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A statistical-hydrodynamical approach to problems of climate and its evolution

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ABSTRACT

As an alternative to computing many long numerical integrations of a set of model equations, we explore here a statistical-hydrodynamical method of calculating the equilibrium statistics of an ensemble of atmospheric states, with fixed but arbitrarily prescribed external conditions. The purely mathematical problems are solved or solvable, and the remaining numerical calculations are within the capacity of existing computing machines.

1. Introduction

Many people have doubtless been struck by the similarities between the problem of climate change and certain problems in the statistical mechanics of systems that respond to changing external conditions. One is also struck, however, by the differences between those problems, namely, that conditions external to the atmosphere are affected by the statistics of the atmosphere itself. In contradistinction from classical statistical-mechanical systems, the interactions between the atmosphere, its external sources of energy and the underlying surface operate in both directions.

On reflection, one also sees that some aspects of the problem that have made it difficult to formulate and analyze the interactions between the atmosphere and its underlying surface are exactly those aspects that characterize the problem and which suggest certain idealizations that can simplify it. In particular, it has often been noted that the depth and extent of semi-permanent ice-sheets, deep-ocean temperature and even sea-surface temperature change imperceptibly or very little during the lifetime of any single atmospheric disturbance. Thus, they respond primarily to average atmospheric conditions over a considerable period of time, probably long enough that a sufficiently long time-average is essentially the average over a very large ensemble of possible atmospheric states.

A corollary is that, since the disparity between the response times of the atmosphere and its underlying surface is very great, the atmosphere is close to a state of statistical equilibrium with its external conditions *at all times*, and adjusts quasi-statically to slowly-changing external conditions. This process of quasi-static adjustment is analogous, for example, to the almost instantaneous adjustment of the statistical properties of a gas enclosed in a cylinder—such as pressure and temperature—as its volume is slowly changed by moving a piston.

In this view, the problem of climate change can be naturally separated into two more or less distinct parts. Imagine, for the moment, that we know the current state of the underlying surface and temporarily regard it as fixed. The first part of the problem is to calculate the *equilibrium* statistics of a large ensemble of atmospheric states for those given external conditions.

Given the equilibrium statistics of the atmosphere, the other part of the problem is to determine whether or not the underlying surface is in energy balance, taking into account the modulation of radiant energy, convective transport of sensible and latent heat from the earth's surface and other

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"valving" mechanisms of the atmosphere. All we wish to know about the latter, of course, are their ensemble statistics. If (as is generally the case) the underlying surface is *not* in energy balance, one then calculates the rate at which its state is changing, and predicts a new set of external conditions (after a time interval which is short compared with the periods of slow variations of climate, but very long in comparison with the periods of atmospheric fluctuations).

Having regenerated the external conditions at a later time, we then repeat the process, recalculating the new equilibrium statistics of the atmosphere, again predicting the changes in external conditions, and so on.

This view of the problem is not, of course, much different from the one that many students of climate dynamics have in the backs of their minds, although it is perhaps more explicitly stated here. The main differences in approach lie in the methods of finding the equilibrium statistics of a large ensemble of atmospheric states for prescribed and fixed external conditions.

A number of research groups have been exploring the statistical structure and behavior of "general circulation models" (GCM), by carrying out long time-integrations or "realizations", starting with a single set of atmospheric initial conditions. Various zonal or running time-averages are generally examined to see if the statistics become stationary in time: if so, it is tacitly assumed that the stationary statistics are representative of averages over a large ensemble of realizations in statistical equilibrium with a particular set of external conditions.

It is not only conceivable, but likely that a realization initiated from a single atmospheric state will evolve into one or a few regimes of flow that are highly persistent, but anomalous when compared with the many regimes prevailing during periods of similar external conditions. Thus, in carrying out any single long time-integration, there is some question about how to interpret the statistics: Does one take the stationary statistics during the first persistent regime as the equilibrium statistics, or those of the second regime, or of all such regimes? Does a long integration from a single set of initial conditions generate all possible regimes for a given set of external conditions?

What one really wishes, of course, are the equilibrium statistics of realizations originating in a large ensemble of initial states. It is presumably possible to construct such statistics, at great labor and expense, simply by carrying out many long time-integrations of the GCM equations, for a wide variety of initial conditions, and by forming the desired ensemble statistics. However, since we need only the statistics and are not concerned with all the details of the atmospheric state, it is natural to ask whether or not it is possible to find the equilibrium statistics of the ensemble of realizations, without constructing each realization or, indeed, any such realizations at all.

An attractive and straightforward alternative to constructing many individual realizations lies in the methods of classical statistical mechanics. Let us imagine that the state of a hydrodynamical system is described by the values of a large, but finite number N of discrete variables—such as the values of temperature and wind components at the nodes of a grid, or the modal amplitudes in a truncated Fourier decomposition of those fields. Those discrete values are regarded as the coordinates of a single point in an N-dimensional "phase-space," so that the state of a realization at a particular instant is represented by that point. The evolution of a realization is then represented by a curve of such points, each point corresponding to a particular time. The parametric equations of motion for the trajectory of a point moving along that curve are just the evolution equations for the discrete variables that describe the state of the system.

We then consider a very large ensemble of realizations—so large that the density of "realization-points" in the phase-space is quasi-continuous even when averaged over small overlapping volumes of that space. If all members of the ensemble originate at the same initial instant, no realizations are subsequently created or destroyed, and the density distribution must satisfy the continuity condition for conservation of realizations. That condition—the so-called Liouville equation—is a homogeneous *linear* partial differential equation in which the independent variables are time and the coordinates in phase-space, and the *sole* dependent variable is the density of realizations. Under the additional requirement that the density distribution be independent of time, the solution of the Liouville equation yields the equilibrium probability distribution for the ensemble. Once the probability distribution is known, all ensemble statistics are readily calculated by evaluating integrals over the phase space.

The basic statistical-mechanical ideas and methods remain to be discussed in more detail in later sections, but the brief description of the general approach given above may at least indicate that it is possible, in principle, to calculate the probability distribution without constructing each member of the ensemble. Whether or not this is a practical approach depends on one's ability to solve the Liouville equation under sufficiently general conditions.

The use of statistical-mechanical methods in studying problems of meteorology and climate, while relatively recent, is not novel. Frederiksen and Sawford (1980), for example, have successfully applied them to find the equilibrium probability distribution and kinetic energy spectrum of an ensemble of two-dimensional nondivergent flows of an unforced, inviscid fluid on the surface of a sphere. More closely related to the present work is Hasselmann's (1976) treatment of a simple climate model, in which the slowly-responding part of the system is subjected to high-frequency stochastic forcing whose statistical properties are designed to simulate those of the atmosphere. Hasselmann, in fact, derives a Liouville (Fokker–Planck) equation that may be regarded as a special form of the key equation in the present work.

In a recent paper, Thompson (1983) has outlined a fairly general method of solving the Liouville equation and has applied it to turbulent two-dimensional flows, driven by randomly-varying sources and sinks of vorticity. As it turns out, the mathematical structure of the equations for that case is virtually identical to that of the equations for multi-level, quasi-geostrophic GCM models. Accordingly, our present purpose is to extend that method to the calculation of the equilibrium probability distribution for models that are known to simulate the most important features of the atmosphere's large-scale structure and behavior.

The physical model we choose for illustrating the methods of formulating and solving problems of this kind is substantially that studied by Phillips (1956) in his classic paper on the general circulation. Thus, we start with the two-level quasi-geostrophic vorticity and thermodynamic energy equations, with forcing induced by arbitrarily specified external conditions (e.g., topography, sea-surface temperature and the like). We also display the truncated spectral form of the quasi-geostrophic equations, which give the rate-of-change of each modal amplitude as a function of all modal amplitudes. From the standpoint of statistical treatment, it is most convenient to choose the modal amplitudes as the discrete variables that describe the state of the system, owing to the explicit dependence of their time-derivatives on other variables in the same set. The modal amplitudes will thus serve as the phase-space coordinates. We conclude this section by discussing the invariants of the discrete system, corresponding to the integral invariants of the continuous system without forcing and dissipation.

In Section 3 we formulate the problem of determining the equilibrium probability distribution as a statistical-mechanical question, deriving the appropriate Liouville equation and associated Fokker–Planck equation for systems whose forcing includes a stochastic component. To ensure that the solution is the unique and physically correct one, we first find the solution for a dissipative system with purely stochastic forcing. This we regard as a reference solution, around which the general solution is constructed. The equilibrium form of the Fokker–Planck equation then becomes a nonhomogeneous linear partial differential equation of elliptic type, in which the only unknown is the departure from the reference solution, induced by topographical and thermal forcing.

The method of solving the Fokker–Planck equation is outlined in Sections 4 and 5. Briefly, the solution is represented as a truncated expansion in eigenfunctions (Hermite polynomials) of the separable part of the Fokker–Planck operator. This representation is shown to lead to a simultaneous system of nonhomogeneous linear algebraic equations, in which the only unknowns are the coefficients in the eigenfunction expansion. Although the matrix of coefficients in this system is characteristically large, its solution is made feasible by computing machines in the class of Cray I and Cyber 205. Once the solution of the matrix is known, the whole problem is essentially solved.

A summary of the main results and conclusions is given in Section 6. At this point, however, it should be made clear that our present purpose is not to put forward a theory of climate: it is, rather, to describe a statistical-hydrodynamical view of the problem, to outline a program of further study and to give the results of the first and most difficult conceptual step in carrying it out.

2. The physical system

As indicated in the Introduction, the illustrative model that we choose to describe the atmosphere's dynamical response to its external conditions is one of the two-level, quasi-geostrophic type. The reasons for this choice are primarily those of simplicity: the number of dependent variables is small, and the quasi-geostrophic equations without forcing and dissipation have simple energy and enstrophy invariants. One should also add, of course, that their simplicity does not impair their ability to simulate the main features of the general circulation, as is evidenced in Phillips' (1956) pioneering work.

The model equations, which are well known and clearly derived in Holton (1972), are:

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + \boldsymbol{\kappa}\cdot\nabla\psi_2\times\nabla(\nabla^2\psi_2) + \beta\frac{\partial\psi_2}{\partial x} - \frac{2f\omega}{p_0} = \kappa\nabla^4\psi_2 + M_2 \tag{1}$$

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \boldsymbol{\kappa}\cdot\nabla\psi_1\times\nabla(\nabla^2\psi_1) + \beta\frac{\partial\psi_1}{\partial x} - \frac{2f(\omega_0-\omega)}{p_0} = \kappa\nabla^4\psi_1 + M_1 \tag{2}$$

$$\frac{\partial}{\partial t}(\psi_1-\psi_2) + \boldsymbol{\kappa}\cdot\nabla\left(\frac{\psi_2+\psi_1}{2}\right)\times\nabla(\psi_1-\psi_2) + \frac{\sigma p_0\omega}{2} = \kappa\nabla^2(\psi_1-\psi_2) - \frac{p_0 C}{2f\rho\theta} - \frac{p_0 H}{2f\rho\theta} \tag{3}$$

in which ψ_1 and ψ_2 are the geostrophic streamfunctions at 750 and 250 mb, respectively, and ω is the total derivative of pressure at 500 mb. Since we adopt the usual "β-plane" approximation, the Coriolis parameter f and its northward derivative β are both taken to be constant. The remaining constants are κ, the coefficient of horizontal "eddy diffusion"; p_0, 1000 mb; ρ and θ, standard values of density and potential temperature at 500 mb; and σ, a standard static stability parameter representative of the layer from 750 mb to 250 mb, given by:

$$\sigma = -\frac{1}{f\rho\theta}\frac{\partial\theta}{\partial p}.$$

The quantities ω_0, M_1, M_2, C and H reflect the effects of the atmosphere's external conditions on its topographical and thermal forcing. Since ω_0 is the total derivative of pressure near the underlying ground or sea surface, it depends on both the slope of the surface and the wind velocity near the surface. Similarly, M_1 and M_2, the net vertical "eddy" fluxes of vorticity at 750 and 250 mb, depend on both the vertical distribution of vorticity and the surface roughness. The net vertical convective flux of sensible heat C depends not only on the surface temperature, but also on the temperature at some higher level in the free atmosphere. Finally H, the rate of direct *in situ* heating by change of phase and by absorption or emission of radiant energy also depends on both the surface characteristics and the vertical temperature distribution. In "parameterized" form, therefore, expressions for quantities of this kind involve the external conditions, but also depend on ψ_1, ψ_2 and ω in a definite way, as yet unspecified. The latter conclusion, of course, also implies that (1)–(3) constitute a closed system of equations in ψ_1, ψ_2 and ω, for prescribed external conditions.

Eqs. (1)–(3) may be put in a simpler and more symmetrical form by introducing new dependent variables—half the sum and difference of ψ_1 and ψ_2. At the same time eliminating ω, we may write the resulting equations in the form:

$$\frac{\partial}{\partial t}\nabla^2\Psi + \boldsymbol{\kappa}\cdot\nabla\Psi\times\nabla(\nabla^2\Psi) + \boldsymbol{\kappa}\cdot\nabla D\times\nabla(\nabla^2 D) + \beta\frac{\partial\Psi}{\partial x} - \kappa\nabla^4\Psi = \frac{M_2+M_1}{2} + \frac{f\omega_0}{p_0} = F \tag{4}$$

$$\frac{\partial}{\partial t}(\nabla^2 D - \alpha_r^2 D) + \mathbf{k}\cdot\nabla\Psi\times\nabla(\nabla^2 D - \alpha_r^2 D) + \mathbf{k}\cdot\nabla D\times\nabla(\nabla^2\Psi) + \beta\frac{\partial D}{\partial X} - \kappa\nabla^2(\nabla^2 D - \alpha_r^2 D) = \frac{M_2-M_1}{2} - \frac{f\omega_0}{p_0} + \frac{2f}{p_0}\frac{\partial p}{\partial\theta}(C+H) = G \tag{5}$$

in which $\Psi = (\psi_2 + \psi_1)/2$ and $D = (\psi_2 - \psi_1)/2$. The constant α_r^2 is $8f/\sigma p_0^2$: it is the reciprocal of the square of the "Rossby radius of deformation". The left-hand sides of (4) and (5) describe the internal dynamics of the atmosphere, whereas the right-hand sides reflect the way in which the topographical and thermal forcing of the atmosphere is affected by conditions external to it.

From the standpoint of treating the statistical behavior of the system, it is most convenient to deal with evolution equations for discrete variables that characterize the state of the system. Our next concern, then, is to recast the model equations (4) and (5) as evolution equations for the modal amplitudes in a discrete spectral representation of Ψ and D. That is, we represent Ψ and D as:

$$\Psi(x,y,t) = \sum_{i=1}^{N} A_i(t) f_i(x,y) \tag{6}$$

$$D(x,y,t) = \sum_{i=1}^{N} B_i(t) f_i(x,y) \tag{7}$$

where the f_i are suitable basis functions, preferably orthogonal to each other, and the A_i and B_i are time-dependent amplitude factors. For periodic flow in an infinite β-plane channel, an appropriate set of orthogonal basis functions are the trigonometric functions, which have the properties that:

$$\nabla^2 f_i = -\alpha_i^2 f_i \tag{8}$$

$$\int_A f_i f_j \, dA = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{9}$$

where the integral is taken over the area A, bounded by the walls of the channel and extending over one full period of the flow.

The procedure for deriving the evolution equations for the A_i and B_i, following Lorentz (1960), is first to substitute the representations (6) and (7) into (4) and (5). We then multiply each of those equations by a particular basis function f_k, integrate each term over A, and make use of (8) and (9), with the result that:

$$-\alpha_k^2 \frac{dA_k}{dt_k} - \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ijk}\alpha_j^2(A_iA_j + B_iB_j) + \beta I_k - \kappa\alpha_k^4 A_k = F_k^* \tag{10}$$

$$-(\alpha_k^2 + \alpha_r^2)\frac{dB_k}{dt} - \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ijk}[\alpha_j^2 A_j B_i + (\alpha_j^2 + \alpha_r^2) A_i B_j] + \beta J_k - \kappa(\alpha_k^4 + \alpha_r^2\alpha_k^2) B_k = G_k^* \tag{11}$$

where

$$I_k = \sum_{i=1}^{N} A_i \int_A f_k \frac{\partial f_i}{\partial x} dA$$

$$J_k = \sum_{i=1}^{N} B_i \int_A f_k \frac{\partial f_i}{\partial x} dA$$

$$F_k^* = \int_A f_k F \, dA$$

$$G_k^* = \int_A f_k G \, dA.$$

Since F and G depend on Ψ and D, F_k^* and G_k^* depend on A_i and B_i. The nonlinear interaction coefficients C_{ijk} are:

$$C_{ijk} = \int_A f_k \mathbf{k} \cdot \nabla f_i \times \nabla f_j \, dA.$$

By integrating by parts and imposing the lateral boundary conditions, it is readily shown that C_{ijk} vanishes if any two indices are equal, remains unchanged under any cyclic permutation of indices and reverses sign under any noncyclic permutation.

Although the statistics of an ensemble of solutions of (10) and (11) may simulate those of the atmosphere fairly well, as indicated by Phillips' (1956) results, individual predictions of the A_k and B_k will undoubtedly contain errors. The latter we simulate as stationary stochastic forcing of each mode independently. With this modification, the evolution equations take the form:

$$\frac{dA_k}{dt} = P_k - \kappa\alpha_k^2 A_k + R_k(t) + \beta\alpha_k^{-2} I_k - \alpha_k^{-2} F_k^* \tag{12}$$

$$\frac{dB_k}{dt} = Q_k - \kappa\alpha_k^2 B_k + S_k(t) + \beta(\alpha_k^2 + \alpha_r^2)^{-1} J_k - (\alpha_k^2 + \alpha_r^2)^{-1} G_k^* \tag{13}$$

in which

$$P_k = -\alpha_k^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ijk}\, \alpha_j^2 (A_i A_j + B_i B_j)$$

$$Q_k = -(\alpha_k^2 + \alpha_r^2)^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ijk} [\alpha_j^2 A_j B_i + (\alpha_j^2 + \alpha_r^2) A_i B_j]$$

The functions $R_k(t)$ and $S_k(t)$ represent stationary stochastic processes that vary randomly with time. Our main and remaining concern, then, is to find the equilibrium probability distribution of an ensemble of solutions of (12) and (13).

The invariants of a dynamical system play a prominent role in the treatment of the statistical behavior of nondissipative, unforced dynamical systems and, as we shall see, are sometimes useful in finding exact probability distributions of an ensemble of forced, dissipative systems. We next consider the invariants of the *nonlinear* part of the model under consideration, as given by (12) and (13), but without the effects of dissipation and forcing. Also omitting the terms containing β, we thus consider the nonlinear system:

$$\frac{dA_k}{dt} = P_k$$

$$\frac{dB_k}{dt} = Q_k.$$

By making use of the symmetry properties of C_{ijk}, it can be shown directly from the definitions of P_k and Q_k (cf. Salmon *et al.*, 1976) that two quadratic invariants of this system are:

$$E_1 = \sum_{k=1}^{N} [\alpha_k^2 A_k^2 + (\alpha_k^2 + \alpha_r^2) B_k^2] \qquad (14)$$

$$E_2 = \sum_{k=1}^{N} [\alpha_k^4 A_k^2 + (\alpha_k^2 + \alpha_r^2)^2 B_k^2]. \qquad (15)$$

The invariant E_1 will be recognized as the total energy of the system—i.e., the sum of total available potential energy and total kinetic energy. The invariant E_2 is the total potential enstrophy: it is, in fact, the sum of two invariants—namely, the potential enstrophies at each of the two levels. Their sum is of particular interest, since only the invariants that are quadratic in both A_k and B_k yield simple analytic solutions for the probability distribution of an ensemble of dissipative, randomly forced flows.

3. Formulation of the statistical-hydrodynamical problem

At this stage, we adopt a crucially different point of view, and focus attention on what is essentially a purely mathematical problem: how can one derive the statistical properties of an ensemble of solutions of (12) and (13) from those equations alone, without constructing their solutions in detail? This is precisely the kind of problem for which statistical mechanics was invented. We shall simply borrow some of the most fundamental ideas from that field and extend them to similar kinds of problems in hydrodynamics. A good general introduction to statistical-mechanical methods is given in Landau and Lifshitz (1958) and Prigogine (1962).

As indicated in the Introduction, the decisive turn of reasoning is to regard the $2N$ discrete variables A_k and B_k as coordinates in a $2N$-dimensional "phase-space", so that a point with coordinates $(A_1, A_2, A_3 \ldots A_N, B_1, B_2, B_3 \ldots B_N)$ corresponds to a particular state of the system. The evolution of a single realization then corresponds to a curve in phase-space, made up of the points corresponding to the state at a succession of times. The speed at which the "realization-point" travels along that curve in the A_k-direction is dA_k/dt (usually written $\dot{A}_k$): thus, the parametric equations of motion of a "realization-point" through the phase-space are just (12) and (13), which express the velocity-components $\dot{A}_k$ and $\dot{B}_k$ as functions of position in phase-space.

Let us next imagine that a very large ensemble of realizations were started at the same initial instant—so large that the density of realizations in phase-space is quasi-continuous, when averaged over small, but finite volume elements centered on $(A_1, A_2, A_3 \ldots B_1, B_2, B_3 \ldots)$. For a sufficiently large ensemble, we thus think of the density of realizations as a continuous function of time and position in the phase-space. We now let $\rho(A_1, A_2, A_3, \ldots B_1, B_2, B_3, \ldots t)$ denote the number-density of realizations.

Since we suppose that all realizations of the ensemble started at the same time, no realizations are subsequently created or destroyed. Accordingly, any net flux of realizations across the faces of a small fixed volume element in the phase-space must be exactly balanced by a change in number density within that volume element. Bearing in mind that realizations can move in all $2N$ directions, we may write the continuity condition for conservation of realizations as:

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} \frac{\partial}{\partial A_k} (\rho \dot{A}_k) + \sum_{k=1}^{N} \frac{\partial}{\partial B_k} (\rho \dot{B}_k) = 0.$$

This equation is, of course, exactly analogous to the continuity equation for conservation of mass in physical space.

The Liouville equation is obtained by substituting the expressions for $\dot{A}_k$ and $\dot{B}_k$, given by (12) and (13), into the equation above, with the result that

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} \frac{\partial}{\partial A_k} \{\rho[P_k - \kappa \alpha_k^2 A_k + R_k + \beta I_k - \alpha_k^{-2} F_k^*]\}$$
$$+ \frac{\partial}{\partial B_k} \{\rho[Q_k - \kappa \alpha_k^2 B_k + S_k + \beta J_k - (\alpha_k^2 + \alpha_r^2)^{-1}$$
$$\times G_k^*]\} = 0 \qquad (16)$$

We suppose that many realizations cross each face of a small, but finite volume-element of the phase space per unit time: thus, the term ρR_k in the curly brackets of (16) is to be interpreted as the expected rate of flux of realizations in the A_k-direction, due solely to the random part R_k of the speeds of many realizations. For simplicity, we also suppose that R_k varies on a time-scale much shorter than the relaxation-time of the density-distribution, so that the random flux of realizations may be calculated from an unchanging density-distribution and the statistics of an ensemble of the R_k. It has been shown by de Groot (1962) that, if R_k has a Gaussian distribution, the random transport of realizations is a diffusive process, described by

$$\rho R_k = -\mu_k \frac{\partial \rho}{\partial A_k}.$$

The "diffusion coefficients" μ_k are given by:

$$\mu_k = \int_0^\infty \langle R_k(t) R_k(t+\tau) \rangle \,\mathrm{d}\tau$$

in which the angle brackets denote the ensemble average.

In exactly the same way, we write

$$\rho S_k = -\nu_k \frac{\partial \rho}{\partial B_k}$$

where

$$\nu_k = \int_0^\infty \langle S_k(t) S_k(t+\tau) \rangle \,\mathrm{d}\tau.$$

Substituting these results in (16), we arrive at a Fokker–Planck equation:

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} \frac{\partial}{\partial A_k} \left[P_k \rho - \kappa \alpha_k^2 A_k \rho - \mu_k \frac{\partial \rho}{\partial A_k} \right.$$
$$\left. + \beta I_k \rho - \alpha_k^{-2} F_k^* \rho \right]$$
$$+ \sum_{k=1}^{N} \frac{\partial}{\partial B_k} \left[Q_k \rho - \kappa \alpha_k^2 B_k \rho - \nu_k \frac{\partial \rho}{\partial B_k} \right.$$
$$\left. + \beta J_k \rho - (\alpha_k^2 + \alpha_r^2)^{-1} G_k^* \rho \right] = 0. \qquad (17)$$

This is the full equation that determines ρ.

It should be emphasized that the coefficients of ρ and its derivatives in the square brackets of (17) are known algebraic functions of the *independent* variables A_k and B_k. Thus (17) is a linear partial differential equation in which the only dependent variable is the density ρ. If we now require that the integral of ρ over the entire phase-space (i.e., over all possible contingencies) be unity, ρ is just the probability distribution for the ensemble of realizations. I.e. if

$$\int \ldots \int \rho \prod_{k=1}^{N} \mathrm{d}A_k \mathrm{d}B_k = 1$$

then the probability that a realization lies within the volume-element $\prod_{k=1}^{N} \mathrm{d}A_k \mathrm{d}B_k$ is $\rho \prod_{k=1}^{N} \mathrm{d}A_k \mathrm{d}B_k$. Accordingly, if (17) can be solved for ρ, it is possible to calculate the expected value of any function of the A_k and B_k. In particular, if we set $\partial \rho / \partial t$ to zero in (17), the solution is the equilibrium probability distribution. Solving (17) is our next concern.

4. The role of integral invariants

In preparation for solution of the general equation (17), it will be useful to consider two special cases. The first, the "classical" case of a nondissipative unforced system, will show the close connection between the invariants of such systems and the equilibrium probability distribution of an ensemble of such systems. The second case is that of a dissipative system with purely stochastic forcing—i.e., without the systematic forcing due to interactions with the underlying surface. Study of this case reveals that, for an infinite class of stochastic forcing spectra, there are exact solutions that have the same form as those for nondissipative, unforced systems.

In the first case, the evolution equations for individual realizations are:

$$\begin{aligned} \dot{A}_k &= P_k \\ \dot{B}_k &= Q_k \end{aligned} \tag{18}$$

We have already seen that two invariants of this system are E_1 and E_2, as given by (14) and (15). The corresponding equilibrium form of the Fokker–Planck equation (17) is:

$$\sum_{k=1}^{N} \frac{\partial}{\partial A_k}(P_k \rho) + \sum_{k=1}^{N} \frac{\partial}{\partial B_k}(Q_k \rho) = 0.$$

Examining the expressions for P_k and Q_k, defined immediately after (13), we see that $\partial P_k/\partial A_k = 0$ and $\partial Q_k/\partial B_k = 0$, simply because C_{ijk} vanishes if any two indices are equal. Thus the equation above becomes:

$$\sum_{k-1}^{N} \left(P_k \frac{\partial \rho}{\partial A_k} + Q_k \frac{\partial \rho}{\partial B_k} \right) = 0. \tag{19}$$

Let us now consider the condition under which any function $E(A_1, A_2, A_3 \ldots A_N, B_1, B_2, B_3 \ldots B_N)$ is invariant. It is

$$\dot{E} = \sum_{k=1}^{N} \left(\frac{\partial E}{\partial A_k} \dot{A}_k + \frac{\partial E}{\partial B_k} \dot{B}_k \right) = 0$$

or, from the evolution equations (18),

$$\sum_{k=1}^{N} \left(P_k \frac{\partial E}{\partial A_k} + Q_k \frac{\partial E}{\partial B_k} \right) = 0. \tag{20}$$

Comparing (19) and (20), we see that one solution of (19) is obviously $\rho = E$. It is also apparent, however, that a much more general solution of (19) is $\rho = \rho(E)$, for any differentiable function ρ. To see this, we substitute the expression $\rho = \rho(E)$ into (19), with the result that the L.H.S. becomes:

$$\frac{\mathrm{d}\rho}{\mathrm{d}E} \sum_{k=1}^{N} \left(P_k \frac{\partial E}{\partial A_k} + Q_k \frac{\partial E}{\partial B_k} \right).$$

In view of (20), this expression vanishes, and $\rho = \rho(E)$ is a solution of (19).

The system of evolution equations (18) has two invariants, E_1 and E_2, and one might surmise that (19) has solutions of the form $\rho = \rho(E_1, E_2)$. To test this conjecture, we substitute $\rho = \rho(E_1, E_2)$ into the L.H.S. of (19), which becomes:

$$\sum_{k=1}^{N} P_k \left[\left(\frac{\partial \rho}{\partial E_1} \right)_{E_2} \frac{\partial E_1}{\partial A_k} + \left(\frac{\partial \rho}{\partial E_2} \right)_{E_1} \frac{\partial E_2}{\partial A_k} \right] + \sum_{k=1}^{N} Q_k \left[\left(\frac{\partial \rho}{\partial E_1} \right)_{E_2} \frac{\partial E_1}{\partial B_k} + \left(\frac{\partial \rho}{\partial E_2} \right)_{E_1} \frac{\partial E_2}{\partial A_k} \right]$$

in which $(\partial\rho/\partial E_1)_{E_2}$ is the partial derivative with E_2 held fixed. This expression may also be rewritten as

$$\left(\frac{\partial \rho}{\partial E_1} \right)_{E_2} \sum_{k=1}^{N} \left(P_k \frac{\partial E_1}{\partial A_k} + Q_k \frac{\partial E_1}{\partial B_k} \right) + \left(\frac{\partial \rho}{\partial E_2} \right)_{E_1} \sum_{k=1}^{N} \left(P_k \frac{\partial E_2}{\partial A_k} + Q_k \frac{\partial E_2}{\partial B_k} \right).$$

But both E_1 and E_2 satisfy (20), whence the L.H.S. of (19) vanishes. I.e., $\rho(E_1, E_2)$ is a solution of (19) for a system with two invariants.

The foregoing reasoning does not reveal which of the general class of equilibrium solutions $\rho(E_1, E_2)$ is the one that is approached in physical actuality: this requires additional thermodynamical arguments, outlined in Frederiksen and Sawford (1980). This point is indeed rather academic, since non-dissipative fluid systems are never encountered except in idealization. As it turns out, the question of non-uniqueness of solutions does not arise in the case of dissipative, steadily forced systems.

The value of studying the simple case of non-dissipative unforced systems lies in understanding the conditions under which the statistical effects of the explicitly nonlinear interactions cancel out exactly, when summed over all modes.

Let us now consider the second special case, namely, that in which the "external" forcing functions, F_k^* and G_k^*, are suppressed. For the moment, we also omit from (17) the terms containing β as a factor. In view of the fact that

$\partial P_k/\partial A_k = 0$ and $\partial Q_k/\partial B_k = 0$, the equilibrium form of (17) reduces to:

$$\sum_{k=1}^{N}\left[P_k\frac{\partial\rho}{\partial A_k}-\frac{\partial}{\partial A_k}\left(\kappa\alpha_k^2A_k\rho+\mu_k\frac{\partial\rho}{\partial A_k}\right)\right]+\sum_{k=1}^{N}\left[Q_k\frac{\partial\rho}{\partial B_k}-\frac{\partial}{\partial B_k}\left(\kappa\alpha_k^2B_k\rho+\nu_k\frac{\partial\rho}{\partial B_k}\right)\right]=0. \quad (21)$$

Let us seek the solution of (21) in the form:

$$\rho=c\exp\left[-\frac{\kappa}{2}(aE_1+bE_2)\right] \quad (22)$$

in which a, b, and c are constants. Now, we have just shown that, if ρ is any differentiable function of the two invariants E_1 and E_2, then

$$\sum_{k=1}^{N}\left(P_k\frac{\partial\rho}{\partial A_k}+Q_k\frac{\partial\rho}{\partial B_k}\right)=0.$$

In particular, this must be true if ρ is the product of exponential functions of the invariants. Let us temporarily suppose that ρ may also be written in the form:

$$\rho=c\exp\left[-\frac{\kappa}{2}\sum_{p=1}^{N}\alpha_p^2\left(\frac{A_p^2}{\mu_p}+\frac{B_p^2}{\nu_p}\right)\right]. \quad (23)$$

If so,

$$\mu_k\frac{\partial\rho}{\partial A_k}=-\kappa\alpha_k^2A_k\rho \qquad \nu_k\frac{\partial\rho}{\partial B_k}=-\kappa\alpha_k^2B_k\rho.$$

In that event, the expressions in the parentheses of (21) vanish. Thus (23) is the solution of (21), provided that there is a choice of constants for which (22) and (23) are the same—i.e., for which

$$aE_1+bE_2=\sum_{k=1}^{N}\alpha_k^2\left(\frac{A_k^2}{\mu_k}+\frac{B_k^2}{\nu_k}\right).$$

From (14) and (15), then,

$$\sum_{k=1}^{N}\{(a\alpha_k^2+b\alpha_k^4)A_k^2+[a(\alpha_k^2+\alpha_r^2)+b(\alpha_k^2+\alpha_r^2)^2]B_k^2\}=\sum_{k=1}^{N}\left(\frac{\alpha_k^2}{\mu_k}A_k^2+\frac{\alpha_k^2}{\nu_k}B_k^2\right).$$

Comparing the coefficients of A_k^2 (and also of B_k^2), we see that this condition is satisfied if

$$\mu_k=\frac{\alpha_k^2}{a\alpha_k^2+b\alpha_k^4}$$

$$\nu_k=\frac{\alpha_k^2}{a(\alpha_k^2+\alpha_r^2)+b(\alpha_k^2+\alpha_r^2)^2}.$$

For any particular choice of a and b, these equations define the spectrum of stochastic forcing: for spectra of this kind, the solution of (21) is (23).*

The main reason for studying the case of dissipative flows with purely stochastic forcing is not merely to show that exact and relatively simple solutions can be found for a special class of forcing spectra, but because (23) will serve as a "reference" solution, around which the general solution of (17) can be constructed.

5. Solution of the general Fokker–Planck equation

To simplify solution of (17), we first symmetrize with respect to the independent variables A_k and B_k. Letting $A_k = \mu_k^{1/2}X_k$ and $B_k = \nu_k^{1/2}Z_k$, and transposing terms, we write the equilibrium form of (17) as:

$$\sum_{k=1}^{N}\left[\frac{\partial}{\partial X_k}\left(\frac{\partial\rho}{\partial X_k}+\kappa\alpha_k^2X_k\rho\right)+\frac{\partial}{\partial Z_k}\left(\frac{\partial\rho}{\partial Z_k}+\kappa\alpha_k^2Z_k\rho\right)\right]=\sum_{k=1}^{N}\left\{\mu_k^{-1/2}P_k\frac{\partial\rho}{\partial X_k}+\nu_k^{-1/2}Q_k\frac{\partial\rho}{\partial Z_k}+\mu_k^{-1/2}\frac{\partial}{\partial X_k}[\beta I_k\rho-\alpha_k^{-2}F_k^*\rho]+\nu_k^{-1/2}\frac{\partial}{\partial Z_k}[\beta J_k\rho-(\alpha_k^2+\alpha_r^2)^{-1}G_k^*\rho]\right\} \quad (24)$$

Under the same transformation of variables, the "reference" solution (23) becomes:

$$\rho_0=c\exp\left[-\frac{\kappa}{2}\sum_{k=1}^{N}\alpha_k^2(X_k^2+Z_k^2)\right].$$

* This solution can also be shown to be unique, although the proof is rather tortuous. It depends on the diffusive character of random forcing in the phase-space, dissipation and the requirement that $\rho\to 0$ as $A_k, B_k\to\pm\infty$.

Let us now put the L.H.S. of (24) into a more compact form, noting that:

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial X_k} = -\kappa \alpha_k^2 X_k \qquad \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial Z_k} = -\kappa \alpha_k^2 Z_k.$$

Substituting from these expressions into the L.H.S. of (24), we see that it can be written as:

$$\sum_{k=1}^{N}\left[\frac{\partial}{\partial X_k}\left(\frac{\partial \rho}{\partial X_k} - \frac{\rho}{\rho_0}\frac{\partial \rho_0}{\partial X_k}\right) + \frac{\partial}{\partial Z_k}\left(\frac{\partial \rho}{\partial Z_k} - \frac{\rho}{\rho_0}\frac{\partial \rho_0}{\partial Z_k}\right)\right]$$
$$= \sum_{k=1}^{N}\frac{\partial}{\partial X_k}\left[\rho_0\frac{\partial}{\partial X_k}\left(\frac{\rho}{\rho_0}\right)\right] + \sum_{k=1}^{N}\frac{\partial}{\partial Z_k}\left[\rho_0\frac{\partial}{\partial Z_k}\left(\frac{\rho}{\rho_0}\right)\right].$$

We now define a new dependent variable, $\phi = \rho/\rho_0 - 1$. Equation (24) then takes the form:

$$\begin{aligned}
\sum_{k=1}^{N}\left[\frac{\partial}{\partial X_k}\left(\rho_0\frac{\partial \phi}{\partial X_k}\right) + \frac{\partial}{\partial Z_k}\left(\rho_0\frac{\partial \phi}{\partial Z_k}\right)\right]
&= \sum_{k=1}^{N}\Bigg\{\mu_k^{-1/2}P_k\frac{\partial}{\partial X_k}(\rho_0\phi) \\
&\quad + \nu_k^{-1/2}Q_k\frac{\partial}{\partial Z_k}(\rho_0\phi) \\
&\quad + \mu_k^{-1/2}\frac{\partial}{\partial X_k}\,[\beta I_k\rho_0\phi - \alpha_k^{-2}F_k^*\rho_0\phi] \\
&\quad + \nu_K^{-1/2}\frac{\partial}{\partial Z_\kappa}\,[\beta J_k\rho_0\phi - (\alpha_k^2+\alpha_r^2)^{-1}G_k^*\rho_0\phi]\Bigg\} \\
&\quad + \sum_{k=1}^{N}\Bigg\{\mu_k^{-1/2}P_k\frac{\partial \rho_0}{\partial X_k} + \nu_k^{-1/2}Q_k\frac{\partial \rho_0}{\partial Z_k} \\
&\quad + \mu_k^{-1/2}\frac{\partial}{\partial X_k}[\beta I_k\rho_0 - \alpha_k^{-2}F_k^*\rho_0] \\
&\quad + \nu_k^{-1/2}\frac{\partial}{\partial Z_k}[\beta J_k\rho_0 - (\alpha_k^2+\alpha_r^2)^{-1}G_k^*\rho_0]\Bigg\}
\end{aligned} \tag{25}$$

It should be emphasized that the second summation on the R.H.S. of (25) does not involve ϕ: it is a known function of X_k and Z_k and is thus the nonhomogeneous part of a second-order partial differential equation that is linear in ϕ. The principal virtue of dealing with the nonhomogeneous equation (25) is that ϕ is then a unique continuation around the "reference" solution ρ_0.

In general, the homogeneous part of (25) does not have separable solutions. As we shall show, however, the eigenfunctions of the L.H.S. of (25) are separable and are, moreover, a complete orthogonal set. This suggests that it would be natural to represent ϕ as:

$$\phi = \sum_p \Lambda_p \phi_p \tag{26}$$

where the Λ_p are coefficients to be determined, the ϕ_p are nonsingular solutions of

$$\sum_{k=1}^{N}\left[\frac{\partial}{\partial X_k}\left(\rho_0\frac{\partial \phi_p}{\partial X_k}\right) + \frac{\partial}{\partial Z_k}\left(\rho_0\frac{\partial \phi_p}{\partial Z_k}\right)\right] = -\lambda_p^2\rho_0\phi_p \tag{27}$$

and the λ_p^2 are the corresponding eigenvalues for the infinite domain of the X_k and Z_k. Let us first observe that, if λ_p^2 and λ_q^2 are distinct eigenvalues and ϕ_p and ϕ_q are the corresponding eigenfunctions, (27) implies that:

$$\sum_{k=1}^{N}\frac{\partial}{\partial X_k}\left[\rho_0\left(\phi_q\frac{\partial \phi_p}{\partial X_k} - \phi_p\frac{\partial \phi_q}{\partial X_k}\right)\right] + \sum_{k=1}^{N}\frac{\partial}{\partial Z_k}\left[\rho_0\left(\phi_q\frac{\partial \phi_p}{\partial Z_k} - \phi_p\frac{\partial \phi_q}{\partial Z_k}\right)\right] = (\lambda_q^2 - \lambda_p^2)\rho_0\phi_p\phi_q.$$

As it turns out, $|\phi_p|$ increases without limit as $|X_k|$ or $|Z_k| \to \infty$, but much more slowly than ρ_0 decreases. Thus the integral of the L.H.S. of the equation above vanishes when taken over the entire phase-space. Since λ_p and λ_q are distinct, we conclude that

$$\int_V \rho_0\phi_p\phi_q\,\mathrm{d}V = 0 \quad (p \neq q). \tag{28}$$

The integral is taken over the hypervolume V, and the element of integration $\mathrm{d}V$ is $\prod_{k=1}^{N}\mathrm{d}X_k\mathrm{d}Z_k$. Moreover, since (27) is homogeneous, the ϕ_p may be normalized, so that:

$$\int_V \rho_0\phi_p\phi_q\,\mathrm{d}V = 1 \quad (p = q).$$

That is, the ϕ_p are an orthonormal set of eigenfunctions.

It remains to find the eigenfunctions ϕ_p, the nonsingular solutions of (27). We seek separable solutions of the form:

$$\phi_p = \prod_{k=1}^{N} f_{pk}(X_k)\,g_{pk}(Z_k).$$

Substituting this expression into (27) and dividing both sides by $\rho_0 \phi_p$, we find that the f_{pk} and g_{pk} are solutions of the ordinary differential equations:

$$\frac{d^2 f_{pk}}{dX_k^2} - \kappa \alpha_k^2 X_k \frac{df_{pk}}{dX_k} = -\lambda_{pk}^2 f_{pk}$$

$$\frac{d^2 g_{pk}}{dZ_k^2} - \kappa \alpha_k^2 Z_k \frac{dg_{pk}}{dZ_k} = -\gamma_{pk}^2 g_{pk} \tag{29}$$

such that $\lambda_p^2 = \sum_{k=1}^N (\gamma_{pk}^2 + \lambda_{pk}^2)$. Under a linear transformation of the independent variables X_k and Z_k, (29) takes the form of Weber's equation, discussed in Kamke (1959). Provided that $\gamma_{pk}^2/\kappa\alpha_k^2$ and $\lambda_{pk}^2/\kappa\alpha_k^2$ are non-negative integers, the solutions are nonsingular and square-integrable (with weight ρ_0): they are, in fact, the Hermite polynomials of degree m, defined as:

$$H_m(x) = \sum_{j=0}^{m/2,\, m-1/2} (-1)^j \frac{m!\, x^{m-2j}}{2^j (m-2j)!\, j!}$$

(m integer)

where the upper limit of the summation is $m/2$ for m even and $(m-1)/2$ for m odd. Thus

$$f_{pk} = H_{m_k}(\kappa^{1/2} \alpha_k X_k)$$

$$g_{pk} = H_{n_k}(\kappa^{1/2} \alpha_k Z_k)$$

and the eigenfunctions ϕ_p are:

$$\phi_p = \prod_{k=1}^N H_{m_k}(\kappa^{1/2} \alpha_k \lambda_k)\, H_{n_k}(\kappa^{1/2} \alpha_k Z_k), \tag{30}$$

The corresponding eigenvalues are given by

$$\lambda_p^2 = \sum_{k=1}^N (\kappa \alpha_k^2 m_k + \kappa \alpha_k^2 n_k)$$

in which m_k and n_k take on nonnegative integer values. For a given combination of the m_k's and n_k's, there corresponds a single eigenfunction and its associated eigenvalue. That combination is denoted by the single index p.

Proof of completeness of the set of Hermite polynomials for piecewise continuous functions is given in Courant and Hilbert (1953).

Returning to the solution of (25), we next substitute the eigenfunction expansion (26) for ϕ in (25), freely reversing the order of summation and differentiation. Then, from (27),

$$\sum_p \Lambda_p \sum_{k=1}^N \left[\frac{\partial}{\partial X_k}\left(\rho_0 \frac{\partial \phi_p}{\partial X_k}\right) + \frac{\partial}{\partial Z_k}\left(\rho_0 \frac{\partial \phi_p}{\partial Z_k}\right)\right]$$

$$= -\sum_p \lambda_p^2 \Lambda_p \rho_0 \phi_p$$

$$= \sum_p \Lambda_p \sum_{k=1}^N \left\{ \mu_k^{-1/2} P_k \frac{\partial}{\partial X_k}(\rho_0 \phi_p) + \nu_k^{-1/2} Q_k \frac{\partial}{\partial Z_k}(\rho_0 \phi_p) + \mu_k^{-1/2} \frac{\partial}{\partial X_k}[\beta I_k \rho_0 \phi_p - \alpha_k^{-2} F_k^* \rho_0 \phi_p] + \nu_k^{-1/2} \times \frac{\partial}{\partial Z_k}[\beta J_k \rho_0 \phi_p - (\alpha_k^2 + \alpha_r^2)^{-1} G_k^* \rho_0 \phi_p] \right\} + W$$

in which W stands for the second summation on the R.H.S. of (25). It is a known function and is the nonhomogeneous part of (25).

Finally, multiplying both sides of the equation above by a particular eigenfunction ϕ_q and integrating over the entire phase-space V, we arrive at:

$$-\sum_p \lambda_p^2 \Lambda_p \int_V \rho_0 \phi_p \phi_q \, dV = -\lambda_q^2 \Lambda_q$$

$$= \sum_p \Lambda_p \sum_{k=1}^N \left\{ \mu_k^{-1/2} \int_V \phi_q P_k \frac{\partial}{\partial X_k}(\rho_0 \phi_p)\, dV + \nu_k^{-1/2} \int_V \phi_q Q_k \frac{\partial}{\partial Z_k}(\rho_0 \phi_p)\, dV + \mu_k^{-1/2} \int_V \phi_q \frac{\partial}{\partial X_k}[\beta I_k \rho_0 \phi_p - \alpha_k^{-2} F_k^* \rho_0 \phi_p]\, dV + \nu_k^{-1/2} \int_V \phi_q \frac{\partial}{\partial Z_k}[\beta J_k \rho_0 \phi_p - (\alpha_k^2 + \alpha_r^2)^{-1} \times G_k^* \rho_0 \phi_p]\, dV \right\} + \int_V \phi_q W \, dV \tag{31}$$

As indicated, the summation on the L.H.S. has only one nonzero term, owing to the orthogonality of the eigenfunctions.

Let us now observe that the summation over k on the R.H.S. of (31) depends only on p and q, such constants as κ, α_r, and β and other known constants required for the parameterization of the external forcing F and G: all of its dependence on

X_k and Z_k has been removed by the integration over V. Accordingly, (31) has the form:

$$\lambda_q^2 \Lambda_p + \sum_p \chi_{pq} \Lambda_p = - \int_V \phi_q W \, \mathrm{d}V \qquad (32)$$

in which the χ_{pq} are known coefficients of the Λ_p's. Hence, if we truncate the eigenfunction expansion at some large value of p, say p_M, and let q range from 1 to p_M, then (32) generates a complete simultaneous system of p_M nonhomogeneous linear algebraic equations, although a very large one. Fortunately, probability distributions are characteristically smooth, and what fine-structure they do possess is unimportant at points far removed from the origin of the phase-space. One might hazard the guess that p_M is 1000–10,000 for models of the current order of complexity. The solution of a 1000×1000 matrix is not a trivial computational problem, but not out of reach of existing computers.

The integrals in (31) can all be evaluated analytically. Some mèthods of transforming them into "inner products" of eigenfunctions are outlined in Thompson (1983): these are particularly useful in identifying quickly the values of k for which the integrals in (31) vanish, owing to the orthogonality of the ϕ_p. The first two integrals under the summation over k, for example, vanish for almost all values of k. All of these reduction methods make use of a convenient property of the eigenfunctions, namely, that

$$\frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = n H_{n-1}(x)$$

At this stage, it is not entirely clear that it is not more economical to solve the basic equation (25) numerically, by finite-differencing with respect to the phase-space coordinates, and applying the relaxation method or other well-known methods for solving nonhomogeneous equations of elliptic type. An attractive feature of relaxation methods is that they are very efficient if the first approximation to the solution of (25) is nearly the true solution: since the external conditions change very slowly, the solution of (25) for external conditions at one time might be expected to provide a good first approximation to the solution for the external conditions at a slightly later time. The purpose of this section is merely to show that there is a straightforward analytical approach to the solution of (25).

Finally, we point out that the solution of (25) for ϕ also determines the probability distribution ρ. From the latter, we can calculate the ensemble average of any function of the modal amplitudes X_k and Z_k by evaluating integrals over the phase-space. I.e.,

$$\langle F(X_k, Z_k) \rangle = \int_V \rho F(X_k, Z_k) \, \mathrm{d}V$$

where the angle brackets denote the ensemble average. In short, the equilibrium statistics of an ensemble of atmospheric states, for fixed external conditions, are calculable from the solution of (25).

6. Summary and conclusions

The main purpose of this paper has been to propose and explore an alternative to the calculation of equilibrium statistics by direct construction of an ensemble of numerical integrations, based on an atmospheric model with fixed external conditions. The approach is an extension of the methods of classical statistical mechanics, via the condition for conservation of realizations in a phase-space of many dimensions (Liouville equation). These methods have been applied to the simplest type of multi-level, quasi-geostrophic model, with dissipation and arbitrarily specified external forcing. With the inclusion of stationary stochastic forcing to simulate the errors inherent in the model, the Liouville equation takes the form of a Fokker–Planck equation, in which the modal amplitudes in a spectral representation of the flow are the independent variables, and the equilibrium probability distribution of the ensemble is the sole dependent variable.

We next discuss the connection between the invariants of a dynamical system and the probability distribution of an ensemble of such systems, in the course of which we derive the exact probability distribution of an ensemble of dissipative baroclinic flows, for a two-parameter family of random forcing spectra. Building on these results, we then outline an analytic procedure for solving the general Fokker–Planck equation, based on an eigenfunction expansion of the probability distribution in phase-space. It is shown to lead to a complete system of nonhomogeneous linear algebraic equations, in which the only unknowns are the coefficients of the eigenfunctions (Hermite polynomials). Once the latter system has been solved, one can construct directly the equilibrium probability distribution and, from it, any desired atmospheric statistics of an infinite ensemble.

The square matrix to be solved is indeed very large (probably 1000–10,000 on a side), but it appears that it can be solved numerically by computing machines of the class of Cray I or Cyber 205, or by projected machines of even greater speed and capacity that will be available soon.

The ways in which these methods can be used in the study of climate and its evolution are discussed at some length in the Introduction.

Finally, it should be added that analytic results (based on a severely truncated eigenfunction expansion of the probability distribution) have been compared with the statistics of a large ensemble of numerical integrations of a dissipative, randomly-forced low-order system. The equilibrium kinetic energy spectra, as calculated analytically and numerically, differ by a few percent—well within the sampling error of the numerical ensemble. These calculations are described in the Appendix to Thompson (1983).

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