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A cyclo-stationary model of sea surface temperatures in the Pacific Ocean

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ABSTRACT

Sea surface temperature anomalies in mid-latitude north Pacific Ocean are compared with a cyclo-stationary stochastic model in which the anomalies are forced by atmospheric "weather" disturbances. The results show that the model explains the observed two-time auto-covariances of the anomalies for almost all (98%) of the studied region.

1. Introduction

In a recent series of publications, stochastic models of climate variability have been considered. In the model proposed by Hasselmann (1976), slow changes of climate were interpreted as the continuous response of climatic variables to random forcing produced by weather disturbances. The model was then applied to some simple climate systems, such as the sea surface temperature anomalies (SST) (Frankignoul and Hasselmann, 1977) and climate variability on longer time scales (10^1–10^4 years) derived from energy balance models (Lemke, 1977).

Stochastic models have been also fitted (in the least squares sense) to observed SST anomalies (Reynolds, 1978) and to sea-ice anomalies (Lemke et al., 1980), thereby achieving statistical verification, as well as yielding information on the statistics of atmospheric forcing and on the feedback governing the response of the climate system.

These comparisons showed good agreement between data and models, and suggested a possible way to improve the models. Generally, in stochastic models the evolution of a climate variable $y(t)$ is governed by a Langevin type equation

$$\dot{y}(t) = -v(t)y(t) + w(t), \tag{1.1}$$

where $w(t)$ is a stochastic process. Earlier analyses were performed assuming $w(t)$ to be delta correlated, gaussian and stationary, while $v(t)$ was taken to be constant.

SST anomalies are often described with mixed-layer models. These models assume a completely mixed, horizontally homogeneous layer of the upper ocean of depth $h(t)$ and temperature $T(t)$, overlying a seasonal thermocline. Supposing that the time variation of $h(t)$ is known, it is possible to derive a heat balance equation for the surface temperature.

$$\partial_t(h(t)T(t)) = f(T(t), T_a, q, U, R), \tag{1.2}$$

where f denotes the sum of three terms: (a) the total flux of latent and sensible heat through the air–sea interface as a function of sea temperature T, air temperature T_a, humdidity q, and wind velocity U; (b) the net balance of short and long wavelength radiation R; (c) the heat exchanged with the layer below.

For small departures from equilibrium, i.e., for small SST anomalies, we can expand the function f with respect to T. We define $T = 0$ as the equilibrium temperature for which $\langle f \rangle = 0$, where angular brackets denote ensemble averages over atmospheric states. If we write

$$f' = f - \langle f \rangle, \tag{1.3}$$

then

$$\partial_t(h(t)T(t)) = (\partial_T\langle f \rangle)|_{T=0}T(t) + f', \tag{1.4}$$

where f', as a function of instantaneous weather variables, can be considered as stochastic for the time scales of variation of T and h. These scales are of the order of weeks to months, while f' varies in time scales of the order of a day. Equations similar to (1.4) were used by Frankignoul (1979), and by Frankignoul and Reynolds (1983). Observations show that the variance of the stochastic forcing f' and the mixed-layer depth vary markedly with the seasons, being larger in winter and smaller in summer. We will therefore assume a periodical time-dependence for the variance of f' and for $h(t)$.

Comparing (1.4) with (1.1) we can write

$$v(t) = -(1/h(t))\{(\partial_T\langle f\rangle|_{T=0} + \partial_t h(t)\}, \quad (1.5a)$$

$$w(t) = f'/h(t). \quad (1.5b)$$

We now expand $v(t)$ in a Fourier series

$$v(t) = a_0 + a_1 \cos(\omega_a t) + b_1 \sin(\omega_a t) + \dots, \quad (1.6)$$

where $\omega_a = 2\pi/12$ month^{-1} is the annual angular frequency, and we make the assumption that we can approximate $v(t)$ by the first three terms in the expansion, which we write as

$$v(t)) \simeq \beta + \beta_1 \cos(\omega_a t + \phi). \quad (1.7)$$

Similar arguments allow us to write the variance of the stochastic forcing

$$\langle w(t)w(s)\rangle = (1/h(t)h(s))\langle f'(t)f'(s)\rangle \quad (1.8)$$

as

$$\langle w(t)w(s)\rangle \simeq (D_0 + D_1 \cos(\omega_a t + \gamma))\,\delta(t-s), \quad (1.9)$$

where we have assumed f' to be a gaussian, delta correlated stochastic process. ϕ and γ denote the phases of the annual cycles of the feedback and forcing variance, respectively.

We propose for SST anomalies the following equation.

$$\partial_t T(t) = -(\beta + \beta_1 \cos(\omega_a t + \phi)T(t) + w(t). \quad (1.10)$$

This equation models a climate variable as being governed by a periodically varying feedback. The SST anomalies are amplified whenever $v(t)$ is negative while they are damped when it is positive.

Eq. (1.10) was analyzed from a formal point of view in a previous paper (Ruiz de Elvira and Lemke, 1982) where the following expression was derived for the two time autocovariances $\hat{C}(t, \tau)$

$$\hat{C}(t, \tau) \equiv \langle T(t+\tau)T(t)\rangle - \langle T(t+\tau)\rangle\langle T(t)\rangle, \quad (1.11)$$

$$\hat{C}(t,\tau) = \exp(-\beta|\tau| - \alpha \sin(\omega_a(t+\tau)+\phi) - \alpha\sin(\omega_a t + \phi)) \times \sum_{n=-\infty}^{\infty} J_n(2i\alpha) \begin{cases} \exp(-in(\omega_a t + \phi)\{A + B\cos(\omega_s t + \gamma) + C\sin(\omega_a t + \gamma)\}; & \tau > 0 \\ \exp(-in(\omega_a(t+\tau) + \phi)\{A + B\cos(\omega_a(t+\tau)+\gamma) + C\sin(\omega_a(t+\tau)+\gamma)\}; & \tau < 0 \end{cases} \quad (1.12)$$

where J_n are Bessel functions of the first kind and

$$\begin{aligned} A &= D_0/(2\beta - in\omega_a), \\ B &= D_1(2\beta - in\omega_a)/((2\beta - in\omega_a)^2 + \omega_a^2), \\ C &= D_1\omega_a/((2\beta - in\omega_a)^2 + \omega_a^2), \\ i &= \sqrt{-1}, \\ \alpha &= \beta_1/\omega_a. \end{aligned} \quad (1.13)$$

2. Procedure

To determine the parameters of the model, we must compare the results of eq. (1.12) with covariances obtained from observations. The data used consists of 28 years of monthly average temperatures in a 5° square grid in mid-to-low latitudes (40° N, 15° N) of the Pacific Ocean (NORPAX data).

For each point on the grid, hereafter called location, we arrange the temperatures into a rectangular matrix

$$T_\mu(t_i) = T_{\mu i}, \quad \mu = 1, 2, \dots, 28; \quad i = 1, 2, \dots, 12, \quad (2.1)$$

where μ denotes the year, i the month and we assign $i = 1$ to January. As we intend to fit a seasonal model we define, for each location and month i, an average temperature

$$\bar{T}_i = \frac{1}{N}\sum_{\mu=1}^{N} T_{\mu i}, \quad (2.2)$$

where the number N of terms in the sum varies with the location. For out data, $21 < N < 28$. We then perform the analysis with the rectangular matrix of temperature anomalies

$$\delta T_{\mu i} = T_{\mu i} - \bar{T}_i \quad (2.3)$$

for each location.

The true covariances of temperature anomalies are estimated by the sample covariances defined by

$$C_{ij} = C(t_i, \tau_j) = \frac{1}{N}\sum_{\mu=1}^{N} \delta T_\mu(t_i + \tau_j)\,\delta T_\mu(t_i). \quad (2.4)$$

A typical sample covariance is shown in Fig. 1. Most interesting is the presence of secondary peaks at time lags $\tau_j \simeq \pm 12$ months. These peaks are not simply due to high values of temperature anomalies in the months of December in all years computed, but indicate a true relationship between these anomalies at time lags of about 12 months. This relationship was confirmed by the calculation of the corresponding correlations.

For $\tau = 0$ data, covariances are a sum of N positive terms; hence, in this case, they are positive for all months t_i. If we expand (1.12), we observe that for $D_1 > D_0$, the last two terms in the curly brackets produce for $\tau = 0$ and some months t_i, depending on the phase γ, negative model covariances. Therefore we must impose the condition $D_1 < D_0$. This condition is also evident if we look at expression (1.9) where D_0 and D_1 were introduced.

Forcing $D_1 < D_0$ imposes positive model covariances for all time lags τ, and therefore our model cannot represent the negative values of data covariances which appear at some time lags $\tau \neq 0$. Moreover, for $D_1 < D_0$, model features are similar to those obtained when $D_1 = 0$. As we are modelling an essentially low-order stochastic process, we must use as few parameters as possible. Of course we do not think that the effects of taking $D_1 \neq 0$ are negligible in the evolution of temperature anomalies. However, these effects are similar to those of a seasonal varying feedback in the temperature autocovariances for $D_1 < D_0$. Keeping

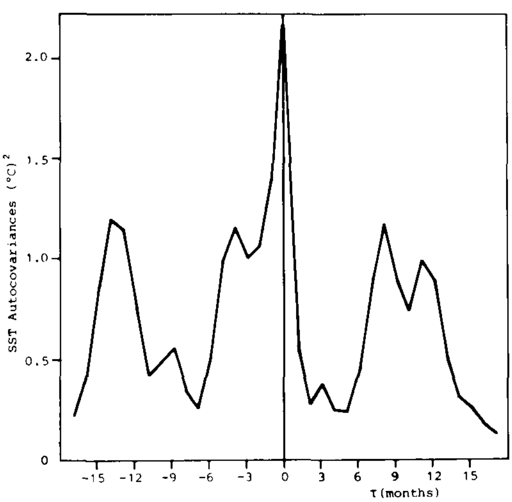

Fig. 1. Data autocovariance $C(t_{12}, \tau)$ at 40° N and 155° E, for $T =$ December as a function of τ.

$D_1 \neq 0$ would add nothing to the model fit, while it would impair its statistical significance.

If the present model is to be an improvement over former ones, it has to represent the secondary peaks we have mentioned above, while maintaining the overall decaying behaviour observed in the data covariances. As was shown in Ruiz de Elvira and Lemke (1982), the modelling of secondary peaks is a result of the cyclostationary stochastic model. To perform the fit between data and model we have therefore given more weight to the peaks of covariances than to their near zero values.

We have run three series of fittings. In the first one, we used three data autocovariances for each location simultaneously, and the same set of parameters was determined for all three. We chose the months of August, December and April. This choice was made to introduce in the calculations months with supposedly different values of the feedback function $v(t)$. We then searched for the set of parameters that minimizes the expression

$$\varepsilon = \sum_{i=1}^{3} \varepsilon_i = \sum_{i=1}^{3} \sum_{j=128}^{18} \{C_{ij}^{\nu}(C_{ij} - \hat{C}_{ij})\}^2, \tag{2.5}$$

while maintaining the secondary peak structure. Here C_{ij} are sample data autocovariances as defined in (2.4) while $\hat{C}_{ij}$ are the model autocovariances. The exponent ν takes into account the weight given to the peaks of the covariances. As there are more terms in the sums (2.5) with near zero values than terms with values markedly different from zero (the peaks), $\nu = 0$ would produce such a set of parameters that the model would better fit the small values of the data covariances. On the other hand, with $\nu > 1$, the model would fit only the values near the peak at $\tau = 0$. A compromise value of $\nu = 1$ was chosen as a working assumption.

In Fig. 2 we show the model autocovariances obtained through this least squares procedure for the location at 20° N and 150° W. Data autocovariances show clearly the secondary peak structure, with the corresponding time lags varying with each month t_i. The model autocovariances follow the same general pattern with alternating maxima and minima. The value of the maxima is controlled by parameter D_0. Lower values of this parameter would produce better fits for the two months of April and August, but then the peak at $\tau = 0$ in the data autocovariances for December

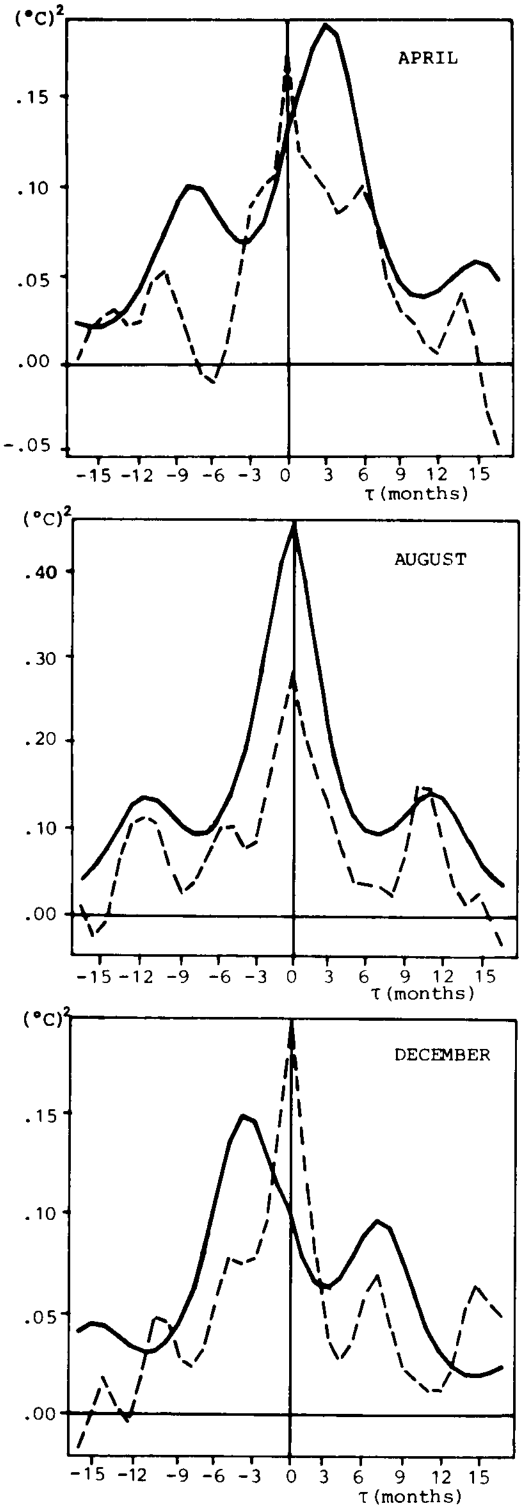

Fig. 2. Autocovariances at 20° N and 150° W. Full lines are model covariances $\hat{C}_{ij}$, dashed lines are data covariances C_{ij}. They are given for three months: April, August and December as functions of lag time τ. The model parameters are: $\beta = 0.10$ month^{-1}, $\beta_1 = 0.25$ month^{-1}, $D_0 = 0.03$ (°C)^2 months^{-1}, $\phi = 1.25$ radians.

would not be fitted at all. Similar problems arise for the other three parameters, and therefore, minimizing ε in (2.5) represents a compromise fit for each particular month.

Seasonal varying stochastic processes need a large number of parameters if each month is to be fitted separately, and the statistical significance of such a fit is rather low. Improved significance for our data may be obtained by fitting the first coefficients in a Fourier series expansion of the autocavariances C_{ij}, as was suggested in Hasselmann and Barnett (1982). In our second run of fittings, we have used the first three coefficients in the Fourier expansion

$$C_{ij} = \sum_{n=0}^{\infty} a_n(\tau_j) \cos(n\omega_a t_i)$$

$$+ \sum_{m=1}^{\infty} b_m(\tau_j) \sin(m\omega_a t_i). \tag{2.6}$$

We have sought the set of parameters that minimize the expression

$$\theta = \sum_{k=1}^{3} \theta_k = \sum_{k=1}^{3} \sum_{j=-18}^{18} \{g_k^{\nu}(\tau_j)(g_k(\tau_j) - \hat{g}_k(\tau_j))\}^2, \tag{2.7}$$

where $g_1 = a_0$, $g_2 = a_1$, $g_3 = b_1$. The g_k are data Fourier coefficients and $\hat{g}_k$ are model Fourier coefficients. Here again we have taken $\nu = 1$ as a working assumption for the weighting function. In Fig. 3 we show the data and model Fourier coefficients obtained through this procedure for the location at 20° N and 150° W. The small peaks present at time lags of about 9 months in the coefficient a_0 would imply higher values of the fitted parameter β_1, which in turn would produce bigger peaks at $\tau = 0$ for the coefficients a_1 and b_1, and therefore a bigger error θ. The result shown is the situation finally reached when θ is a minimum, and it is evidently a compromise between fitting coefficient a_0 and fitting coefficients a_1 and b_1.

We must point out that another minimum (local) θ' exists for a value of the parameter β considerably larger than that shown in Fig. 3. We think it is interesting to indicate the reason for it. For β values three or four times greater than those obtained using (2.7), model autocovariances (1.12) approach those given by a first-order autoregressive model. The decay is then nearly exponential and the secondary peaks almost disappear.

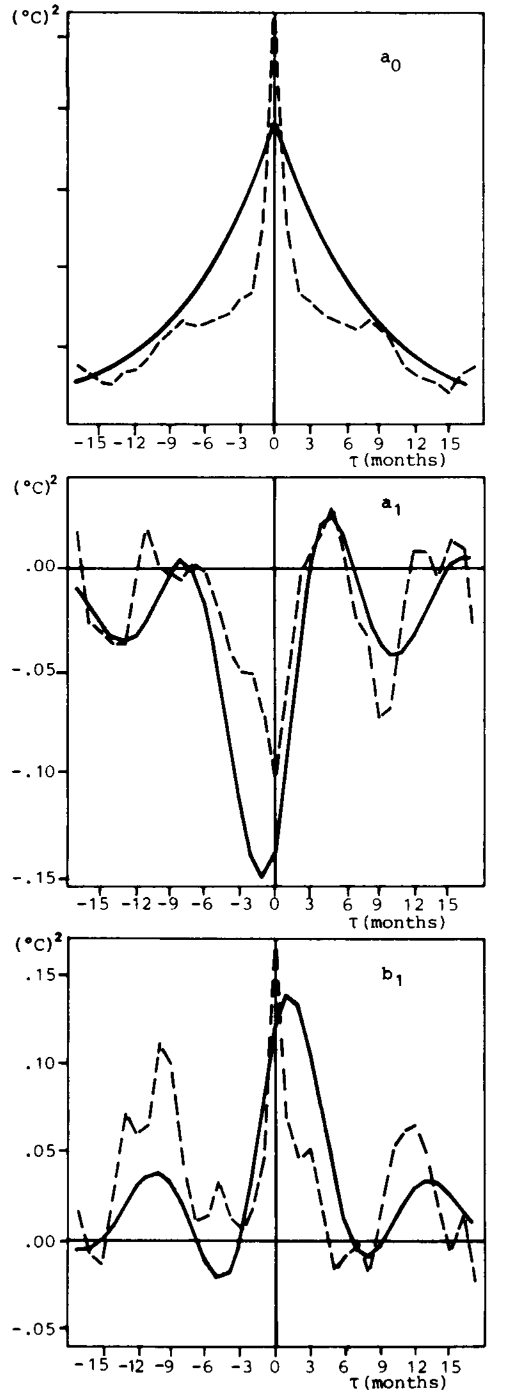

Fig. 3. Fourier coefficients $a_0(\tau)$, $a_1(\tau)$, $b_1(\tau)$, at 20° N and 150° W, as functions of lag time τ. Full lines are model coefficients, dashed lines are data coefficients. The model parameters are: $\beta = 0.12$ month^{-1}, $\beta_1 = 0.10$ month^{-1}, $D_0 = 0.12$ (°C)^2 month^{-1}, $\phi = 2.10$ radians.

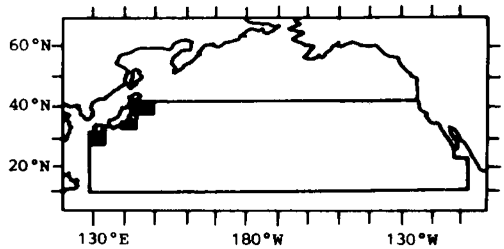

Fig. 4. Central Pacific region. The dashed squares indicate where the model is not valid.

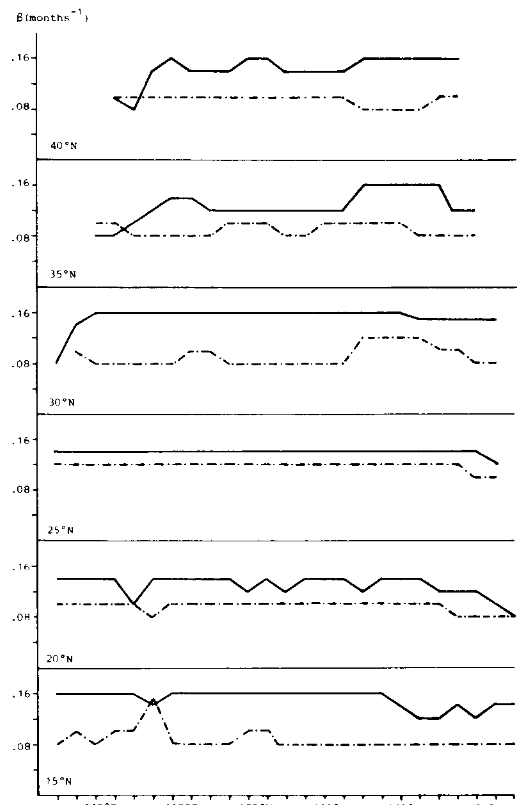

Fig. 5. Spatial variation of parameter β. Full lines correspond to the fit of Fourier coefficients. Dashed lines correspond to the three months fit.

In this case, model coefficients $\hat{a}_1$ and $\hat{b}_1$ are nearly zero, but the model coefficient $\hat{a}_0$ follows rather closely the data coefficient a_0 for time lags τ_j, $-8 < \tau_j < 8$ months, thus producing a local minimum. This minimum could be eliminated in general by using rather involved weighting functions. We think this would introduce too much complexity into what is an essentially simple problem and therefore we stick to the simple weighting function of (2.7).

Considering that the fits were performed with data autocovariances subject to sampling errors, and taking into account the fits for all other locations (which in general follow the pattern shown in Figs. 2 and 3), we conclude that we can satisfactorily describe some interesting features of air sea interaction in the ocean keeping our model simple enough.

To perform a statistical significance test for the parameters and locations where this test is satisfied, we made the hypothesis that the model fits the true covariances perfectly. However, as we use samples of finite size, the model would never fit the measured covariances exactly and therefore ε and θ will never be zero. As is proven in the Appendix, we can find lower limits for the expected values of mean and variance of both ε and θ whose distribution is approximatelly Gaussian. If we denote these limits by $\langle\varepsilon\rangle$, σ_ε^2, $\langle\theta\rangle$, σ_θ^2, we can define two critical errors $\bar{\varepsilon}$ and $\bar{\theta}$:

$$\bar{\varepsilon} = \langle\varepsilon\rangle + 1.65\,\sigma_\varepsilon,$$
$$\bar{\theta} = \langle\theta\rangle + 1.65\,\sigma_\theta, \tag{2.8}$$

such that the probability that either $\varepsilon < \bar{\varepsilon}$ or $\theta < \bar{\theta}$ is 95%. For each fit, the model was considered to be statistically significant, with a 95% confidence limit, if in each case $\varepsilon < \bar{\varepsilon}$ or $\theta < \bar{\theta}$.

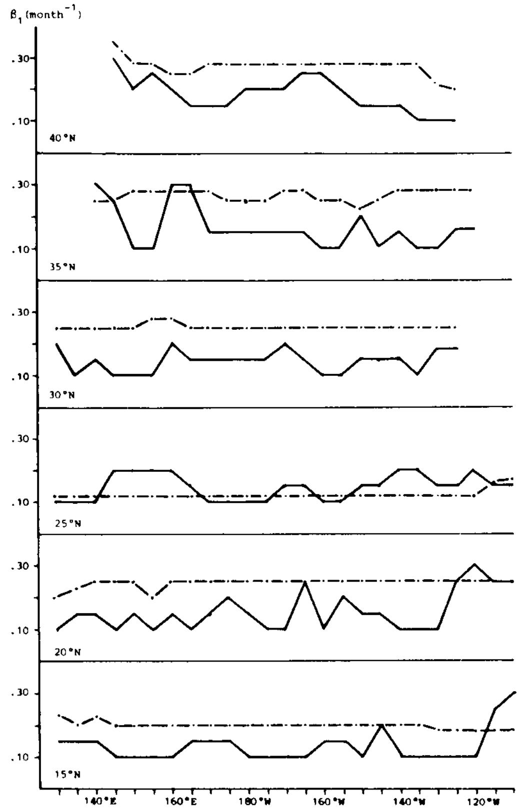

Fig. 6. Spatial variation of parameter β_1. Full lines correspond to the fit of Fourier coefficients. Dashed lines correspond to the three months fit.

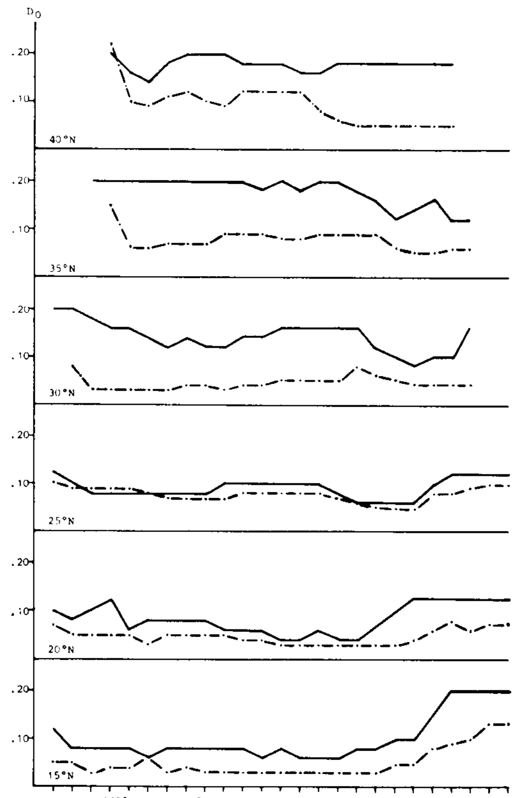

Fig. 7. Spatial variation of parameter D_0. Full lines correspond to the fit of Fourier coefficients. Dashed lines correspond to the three months fit.

3. Results

The boxed in region in Fig. 4 indicates the 5° square grid for which data autocovariances were computed. The dashed regions are those in which *neither of either* fit is significant. As we can see, the model is *not* valid (statistically significant) only near the costs of Japan, where advection (Kuroshio) is probably more important than seasonal varying feedback coefficients. In the remaining squares (98% of the area in study), the model is valid.

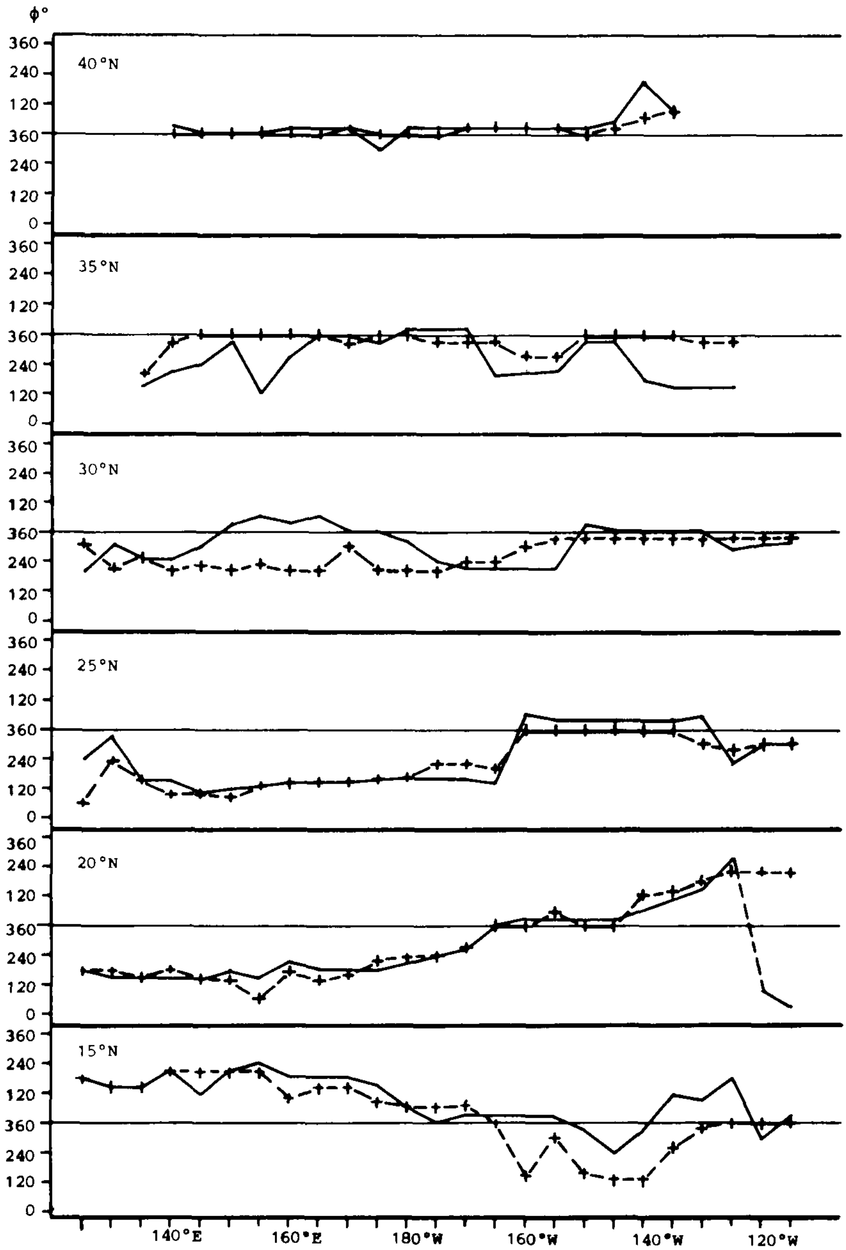

Fig. 8. Spatial variation of parameter ϕ. Full lines correspond to the fit of Fourier coefficients. Dashed lines correspond to the three months fit.

In Figs. 5 to 8, we present the value of the four parameters for the two fits, for each latitude as functions of longitude.

The overall decay parameter β is in general lower in the three months fit than in the fit of the Fourier coefficients. To get a minimum θ in this last fit, it is necessary to use as large β and β_1 as is possible because the biggest source of errors is the difference between model and data for time lags τ, $-8 < \tau < 8$, in the fit of coefficient a_0. The sources of error in the three months fit are more evenly distributed in τ, and therefore we obtain lower β values. In general, values of β are around 0.12 month^{-1}, and are therefore nearly equal to those obtained by Reynolds (1978) using a non cyclo-stationary stochastic model.

The parameter β_1 controls the secondary peak structure which is revealed more clearly by the non-zero values of the Fourier coefficients a_1 and b_1. The only marked spatial variation of this coefficient occurs at 40° N, were β_1 values are decreasing eastward. Far off the coasts, β_1, which represents the intensity of the seasonal modulation of the feedback function, decreases southward. Taken together with β values, we see that while at 40° N and for almost all eastern locations there are some months where the temperature anomalies are being amplified, no such amplification takes place at 15° N.

The parameter D_0 controls the intensity of the stochastic forcing. Here the values obtained in both fittings are nearly equal for the three southern latitudes, while for the northern they are bigger for the fit of the Fourier coefficients. It seems that the three months chosen for the covariance fit underestimate the forcing intensity at these latitudes. The general tendency for D_0 is to increase eastwards at low latitudes, while decreasing in the same direction at mid-latitudes.

4. Summary

We have shown that the cyclo-stationary model introduced by Ruiz de Elvira and Lemke (1982) represents significantly the autocovariances of SST anomalies in the mid-Pacific. Almost all autocovariances in the region studied could be explained by this model.

As an improvement over former models, the present model has the ability to represent the secondary peaks present in the data autocovariances at time lags of about 12 months. Forcing was found strong in the mid-west and low-east Pacific, with quiet areas in the region at 15° N–25° N and 160° W–140° E.

The objective of this work was to test a general climate model by means of comparison of its solution with observational data in a particular case (air–sea interaction). The results obtained for the autocovariances of SST anomalies seem to confirm the possibility of representing climate variables by means of stochastic processes governed by Langevin equations with periodic coefficients.

Improved results for the statistical analysis of the data should include the study of cross covariances between temperature anomalies at different locations. For this study, the inclusion of advection effects is necessary. A more complete seasonal model including advection is underway and we hope to report on it shortly.

A further test of the model will be performed for another climate variable, the sea-ice anomalies for which the underlying assumption of well separated time scales of the model is reasonably satisfied.

5. Acknowledgements

It is a pleasure to thank Prof. K. Hasselmann and K. Herterich for suggesting this topic and for many helpful discussions. We also want to thank the referees of this work for interesting suggestions.

6. Appendix

In this appendix we determine the lower limits of the first and second moments of the probability distribution of ε and θ as defined in (2.2) and (2.4).

To begin, let $\bar{C}_j$ be the strue covariance, corresponding to a seasonal time t, and a lag time τ_j:

$$\bar{C}_j = T(t + \tau_j)T(t)\rangle - \langle T(t)\rangle\langle T(t + \tau_j)\rangle, \qquad (A.1)$$

and $\bar{a}_k$ the true parameters of the model, such that

$$\bar{C}_j = \hat{C}_j(\bar{a}_k), \qquad (A2)$$

where $\hat{C}_j(\bar{a}_k)$ is the model.

Let C_j be the measured covariance and define

$$\delta C_j = C_j - \bar{C}_j = C_j - \hat{C}_j(\bar{a}_k), \qquad (A.3)$$

$$\delta a_k = a_k - \bar{a}_k, \qquad (A.4)$$

where a_k is the fitted set of parameters. From (A.3) and (A.4) we can write:

$$\hat{C}_j(a_k) = \hat{C}_j(\bar{a}_k) - \delta\hat{C}_j(a_k), \qquad (A.5)$$

where

$$\delta\hat{C}_j(a_k) = \sum_{k=1}^{q} (\partial\hat{C}_j(a_k)/\partial a_k)\,\delta a_k. \quad \text{(A.6)}$$

We use a weighted least squares fitting with

$$\varepsilon_1 = \sum_{\tau_j=-18}^{\tau_j=18} (C_{ij}(C_{ij} - \hat{C}_{ij}(a_k)))^2, \quad t = t_1. \quad \text{(A.7)}$$

To simplify the notation we will rewrite (A.7) as

$$\varepsilon = \sum_j (C_j(C_j - \hat{C}_j(a_k)))^2. \quad \text{(A.8)}$$

With the help of (A.5) and (A_3) we can rewrite the last expression as

$$\varepsilon = \sum_j C_j^2(\delta C_j - \sum_{k=1}^{q} (\partial\hat{C}_j(a_k)/\partial a_k))^2. \quad \text{(A.9)}$$

Expression (A.8) must satisfy

$$\partial\varepsilon/\partial a_k = \sum_j 2C_j^2(C_j - \hat{C}_j(a_k)) \times (\partial\hat{C}_j(a_k)/\partial a_k) = 0. \quad \text{(A.10)}$$

In this expression we eliminate $(C_j - \hat{C}_j(a_k))$ by using (A.3), (A.5) and (A.6)

$$\sum_j C_j^2(\delta C_j - \sum_{l=1}^{q} (\partial\hat{C}_j/\partial a_l)(\partial\hat{C}_j/\partial a_k)\,\delta a_l) = 0. \quad \text{(A.11)}$$

If we now define

$$m_{kl} = \sum_j (\partial\hat{C}_j/\partial a_k)(\partial\hat{C}_j/\partial a_l)C_j^2, \quad \text{(A.12)}$$

we can write

$$\delta a_l = \sum_j \sum_{k=1}^{q} m_{kl}^{-1}(\partial\hat{C}_j/\partial a_k)C_j^2,\delta C_j. \quad \text{(A.13)}$$

If we further define

$$R_{ij} = \delta_{ij} - \sum_{l=1}^{q}\sum_{k=1}^{q} C_i C_j(\partial\hat{C}_i/\partial a_l)(\partial\hat{C}_j/\partial a_k)\,m_{kl}^{-1}, \quad \text{(A.14)}$$

we can write

$$\varepsilon = \sum_i \sum_j R_{ij}C_j\,\delta C_j C_i\,\delta C_i. \quad \text{(A.15)}$$

We must now determine the mean and variance of the errors ε. As the SST anomalies have long relaxation times, the sampling errors δC_i and δC_j are correlated. If we write

$$C_j = \frac{1}{N}\sum_{\mu=1}^{N} \delta T_\mu(t)\,\delta T_\mu(t+\tau_j), \quad \text{(A.16)}$$

$$C_i = \frac{1}{N}\sum_{\mu=1}^{N} \delta T_\mu(t)\,\delta T_\mu(t+\tau_i), \quad \text{(A.17)}$$

$$\sigma^2 = \frac{1}{N}\sum_{\mu=1}^{N} (\delta T_\mu(t))^2, \quad \text{(A.18)}$$

$$\sigma_j^2 = \frac{1}{N}\sum_{\mu=1}^{N} (\delta T_\mu(t+\tau_j))^2, \quad \text{(A.19)}$$

$$\sigma_i^2 = \frac{1}{N}\sum_{\mu=1}^{N} (\delta T_\mu(t+\tau_i))^2, \quad \text{(A.20)}$$

$$C_{ji} = \frac{1}{N}\sum_{\mu=1}^{N} \delta T_\mu(t+\tau_j)\,\delta T_\mu(t+\tau_i), \quad \text{(A.21)}$$

then it is possible to prove that

$$\langle\delta C_j\rangle = \langle\delta C_i\rangle \simeq 0 + \mathcal{O}(1/N) \quad \text{(A.22)}$$

$$\langle\delta C_j\delta C_i\rangle \simeq 2\sigma C_{ji}/(N\sqrt{\sigma_j\sigma_j}) + \mathcal{O}(1/N^{1.5})$$
$$\langle(\delta C_j)^2\rangle \simeq 2\sigma\sigma_j/n + \mathcal{O}(1/N^{1.5}). \quad \text{(A.23)}$$

Using these results we can write

$$\langle\varepsilon\rangle \simeq \sum_j\sum_i 2R_{ij}C_iC_jC_{ij}\sigma/(N\sqrt{\sigma_i\sigma_j}), \quad \text{(A.24)}$$

and

$$\sigma_\varepsilon^2 \simeq \sum_j 2R_{ij}^2C_j^4(2\sigma\sigma_j/N)^2 + 2\sum_{i,j}\sum_{k,n} R_{ij}R_{kn}C_iC_jC_kC_n \times ((4\sigma^2/N^2\sqrt{\sigma_j\sigma_i\sigma_k\sigma_n})(C_{in}C_{jk} + C_{ik}C_{ij})). \quad \text{(A.25)}$$

To obtain $\langle\theta\rangle$ and σ_θ^2, we must only perform the corresponding averages following definition (2.7). These moments were computed for all locations and then introduced in the definitions (2.8) and (2.9) to perform the significance test.

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