottom topography. The fluid is spun up for two hours with $\omega = 0$. Then the differential driving is turned on to a value outside the neutral curve so that the flow should be axisymmetric. After a two-hour waiting time, data from the probes are digitized and recorded on tape for about 10 h. The computer then increases the driving rate ω slightly (typically 5%, although as small as 0.5% near a transition point from one type of motion to another.) After another waiting period, data is again recorded. Thus horizontal lines are traced out on the F–Q plane, going from the axisymmetric region at large absolute Q to the highly non-linear region near $Q = 0$. Although the quasi-geostrophic equation (3) is invariant with respect to the sign of Q, experiments were carried out for both positive (co-rotating lid) and negative (counter-rotating lid) Rossby numbers.

Fig. 3 shows a regime diagram built up from many runs, each about 2 weeks long. For reference, quasi-geostrophic stability curves are shown for wavenumbers 1 and 2. The region of parameter space between these two neutral curves at low F is the only region for the f-plane where one might expect chaotic single-wave behavior. At larger F, the flows seem to contain many zonal wavenumbers. For positive Rossby number ($Q > 0$), the axisymmetric flow becomes unstable near the theoretical neutral curve and equilibrates to a steady wave. As Q is decreased, instead of bifurcating to a chaotic flow, the motion becomes periodic first. At low Q, the periodic regime gives way to chaos. At very small F, steady flow is observed down to $Q \simeq 0.007$

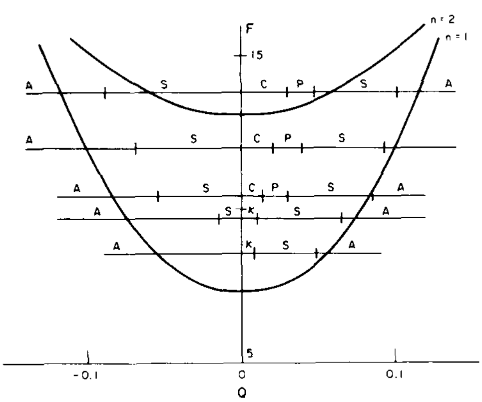

Fig. 3. Experimental regime diagram. Solid curved lines show wavenumber $n = 1$ and $n = 2$ quasigeostrophic neutral curves. Horizontal lines show experimental tracks. A = axisymmetric flow (no waves). S = steady waves. P = periodic wave amplitudes. C = chaotic. K = high frequency K–H instability near outer sidewall.

whence a Kelvin–Helmholtz instability occurs near the outer sidewall at $r = 1$. This is a consequence of the high shear across the interface required to get to very small Q. There might be behavior akin to that described by the quasi-geostrophic theory in the small K region of Fig. 3, but because the depth cannot be made larger, it is not possible to investigate it at the smaller ω required to avoid the K–H instability.

Notice that the negative Q behavior is substantially different. There is no chaos or periodic motion, and the axisymmetric-wave transition is not well described by the quasi-geostrophic theory. We shall come back to these observations in Section 4.

To investigate the nature of the transition to chaos observed for $Q > 0$ in more detail, it is useful to perform phase space and spectral analyses on the interface height measurements. As described in Farmer et al. (1982), a phase-space reconstruction of the single time series can be made that allows comparison with trajectories as in Fig. 2, as well as with other theoretical notions. Fig. 4a shows a typical time trace of wave-height $H(t)$ in the periodic region of Fig. 3. Units are uncalibrated and appear in the figure directly as output from the analog to digital converter. One can reconstruct a projection of the underlying phase space attractor (the set to which trajectories contract after an initial transient) simply by plotting $H(t)$ *versus* $H1(t) \equiv H(t + \tau)$, where τ is a lag time chosen to be less than the shortest period in the signal. Varying τ changes the shapes of the projected attractors, but not their topological equivalence. A limit cycle stays a limit cycle, etc.

Fig. 4b show the attractor for the same parameters of Fig. 4a, but for a longer time series. The orbits fall onto a torus (which would fill in if still longer data set had been used). This indicates the presence of two incommensurate frequencies in the wave-height time series, an observation verified by spectral analysis as well. A Poincaré cross section through the attractor is made by plotting $H1 \equiv H(t + \tau)$ *versus* $H2 \equiv H(t + 2\tau)$ for those points that cut a specified value of H in a long record. This is shown in Fig. 4c. Apart from instrumental noise arising primarily from microscopic dirt on the platinum probe wires, the points outline the skin of a torus. In the chaotic region, the wave-height vacillations fluctuate randomly and the underlying torus becomes fuzzy (see also Farmer et al. (1982)). This is shown in Fig.

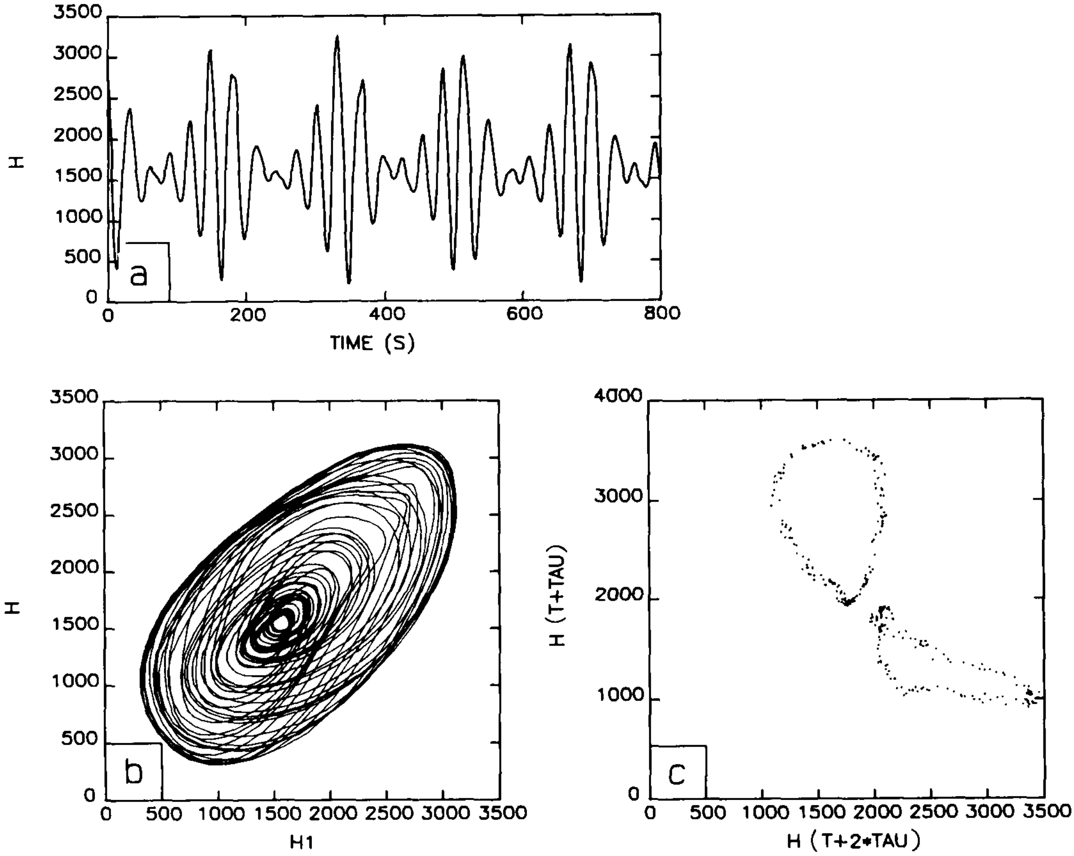

Fig. 4. Waveheight measurements in the periodic regime for $F = 11.8$, $Q = 0.028$. (a) Waveheight versus time. (b) Projection of the phase-space attractor (see text). (c) Poincaré section through $H = 2000$ of (b).

5a, b. The wave-height spectrum, in Fig. 5c is broad-band with a wave-peak at about 0.23 rad s^{-1}, and some distinct low-frequency peaks at 0.05 rad s^{-1} and smaller.

Now linear instability theory suggests that waves should drift around the tank at a fixed phase speed. Weakly non-linear oscillations on the f-plane do not produce changes in the phase speed. Thus, one of the frequencies in the wave-height field is removable by a rotation of the probe coordinates. This of course is hard to do physically, and filtering the large wave-height signal out of the time series is also difficult. However, the non-linear single-wave calculations also feature a strong energy interaction between the wave and the zonal flow. Thus, it is advantageous to avoid complications with the drift frequency by looking at the zonal flow, as measured by the zonal height probe on the axis of rotation. In this data set, the wave-drift fluctuations are not present. In this case one can see some details in the transition scenario not obvious in the wave-height measurements. This is one advantage of the full cylinder experiment over the annulus configuration.

Fig. 6 shows a set of zonal height time series in the P and C regions of Fig. 3. There is quite a bit of instrumental noise on these signals as the zonal height fluctuations are relatively small. Nonetheless, one can see that there is some structure on the left-hand edge of the P region. In particular, Fig. 6b shows that the originally periodic zonal-height modulation has become "period-4", repeat-

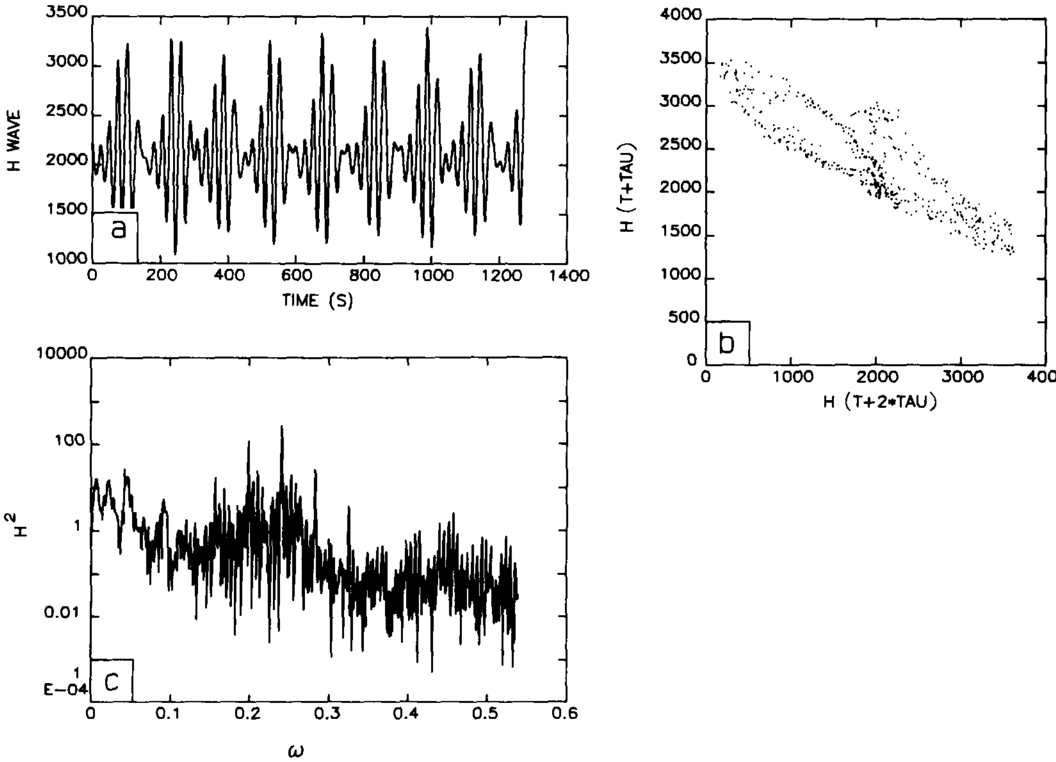

Fig 5. Wave responses in the chaotic region for $F = 11.8$, $Q = 0{\cdot}024$. (a) Time trace. (b) Section through the waveheight attractor. (c) Power spectrum of waveheight.

ing itself every fourth cycle. A period doubling cascade occurs just before the chaos. This is seen a little more clearly in the frequency spectra of Fig. 7. The sub-harmonics emerge prior to a general lifting of the background noise level and broadening of the spectral features. There is first the vacillation frequency f_0 in Fig. 7a. After a bifurcation, a new peak with frequency $f_0/2$ appears. Then, in Fig. 7c, a frequency $f_0/4$ becomes significant. Because of the instrumental probe noise, further bifurcations are not visible in the very narrow Q range before chaos sets in.

Fig. 8 shows the projected zonal height attractors. One can see the banded structure of the period 4 limit cycle, and the relatively simple looking chaotic attractor of Fig. 8c. In comparison, the noisy wave-height torus that generated Fig. 5c looks like a pile of spaghetti. That the chaotic motions may be of somewhat low fractal dimension is suggested by cross sections through the chaotic attractors. One is shown in Fig. 9. Rather than scattering over the whole plane, the points tend to lie near a curve. This is typical of chaos arising from low-order sets of ordinary differential equations.

It should be mentioned that the period doubling route to chaos has been observed in a number of other fluid systems (Swinney (1983) presents a nice review). Notable are the convection experiments of Libchaber et al. (1983) where 5 period doubling bifurcations are seen in a very quiet low Prandtl number convection experiment where higher modes are suppressed by the application of a magnetic field. Feigenbaum (1983) reviews his earlier theory of the period doubling cascade. He suggests that hyperbolic (unstable) orbits in a forced-dissipative non-linear system must, when they become large enough, be folded back and re-injected into the interior of the attractor (as in Fig. 8c). This exponential growth and folding can be modelled by

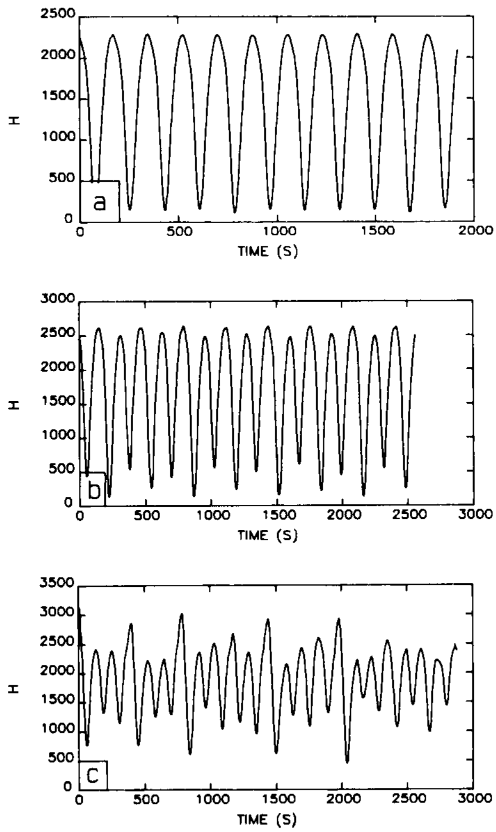

Fig. 6. Zonal flow time traces $F = 11.8$. (a) $Q = 0.0298$. (b) $Q = 0.0265$. (c) $Q = 0.0226$.

a quadratic iterative mapping that has universal properties. In particular, he shows that after a large number of bifurcations, the ratio of the parameter space widths (here ΔQ) of successive period-2^m orbits should approach a universal number 4.669 .... In our experiments, this ratio of the period-2 width to the period-4 width is about 2.7. One should not attach much significance to this since the period-4 cycle is certainly not asymptotic, and the extreme difficulty in measuring the precise location of the bifurcation into and out of the period-4 cycle puts very large error bars on this number.

The above discussion has concentrated on the case where $F = 11.8$. Similar behavior is observed at the left edges of the other periodic regions in Fig. 3.

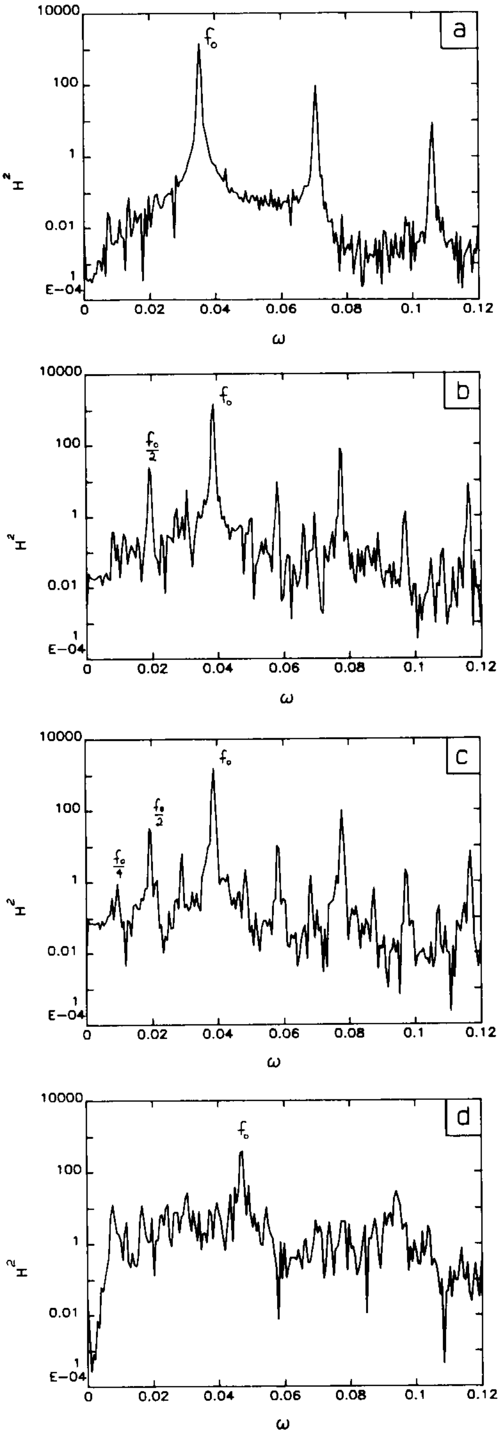

Fig. 7. Zonal flow spectra for $F = 11.8$. (a) $Q = 0.0298$. (b) $Q = 0.0268$. (c) $Q = 0.0265$. (d) $Q = 0{\cdot}0226$.

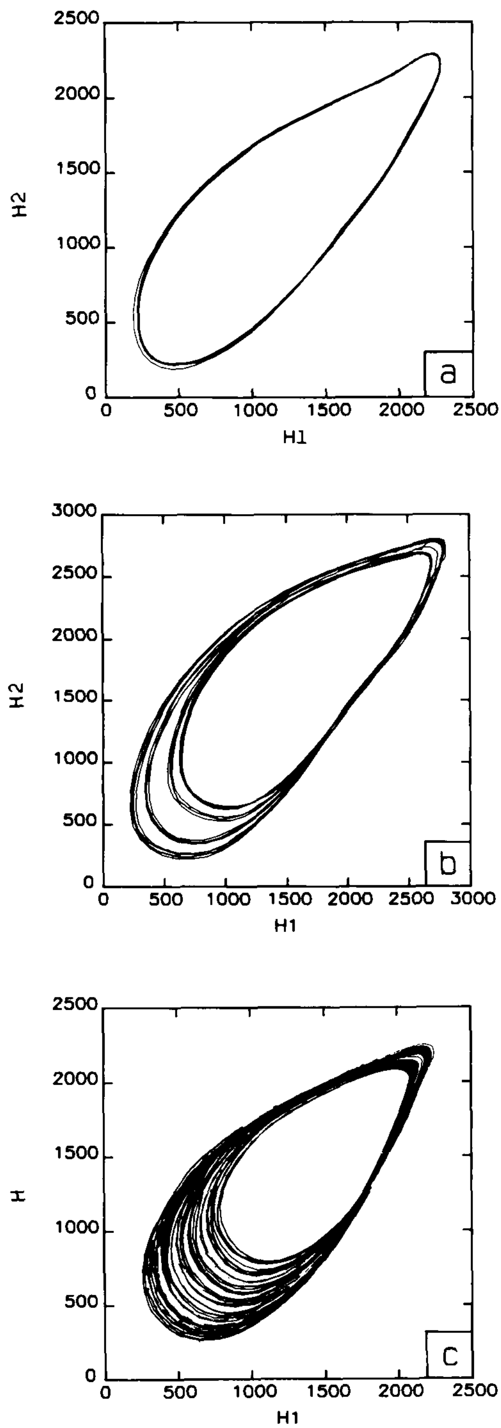

Fig. 8. Zonal flow projected attractors for (a) $Q = 0.0298$. (b) $Q = 0.0265$. (c) $Q = 0.0245$.

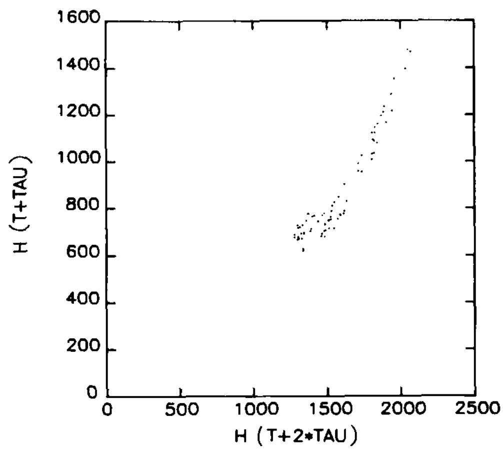

Fig. 9. Poincaré section through $H = 1000$ of Fig. 8c.

4. Summary and conclusions

It has been shown that the transition to chaos on the f-plane is via a period-doubling cascade, at least in the $n = 1$ unstable region where Kelvin–Helmholtz instability does not occur first. This is, of course, contrary to the weakly non-linear and single-wave quasi-geostrophic models which predict a bifurcation directly from steady waves into chaos. The present results suggest that there are physical processes not included in these models which limits their applicability to a very small area of parameter space, which is probably not accessible experimentally. Said another way, the models appear to be structurally unstable, in that the addition of ostensibly small effects like wave-wave interactions or ageostrophic accelerations, cause the Lorenz attractor and attendant bifurcations to disappear, to be replaced by a different transition scenario.

The measurements at negative Q indicate that ageostrophic effects are significant even when the Rossby number is less than 0.1, as it is for the steady (S) to periodic (P) transitions. There are several possible mechanisms by which ageostrophy might influence the experiment. If the Ekman layers are affected by centrifugal (or non-linear) terms, the basic state may be altered. The equilibrium interface then may not line up directly with the surface parabolae, and then one might expect a sort of $\pm\beta$-effect to occur. It has already been found (Hart, 1984) that the transition

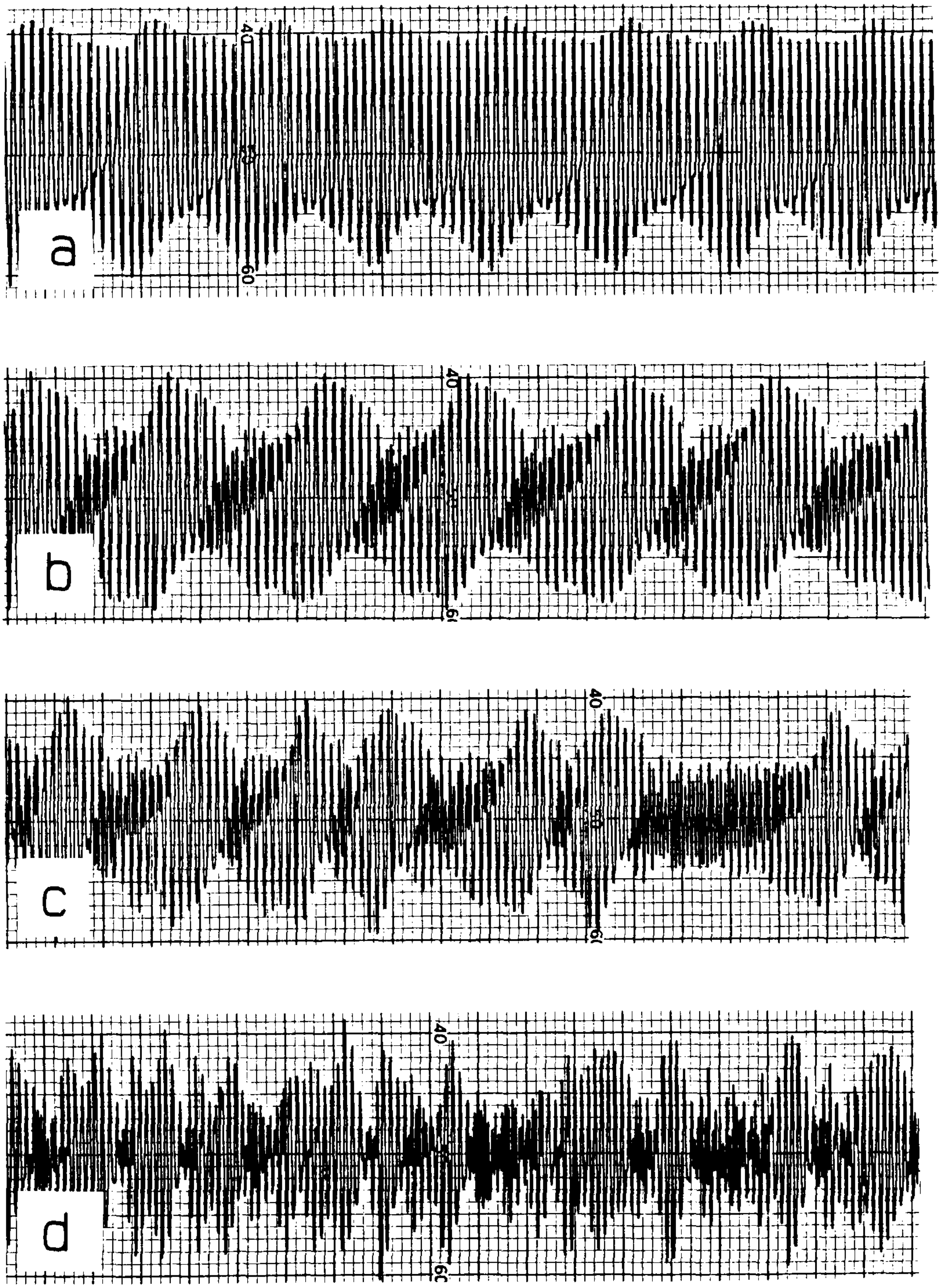

Fig. 10. Waveheight time traces for $F = 19.3$. (a) $Q = 0.102$. (b) $Q = 0.089$. (c) $Q = 0.074$. (d) $Q = 0.058$.

scenario is via quasi-periodic motion for flat topography where this $\pm\beta$-effect is dominant. Another possibility is that interior centrifugal acclerations may affect the interior linear and non-linear stability properties of the system. These ideas, as well as the effects of quasi-geostrophic wave-wave interactions on the transition to chaos, need to be investigated.

One might hope that in investigating what happens at larger Froude number F, where transition to chaos occurs at larger Q (or smaller Rossby number), geostrophic theory would do better. Indeed, experiments show that the neutral curves at larger F approach the quasi-geostrophic values for both positive and negative Rossby number. Unfortunately at these larger values of F, many azimuthal wavenumbers are linearly unstable and compete actively in the transition region, and the dynamics are thus further complicated. Fig. 10 shows some typical wave-height time traces. The higher frequency bursts indicate the presence of wavenumber two, and it is clear that there is important interplay between wavenumbers 1 and 2 in the progression from amplitude vacillation to frequency vacillation to chaos as shown in this figure. We need to break away from the single-wave models and generate some theoretical understanding of non-linear, almost inviscid baroclinic waves that involve multiple wave interactions. In addition to being of more geophysical interest, such models might be more easily tested in the laboratory since the requisite values of the friction parameter for complex dynamics are not so small as to lead to Kelvin–Helmholtz or boundary layer instability.

5. Acknowledgements

The author wishes to thank Dr. Joseph Pedlosky for helpful discussions. Mr. Atsushi Yoshida provided Fig. 2 which will form part of his Ph.D. thesis, a numerical study of quasi-geostrophic baroclinic transition. This research was sponsored by the National Science Foundation, grant ATM-8111718.

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