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An explicit solution of the linear thermocline equations

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ABSTRACT

A simple analytic solution of the linear thermocline equations is obtained for the case where the forcing is purely by Ekman pumping, and the results are displayed graphically. The motivation was purely to find a very simple solution to use as a departure point for discussion of the ocean circulation and its associated temperature. However, the calculated temperature field turns out to display a surprising degree of reality in that (i) isotherms slope up toward the east in the subtropical gyre (but down in the subpolar gyre), (ii) the surface temperature increases towards the west in the subtropical gyre (but decreases in the subpolar gyre), (iii) the maximum vertical temperature gradient is subsurface in the subtropical gyre (the gradient is greatly reduced in the subpolar gyre), (iv) the surface temperature decreases towards the poles, except at low latitudes, (v) the meridional surface temperature gradient is largest in the subpolar gyre, and (vi) surface isotherms spread out longitudinally near the eastern boundary. The surprising result is that all these features are produced by wind forcing alone, and thus raise the question as to what extent the same is true of the actual temperature field of the ocean.

1. Introduction

The large-scale interior circulation of the ocean is governed according to classical theory by the so-called thermocline equations (Pedlosky, 1979; Gill, 1982) in which the motion is in exact geostrophic and hydrostatic balance, but where the temperature equation contains both advection and vertical diffusion terms. The history of the study of this equation is rather curious. The first study by Stommel and Veronis (1957) was of the linearised form in which vertical advection of temperature against the specified mean vertical gradient is balanced by vertical diffusion. They found the solution for the case of periodic variation with longitude and this displayed some very interesting features, such as the vertical depth of penetration of the solution from the surface, and the phase shift towards the west with depth. The non-linear version was introduced by Robinson and Stommel (1959) and Welander (1959). The only way to make this version tractable analytically was to look for special forms of solution. The special forms, unfortunately, required certain coincidences in physical balances and were not capable of satisfying the full boundary conditions. It is doubtful whether the effort expended in finding ingenious solutions to these equations was rewarded by a corresponding increase in understanding of the dynamics of ocean circulation. These special solutions are discussed by Veronis (1969, 1981).

Stommel (1984), in presenting his very stimulating Crafoord Prize lecture in Stockholm, discussed the general impasse that had been reached in thermocline theory and the new ideas that are beginning to lead us out of it. One idea (Rhines and Young, 1982; Holland, et al., 1984) is that eddies in the ocean can be responsible for producing circulation in sub-surface layers and produce regions of homogeneous potential vorticity. Another is the idea of the ventilated thermocline (Luyten et al., 1983), namely that water is pumped into ocean layers at high latitudes where they outcrop and that

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this forces motion in them at lower latitudes where they are sub-surface. These concepts have led to renewed interest in the thermocline equations and the development of special types of non-linear non-diffusive solutions (e.g. Pedlosky and Young, 1983) which can satisfy realistic boundary conditions but are not uniquely determined without additional physical assumptions.

The Crafoord lecture made me think it would be worthwhile to calculate and display explicit solutions of the *linear* diffusive thermocline equations for two reasons. Firstly, if one is to understand what features of the new solutions are unique to them, it is necessary to know what a linear model can produce. For instance, can a linear diffusive solution produce a beta-spiral? Secondly, diffusive effects are bound to play a rôle in reality and it is useful to assess their importance, for reasonable values of diffusivity, in the context of a linear model. For instance, what thermocline depth is predicted by a linear diffusive solution? Thus the object of this paper is to calculate such a solution, discuss its physics and present the solution graphically.

2. The modal approach

One convenient way of representing linear non-diffusive solutions of the *transient* problem is in terms of vertical normal modes, and McCreary (1981) has shown how this approach can be generalised to the diffusive case. Although the approach involves certain compromises in the physics, which led me to abandon it as a way of tackling the problem, it does lead to certain insights, and these will be described in this Section.

First consider the non-diffusive problem following Anderson and Gill (1975). Suppose the ocean is driven by a wind which forces the ocean only in an upper mixed layer, this homogeneous layer overlying a continuously stratified region below. The transient problem can be solved by expanding both perturbation pressure and the forcing function (which is non-zero only in the mixed layer) in terms of vertical normal modes (see Gill 1982, Section 6.11). For scales large compared with the Rossby radius (as is assumed in deriving the thermocline equations), the contribution η of each mode to the perturbation pressure satisfies an equation of the form (see Gill (1982) Section 12.4)

$$-\frac{f^2}{c^2}\frac{\partial \eta}{\partial t} + \beta\frac{\partial \eta}{\partial x} = \text{forcing}, \tag{2.1}$$

where f is the Coriolis parameter, β its gradient, x distance eastwards and c is the separation constant (wave speed) for the mode in question. Eq (2.1) is that for forced long planetary waves. Since long planetary waves propagate westwards, the only boundary condition required is that of no motion normal to the eastern boundary. If this is located at $x = 0$, the condition is

$$\left.\begin{aligned} &\partial\eta/\partial y = 0 \\ &\text{i.e.,}\quad \eta = \eta_B(t) \end{aligned}\right\} \text{ at } x = 0 \tag{2.2}$$

The solution of (2.1) can be expressed as the sum of two parts, a direct wind-forced part and a boundary-forced part. The former is defined as the part with $\eta = 0$ at the eastern boundary. Because of the assumption that the scale is large compared with the Rossby radius, there are no coastal Kelvin waves as such, but the boundary perturbation pressure changes with time in excatly the same way at each latitude. This could be due, for instance, to a coastal Kelvin wave propagating up from the equator since, in the present approximation, this wave is assumed to move so quickly (or be so long) that changes are virtually simultaneous at each latitude. In this paper, the *boundary-forced part*, or its equivalent, *will be ignored.*

The full solution is obtained by summing the normal modes. The higher modes take longer to arrive at a given point, so the structure in the solution which has a small vertical scale can only be built up slowly. At $t = \infty$, all the modes are in equilibrium and then, as Anderson and Gill (1975) demonstrated, the motion is entirely confined to the mixed layer. The solution is singular because the horizontal velocity components and the perturbation pressure are uniform in the mixed layer but zero below. This corresponds to *infinite shear* at the base of the mixed layer and hence *infinite horizontal temperature gradient* from the thermal-wind equation. Of course, at any finite time, the solution is well-behaved but values will get large as $t \to \infty$.

Now consider the effects of vertical diffusion. Intuitively this will remove the singularity because the step will be diffused out. One way of modelling how this happens is to retain the modal expansion technique, following McCreary (1981), but accept-

ing the compromises required for the expansion to be valid. These compromises are (a) the boundary conditions at the horizontal boundaries have to take a special form and (b) the vertical diffusion of heat needs to have a special form.
If these are accepted, (2.1) becomes

$$-\frac{f^2}{c^2}\left(\frac{\partial \eta}{\partial t}+\frac{K_0}{c^2}\eta\right)+\beta\frac{\partial \eta}{\partial x}=\text{forcing}, \tag{2.3}$$

where K_0 is a constant equal to the diffusivity K times N^2, where N is the buoyancy frequency. Thus if the forcing in (2.3) is independent of x, the steady-state solution has

$$\eta \propto 1-\exp(\mu x) \tag{2.4}$$

where

$$\mu = K_0 f^2/\beta c^4. \tag{2.5}$$

Since the high modes decay the most rapidly with x, all the *fine vertical structure is confined to the neighbourhood of the eastern boundary* and the structure becomes more spread out in the vertical as one proceeds westwards.

It is an easy matter to construct solutions by the McCreary technique, and some were calculated for an ocean with a homogeneous surface-mixed layer. However, there were some unsatisfactory features of this solution associated with the assumed form of diffusion, particularly that it is required to have the form

$$(K\theta)_{zz} = K_{zz}\theta + 2K_z\theta_z + K\theta_{zz}$$

instead of

$$(K\theta_z)_z = K_z\theta_z + K\theta_{zz}$$

The former structure has an inappropriate term K_{zz} which causes problems at the base of the mixed layer where K jumps from a finite value to infinity. For this reason, the McCreary approach was not pursued further, and a more direct method was used.

The direct method is discussed in the following sections. Although it requires N to be uniform (when the above criticism does not apply), it is superior to the modal expansion in that (a) the boundary conditions are not required to have an artificial form, and (b) an infinite expansion is not required. Basically it shows how diffusion spreads out the singularity found by Anderson and Gill (1975). A distinctive property is the *thickening of the thermocline towards the west*, and the argument above about the decay scale of modes is quite helpful for understanding why this should be so. It also helps relate thermocline theory to the theory of normal modes.

3. The equations

The thermocline equations apply for scales large compared with the internal Rossby radius, this assumption implying that the motion can be regarded as being in exact geostrophic balance. Thus if ϕ is latitude, λ longitude, z distance upwards, a the earth's radius and Ω its rotation rate, the horizontal velocity components u (eastward) and v (northward) are given by

$$2\Omega \sin\phi\, u = -a^{-1}\,\partial p/\partial\phi, \tag{3.1}$$

$$2\Omega \sin\phi\, v = a^{-1}\sec\phi\,\partial p/\partial\lambda, \tag{3.2}$$

where p is $(\text{density})^{-1}$ times the perturbation pressure. The incompressibility condition is

$$\frac{\partial u}{\partial \lambda}+\frac{\partial}{\partial \phi}(v\cos\phi)+a\cos\phi\frac{\partial w}{\partial z}=0, \tag{3.3}$$

and the hydrostatic approximation gives

$$\frac{\partial p}{\partial z}=\alpha g\theta, \tag{3.4}$$

where α is the thermal expansion coefficient (here assumed constant), g is the acceleration due to gravity and θ is the temperature perturbation from a state of uniform stratification in the vertical. The linearised form of the temperature equation is simply

$$N^2 w = \alpha g K\,\partial^2\theta/\partial z^2, \tag{3.5}$$

where the diffusivity K will be assumed to be a constant independent of λ, ϕ or z.

From (3.1) and (3.2), the horizontal divergence can be calculated, namely

$$\frac{\partial u}{\partial \lambda}+\frac{\partial}{\partial \phi}(v\cos\phi)=\frac{-\cos\phi}{2\Omega\, a\sin^2\phi}\frac{\partial p}{\partial \lambda}=-\frac{\cos^2\phi}{\sin\phi}v, \tag{3.6}$$

and thus, using (3.3)

$$a\frac{\partial w}{\partial z}=\cot\phi\, v=\frac{1}{2\Omega\, a\sin^2\phi}\frac{\partial p}{\partial \lambda}. \tag{3.7}$$

This is a form of the vorticity equation. Finally, substituting for w from (3.5) and then for θ from (3.4), a single equation for p is derived, namely

$$2\,\Omega\, a^2 \sin^2\phi\, K\, \partial^4 p/\partial z^4 = N^2 \partial p/\partial\lambda. \tag{3.8}$$

This is the linearised thermocline equation for which solutions periodic in λ were calculated by Stommel and Veronis (1957). The most complete discussion of solutions of this equation in the context of ocean circulation is given by Pedlosky (1969) who considers the problem of matching solutions with the surface and side-wall boundary layers. Matching with the surface Ekman layer leads to the condition that w is specified at the top of the thermocline (equal to the Ekman suction velocity) and that the surface thermal condition (specified temperature or specified heat flux) is carried through to the thermocline régime. With these conditions at $z = 0$, the thermocline equations can be solved as the sum of a direct wind-forced solution with $p = 0$ at the eastern boundary $\lambda = 0$ and a boundary-forced solution with p a given function of z at $\lambda = 0$. For a closed basin, Pedlosky shows how this latter part is determined from a global condition and knowledge of the side-wall layer structure. The modal analysis of section 2 suggests this boundary-forced part would in reality be associated with the steady state end product of waves propagating around the boundaries. In this paper, these contributions to the solution will be ignored. Pedlosky (1969) has shown how the solution to be studied here can be written down as a Fourier integral with respect to the vertical wavenumber, and Barcilon (1978) has discussed some of its properties. However, an explicit form can be obtained in terms of a similarity variable using the special functions of Gill and Smith (1970). This makes it a relatively simple matter to calculate solutions and display temperature sections, velocity sections, etc., so that their properties can be compared with observations. The explicit solution will be derived in Section 4.

4. Explicit solutions

For discussing solutions of (3.8), it is convenient to replace longitude λ by a distance x *westwards* from the eastern boundary $\lambda = 0$, i.e.

$$x = -\lambda a \cos\phi. \tag{4.1}$$

Then solutions of (3.8) can be expressed in terms of x and the similarity variable

$$\xi = -rzx^{-1/4}, \tag{4.2}$$

where r is a parameter given by

$$r^4 = \frac{N^2 \cos\phi}{2\,\Omega\, K\, a \sin^2\phi}. \tag{4.3}$$

In (4.2) it is useful to think of $-z$ as distance *downwards* from the surface so that ξ is positive in the domain of interest where x and $-z$ are both positive. The parameter r^{-1} plays the rôle of a vertical scale, but in the space of the similarity variable rather than physical space. Thus it is $r^{-4/3}$ rather than r^{-1} which has the dimensions of length.

The solutions of (3.8) have the general form

$$p = p_0(\phi)\, x^{n/4}\, L_n(\xi), \tag{4.4}$$

where L_n is a linear combination of the special functions described by Gill and Smith (1970). These have the property that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^m L_n(\xi) = L_{n-m}(\xi), \tag{4.5}$$

so it follows that from (3.4), (3.5) and (3.7) that θ, w and v have the form

$$\begin{aligned} \theta &= \theta_0(\phi)\, x^{(n-1)/4} L_{n-1}(\xi) \\ w &= w_0(\phi)\, x^{(n-3)/4} L_{n-3}(\xi) \\ v &= v_0(\phi)\, x^{(n-4)/4} L_{n-4}(\xi), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \theta_0 &= N^2 w_0/\alpha g K r^2, \\ p_0 &= -N^2 w_0/K r^3, \\ v_0 &= -\mathrm{ar} w_0 \tan\phi. \end{aligned} \tag{4.7}$$

The value of n and the choice of linear combination of special functions depends on the boundary conditions. Here it will be supposed that the two surface conditions applied at $z = 0$ are those of prescribed longitude-independent Ekman pumping velocity w and of prescribed longitude-independent vertical heat flux F. Then the solution can be expressed as the sum of two parts — a pumping-forced part with zero surface flux, and a flux-forced part with zero surface pumping.

For the pumping-forced part, n must equal 3 in order for w to be longitude-independent, and the linear combination L_n of special functions required for F to vanish at the surface is

$$L_n(\xi) = Jo_n(\xi) - Ko_n(\xi), \tag{4.8}$$

where Jo and Ko are defined by Gill and Smith (1970). Thus

$$p = p_0\, x^{3/4}\, L_3(\xi),$$

$$\theta = \theta_0\, x^{1/2}\, L_2(\xi),$$

$$w = w_0\, L_0(\xi),$$

$$v = v_0\, x^{-1/4}\, L_{-1}(\xi). \tag{4.9}$$

This will be the only solution considered in this paper, and its properties will be discussed in Section 5. However, it is appropriate to note that, for the flux-forced part, n would be equal to 2 and L_n would be replaced by M_n where

$$M_n(\xi) = Jo_n(\xi) + Ko_n(\xi). \tag{4.10}$$

Properties of the functions L_n and M_n, and means of computing them, are given in the Appendix.

5. The wind-forced solution

The wind-forced or pumping-forced solution is given by (4.9). At any given point, the vertical profiles are given by the L functions and these are displayed in Fig. 1. With the exception of L_1, they decrease in magnitude away from the surface where the functions have their largest value. The perturbation temperature gradient, however, is zero at the surface because of the flux boundary condition and has its largest value subsurface at $\xi = 1.234$, i.e. at

$$-z = 1.234\, r^{-1}\, x^{1/4}. \tag{5.1}$$

Much of the interest in this solution and with the associated physics comes purely from a discussion of the *signs* of the perturbation field near the surface. The starting point for any such discussion is the sign of w because this is given, and the task is to find the signs of the other fields and the reasons for those signs. To fix ideas, consider the *subtropical gyre* where *w is negative* at the surface. Consider first the field of motion. Since w vanishes at depth, it may be anticipated (as Fig. 1 indeed shows) that $\partial w/\partial z$ is negative near the surface. In other words, the downward motion at the surface causes shortening in the vertical so that the *horizontal motion is divergent*. For exactly geostrophic motion, the divergence is proportional to v (and the opposite sign) thus giving eq. (3.7) relating v to $\partial w/\partial z$. Alternatively (3.7) can be regarded as the vorticity equation which requires the *near-surface velocity* in the subtropical gyre to be *equatorward* because vortex lines are being shortened. The result for the vertically integrated flow (which follows from the integral of (3.7)) is quite familiar as Sverdrup's result, so it is not surprising that the near-surface velocity has the same sign. In the limit as K tends to zero, the *thermocline thickness* tends to zero but the transport does not alter. Hence the velocity v itself tends to infinity.

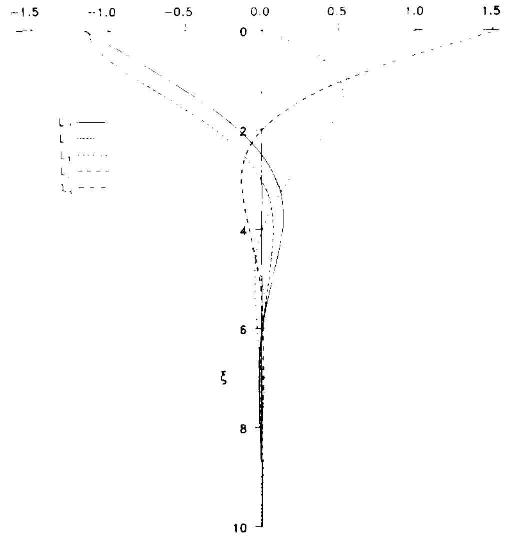

Fig. 1. The special functions L_{-1} (solid), L_0 (close small dashes), L_1 (spaced small dashes), L_2 (dash-dot) and L_3 (large dashes).

Now consider the sign of θ, the surface temperature perturbation. From (4.7), θ_0 is negative while Fig. 1 shows that L_2 is negative, so that by (4.9), θ is positive. To understand this result, consider the heat equation which relates θ and w. It is helpful first to consider the time-dependent version of this equation in the absence of diffusion, namely

$$\frac{\partial \theta}{\partial t} + w\frac{\partial T}{\partial z} = 0. \tag{5.2}$$

This shows that *when there is a downwelling, the fluid warms up*. This occurs because of the stable temperature gradient, which means that the fluid above is always warmer, and so downward motion leads to warming. In the linear régime, there is no limit to this warming since the perturbation temperature range is infinitesimal compared with the undisturbed temperature range. Putting this another way, θ has an effectively infinite range of temperatures to draw from.

In the linear régime, the physical effect which limits the value θ can reach is diffusion. When this effect is added to (5.2) and the solution is allowed to reach a steady state, the computed solution (4.9) is obtained. The term involving w in (5.2) and/or (3.5) can be regarded as a *heat source* term where there is downwelling. It is not, however, a heat source which can be prescribed independently, as the w field is part of the solution. Since diffusion is the limiting process, it is not surprising that θ_0, and hence the magnitude of the temperature perturbation, tends to infinity as K tends to zero.

Another sign of interest is that of the pressure perturbation. This sign follows from (4.7), (4.9) and the value of $L_3(0)$ given in the Appendix. It is *positive* in the subtropical gyre. The result is a natural consequence of hydrostatics, for if the ocean warms it expands and the surface rises, so the perturbation pressure is positive. Another way of arriving at the sign of p, and hence checking consistency, is from the geostrophic relation between v and p. Equatorward flow requires pressure *rising towards the west*, and since $p = 0$ at the eastern boundary, p must be positive near the surface in the entire gyre.

The perturbation temperature gradient is zero at the surface in accordance with the flux condition, but is positive below to be consistent with the elevated surface temperature. Thus, the stability (or equivalently, the buoyancy frequency) is enhanced in the subtropical gyre, and conversely *is reduced in the subpolar gyre.* The maximum magnitude of the perturbation is given by

$$\frac{\alpha g \partial\theta/\partial z}{N^2} = 0.531 w_0 \left(\frac{2\Omega a \sin^2\phi}{\cos\phi} \frac{x}{N^2 K^3} \right)^{1/4}. \quad (5.3)$$

The ratio given by this formula is of special interest as it is a measure of non-linearity. For estimating magnitudes in this paper, we take as standard values $2\Omega = 1.47 \times 10^{-4}\ \mathrm{s}^{-1}$, $a = 6.4 \times 10^6$ m, $g = 10\ \mathrm{m\ s}^{-2}$, $\alpha = 2 \times 10^{-4}\ K^{-1}$ while the undisturbed temperature gradient is assumed to be 12.5° K per km, giving $N^2 = 2.5 \times 10^{-5}\mathrm{s}^{-2}$. A wide range of values for w_0, K and x could be considered reasonable. As an example, suppose $w_0 = 10^{-6}$m s^{-1} (30 m/yr), $K = 10^{-4}\mathrm{m}^2\mathrm{s}^{-1}$ and $x = 1000$ km. Then the value of the ratio (5.3) is around 1.3 in mid-latitudes, so clearly non-linearity should not be ignored in this case. Note, however, the *extreme sensitivity to the values of* w_0 *and* K. This is brought out by inverting (5.3) to give the distance from the coast at which the ratio is unity, namely

$$x = \frac{12.6\, N^2 K^3 \cos\phi}{2\,\Omega\, a \sin^2\phi\, w_0^4}. \quad (5.4)$$

In the sub-polar gyre, this is where the linear solution would make the stability vanish. For the above values of w_0 and K, this distance is only 300 km, but if K is increased by a factor of 5 or w_0 is reduced to $3 \times 10^{-7}\ \mathrm{ms}^{-1}$, the distance becomes 36,000 km instead, and so gives a very different view of the importance of non-linearity.

Returning to the linear solution, it is useful to look at magnitudes of surface values of ϕ, p and v which follow from (4.9) and the values in Appendix A. The values turn out to be quite reasonable, e.g., at mid-latitudes for $x = 1000$ km, $w_0 = 10^{-6}\ \mathrm{m\ s}^{-1}$ and $K = 10^{-4}\ \mathrm{m}^2\mathrm{s}^{-1}$, θ values are 8–9° K, elevation values are about 60 cm and meridional velocity values are about 6 cm s^{-1}.

There is one field not given explicitly by (4.9) and that is zonal velocity u. Its structure is determined by (3.1) and (4.9), which give

$$2\Omega \sin\phi\, u = -\frac{1}{a}\frac{\mathrm{d}p_0(\phi)}{\mathrm{d}\phi} x^{3/4} L_3(\xi). \quad (5.5)$$

Thus the variation of u with depth has the same form as p. However, the variation with latitude is different, so to fix ideas, w_0 will be given a specific form

$$w_0(\phi) = w_E \sin\left[\frac{9}{2}\left(\phi - \frac{1}{4}\pi\right)\right]. \quad (5.6)$$

This is a classical form with zero at 45° latitude, a maximum downward value at 25° latitude and a maximum upward value at 65° latitude. The surface value of u is eastward between 25° and 65° latitude and westward immediately outside this range. Because of interest in the beta-spiral (Stommel, 1984), it is worthwhile to calculate how the velocity vector rotates with depth. Since u (i.e. L_3) falls off more quickly with depth than v (i.e. L_{-1}), it follows that the spiral is clockwise where the product uv is positive at the surface and negative otherwise. Thus between 20° and 45° latitude, *the velocity vector rotates clockwise as depth increases* (the classical sign for the subtropical gyre). Equatorwards of 25° or between 45° and 65°, the rotation is anticlockwise.

Now consider the structure of the solution in the zonal vertical section. Fig. 2 shows four fields in the x, z plane in non-dimensional form corresponding to (a) p or u, (b) θ, (c) $\delta\theta/\delta z$ or heat flux and (d) v. The field for w is not shown as the contours are simply lines $-z \propto x^{1/4}$. Properties to note are: (i) that v has a mild singularity at the eastern boundary; one could interpret this tendency for stronger meridional currents near the boundary as a form of eastern boundary current, but there is no boundary-layer scale in the present model; (ii) $\partial\theta/\partial z$ has a subsurface extremum as already noted; (iii) θ has the structure to be expected from the facts that the surface gradient is zero and that it varies like the square root of x at the surface; (iv) p

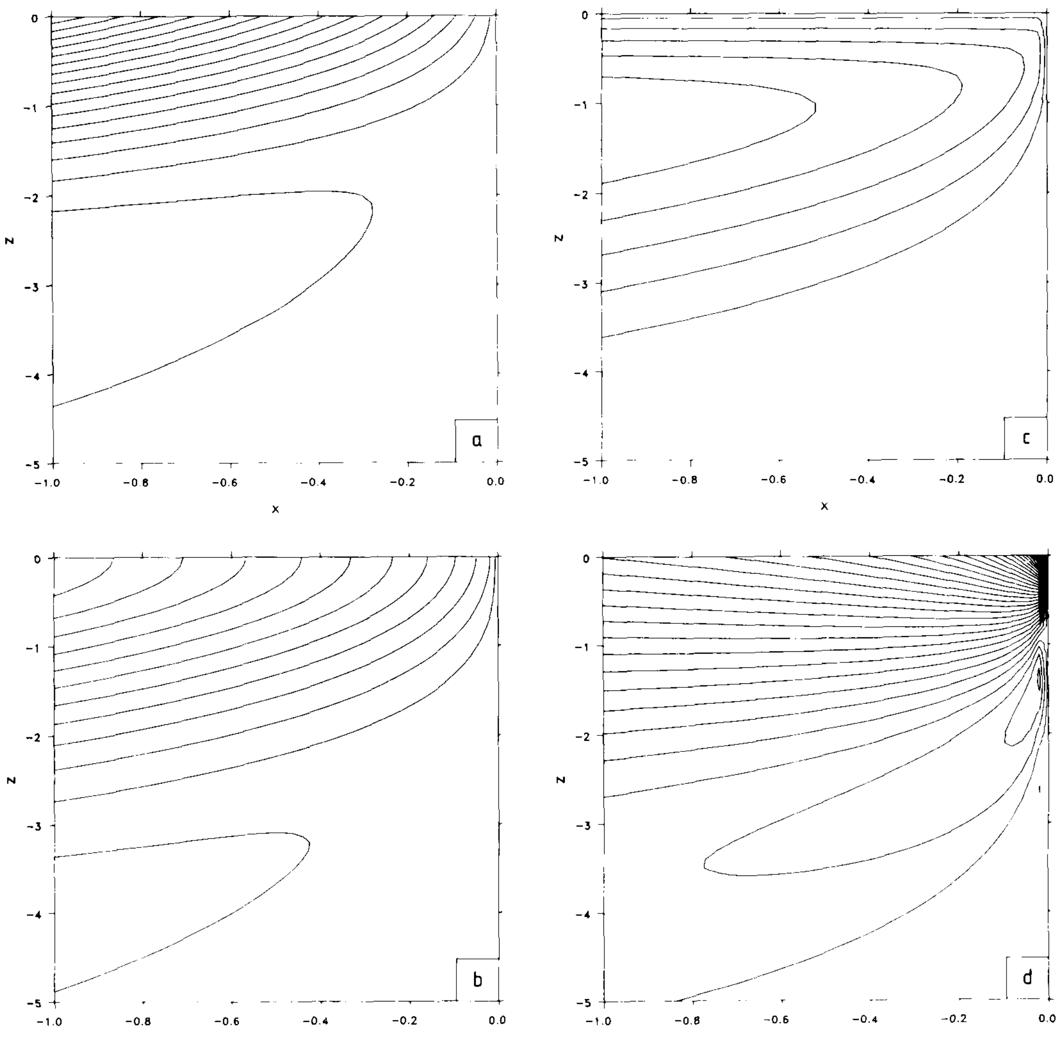

Fig. 2. Vertical sections in the zonal plane of (a) pressure perturbation p or zonal velocity u, (b) temperature perturbation θ, (c) vertical temperature gradient $\partial\theta/\partial z$ or vertical diffusive heat flux, (d) meridional velocity v. In each case, contours are of the non-dimensional quantity and are shown for the values ± 0.05, ± 0.15 etc.

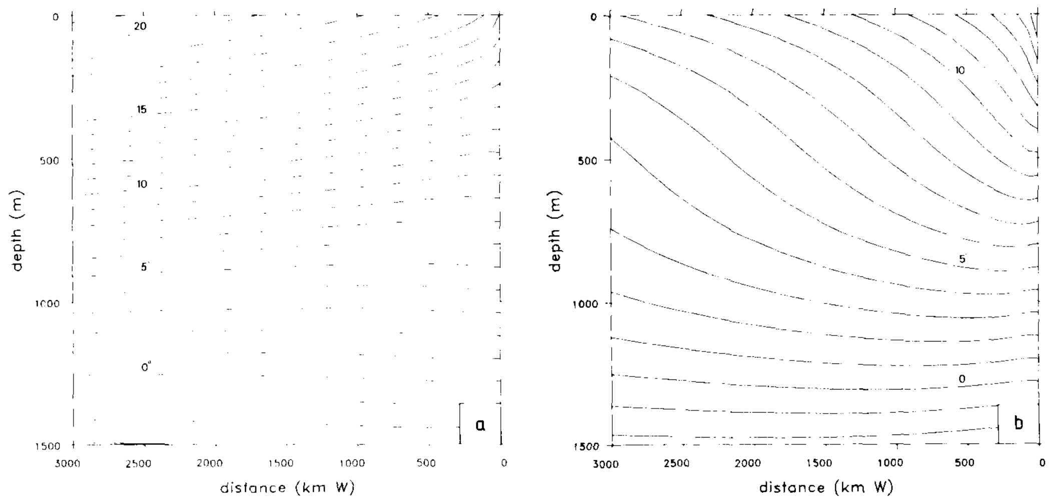

Fig. 3. Vertical temperature sections at (a) 35°N, and (b) 55°N as given by eq (5.7). The prescribed vertical velocity at the surface has the form (5.3) with $w_E = 1.4 \times 10^{-6}$ m s^{-1} (44 m yr^{-1}) and the diffusivity K is 3×10^{-4} m^2 s^{-1}.

or u contours are as expected for a function with a maximum at the surface and varying like $x^{3/4}$ on the surface.

The quantity one can most readily compare with observation is temperature, and for this purpose the mean gradient needs to be added on. Fig. 3 shows the results at 35° and 55° latitude in the form

$$T = 16 - 12.5z + \theta. \tag{5.7}$$

The diffusivity in this case is 3×10^{-4} m^2 s^{-1} and w_0 is given by (5.6) with $w_E = 1.4 \times 10^{-6}$ m s^{-1} (corresponding to $w_0 = \pm 10^{-6}$ m s^{-1} at 35° and 55° latitude). It is perhaps surprising how many realistic features can be seen in the results of a model which (a) is linear with a constant diffusivity and (b) is purely wind-driven. (Both the flux-driven and the boundary-driven components of the solution are ignored.) Consider first the section through the sub-tropical gyre at 35° latitude. Realistic features are (i) the isotherms slope up toward the east, and thus (ii) the surface temperature increases towards the west, and (iii) the maximum temperature gradient is subsurface. Furthermore, the scales are reasonable. In the subpolar gyre at 55° latitude, on the other hand, (i) isotherms slope up toward the west, (ii) the surface temperature decreases towards the west, and (iii) there is a minimum subsurface temperature gradient. In fact, the temperature gradient will have negative values far enough from the coast with this solution. This distance is given by (5.4).

6. Discussion

An obvious question to ask is about effects of non-linearity and whether the linear solution can be used as a starting point for understanding such effects. One can at least calculate the non-linear terms which were ignored in the linear approximation and assess their effects on the flow. With the above solution, it is possible to go much further because the linear solution can be taken as the first term in an amplitude expansion which is also an expansion in powers of $x^{1/4}$. The first non-linear correction to the fields was in fact calculated but it proved to be a lot of work; the one extra term by itself did not seem very useful, and both author and referees of an earlier version agree that little insight was gained. Perhaps the only interesting result had to do with the signs of temperature effects and these are discussed below.

The full three-dimensional structure of the linear solution can be appreciated if a horizontal section is added to the vertical sections already displayed. Fig. 4 shows such a section, and presents surface pressure and surface temperature contours. The two fields have a very similar structure which is well-known from Sverdrup dynamics. Particles in the subtropical (anticyclonic) gyre move equatorwards along the pressure contours (marked by broken lines in Fig. 4) and downwards. The motion is faster near the surface and vortex lines get shorter with time. The converse applies in the

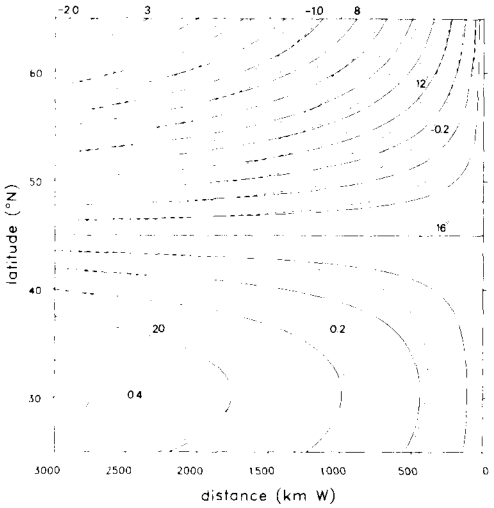

Fig. 4. Surface temperature contours (solid, contour interval 1 °C) and surface elevation (broken, contour interval 0.1 m) for $w_E = 1.25 \times 10^{-6}$ m s^{-1} and $K = 3 \times 10^{-4}$ m^2 s^{-1}.

sub-polar (cyclonic) gyre (polewards of 45°) where the motion is polewards and upwards and vortex lines are stretched.

For assessing the sign of non-linear effects, the small angle between isotherms and pressure contours is critical. It can be seen from Fig. that in the subtropical gyre, the *horizontal component* of the motion is toward warmer water (it is towards colder water in the subpolar gyre), thus giving a cooling tendency which reduces the magnitude of the temperature perturbation. Even when the non-linear vertical advection effect is added, it is found that, with the exception of the small region equatorward of 28°, the non-linear effect is to *reduce* the magnitude of both θ and p at the surface.

To return to the linear solution, the most interesting questions to ask are about what features of the real circulation it helps us to understand. The motion field largely comes from Sverdrup dynamics so that need not be discussed here. What is really interesting is the thermal field, because the linear solution gives warm water in the subtropics and coldwater at higher latitudes *purely because of the sign of the Ekman pumping.* The heat flux into the ocean in the model has no variations with latitude at all. Why should the wind-forced temperature field in Fig. 4 be so realistic? Not only does it show lower temperatures toward the poles, but it also shows tighter meridional gradients in the sub-polar gyre as observed, and a latitudinal spreading of isotherms near the eastern boundary as observed. This raises the question as to what extent such features in the real surface temperature field are due to wind forcing rather than heating or cooling. Perhaps it is amusing to end this paper with this intriguing question, because it raises interesting possibilities for further research.

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8. Appendix. The special functions

From Gill and Smith (1970) and the definitions (4.8) and (4.10), the functions L_n and M_n are given by

$$L_n = P_n - \pi^{-1}\,[H_{n,n+1} - H_{n,\,n+2}],$$

$$M_n = P_n - \pi^{-1}\,[H_{n,\,n+2} - H_{n,n+3}],$$

where P_n is a polynomial which is zero for $n < 0$ and is given for small n by

$$P_0 = 1,\ P_1 = \xi,\ P_2 = \tfrac{1}{2}\xi^2,\ P_3 = \xi^3/3!.$$

The functions $H_{n,m}$ are generalised hypergeometric functions which can be evaluated from the expansions

$$H_{n,m} = \frac{\Gamma\left(\dfrac{m-n}{4}\right)\xi^m}{m!}\left(1 + \frac{(m-n)\frac{1}{4}\xi^4}{(m+1)(m+2)(m+3)(m+4)}\left(1 + \frac{(m-n+4)\frac{1}{4}\xi^4}{(m+5)(m+6)(m+7)(m+8)}\left(1 + \ldots,\right.\right.\right.$$

where m is the value modulo 4, i.e. the functions are evaluated using m = 0, 1, 2 or 3. The power series expansions are quite adequate for values of interest, e.g., on an IBM 3081 using double precision, the series expansion gave sensible values up to about $\xi = 18$, where the values were less than 10^{-6}. For larger ξ, the values started to increase. The asymptotic expansions, if required, are given by Gill and Smith (1970). The first term is

$$L_n \sim -\frac{2}{\sqrt{3\pi}}\left(\frac{\xi}{4}\right)^{-(n+2)/3}$$

$$\exp\left(-\frac{3}{2}\,\frac{1}{4}\xi^{4/3}\right)\sin\theta_n,$$

where

$$\theta_n = \frac{3\sqrt{3}}{2}\left(\frac{\xi}{4}\right)^{4/3} - \frac{2n\pi}{3} - \frac{\pi}{12}.$$

The same expression holds for M_n, but with $-\sin\theta_n$ replaced by $\cos\theta_n$. Other useful formulae are (4.5) and the recurrence formula (which comes from the differential equation satisfied by L_n):

$$4\,L_{n-4} = \xi L_{n-1} - nL_n\,.$$

This applies to M as well. The values of L_{-1}, L_0, L_1, L_2, L_3 at $\xi = 0$ are -1.154, 1, 0, -1.128, 1.539, respectively, whereas for M_{-2}, M_{-1}, M_0, M_1, M_2 are -0.564, 0, 1, -1.560, 1.128. The maximum value of L_1 is 0.531 at $\xi = 1.234$.

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