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Energy analysis of a recent approximation to the atmospheric primitive equations

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ABSTRACT

An approximation to the thermodynamic equation was introduced by Holton which was motivated by an energy principle involving a quadratic expression in temperature deviation, which he identified as available potential energy. The approximation consists of neglecting the component of adiabatic warming/cooling due to the temporally and horizontally variable part of the specific volume on a constant pressure surface, $W\kappa T^*$. If the approximation is made, the system lacks a total energy principle involving either total potential energy or available potential energy (except in one special case). Although this approximate form of the primitive equations does not cause substantial error in a numerical simulation of the annual cycle of a zonally-symmetric circulation, there appears to be no significant advantage to its application. Use of the standard set of primitive equations is suggested.

1. Introduction

In his classical paper entitled "Energy and Numerical Weather Prediction", Lorenz (1960) emphasized that approximate sets of dynamical equations which are used in numerical models of atmospheric circulation systems should satisfy certain energy conservation constraints. In developing valid filtering approximations, Lorenz relied primarily on the principle that the sum of kinetic energy (KE) and total potential energy (TPE), which consists of potential energy (P) plus internal energy (I), integrated over the entire atmospheric domain being modeled, must be conserved when diabatic heating, mechanical dissipation, and boundary fluxes are neglected. In this paper it is shown that several model calculations (e.g., Holton, 1976; Holton, 1979; Holton and Wehrbein, 1980; Lordi et al., 1980; Hsu, 1980; Dunkerton et al., 1981, Bridger and Stevens, 1982) of stratospheric circulations which have been reported in the literature of the past few years in fact do not conserve total energy. The common approximation under question in these primitive equation models is the neglect of a component of the adiabatic warming/cooling in the thermodynamic energy equation.

The issue is somewhat analogous to the note and correspondence of Phillips (1966, 1968) and Veronis (1968). Phillips insisted on the impropriety of a set of dynamic equations which do not possess an exact angular momentum-conserving set, erring by a very small geometric factor of order $(r - a)/a \leq 10^{-2}$, where a is the mean radius of the earth and r is the radius from earth's center to an air parcel. From a similar point of view, it seems important that a dynamically approximate set of governing equations allow motions which, in the absence of external sources and sinks, conserve the total energy of the system. Indeed, when originally introducing the approximate set of equations considered in this paper, Holton (1975) stated, "scaling arguments alone do not suffice to determine the suitability of a model. We must additionally be certain that in the absence of energy sources or sinks the model is energy conserving. In some cases this constraint will require retention of terms which would be omitted on scale considerations." It is just this requirement of energy conservation

which the present paper intends to reinforce by examining the approximate set of equations proposed by Holton.

In Section 2, the energetics of the standard primitive equations are reviewed. Section 3 examines the energy conservation principles that result from the approximation introduced by Holton. It is shown that total energy is not conserved. Section 4 analyzes the effects of the approximation on the circulation and energy cycle in a numerical simulation of zonally-averaged circulations.

2. Energetics of a primitive equations model

Following Holton (1975, p. 29) with slightly different notation, the (hydrostatic) primitive equations in "log p" coordinates can be written:

$$\frac{Du}{Dt} - \frac{uv}{a}\tan\phi - fv + \frac{\partial\phi}{a\cos\phi\,\partial\lambda} = 0 \tag{1}$$

$$\frac{Dv}{Dt} + \frac{u^2}{a}\tan\phi + fu + \frac{\partial\phi}{a\,\partial\phi} = 0 \tag{2}$$

$$\frac{DT}{Dt} + WkT = J/c_p \tag{3}$$

$$\nabla_3\cdot(\rho\mathbf{v}) = 0 \tag{4}$$

$$\frac{\partial\Phi}{\partial Z} - RT = 0 \tag{5}$$

The symbols in (1)–(5) have the following definitions:

t = time
λ = longitude
ϕ = latitude
p = pressure
$Z = \ln(p_s/p)$, where p_s is a constant reference pressure
$\rho = p/RT_s$ pseudo-density, actually proportional to pressure but with density units, where T_s is a constant
u = eastward velocity component
v = northward velocity component
W = upward velocity component (DZ/Dt)
$\mathbf{v} = (u, v, W)$

$$\frac{D}{DT} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_3 = \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + W\frac{\partial}{\partial Z}$$

$f = 2\Omega\sin\phi$
R = gas constant for dry air (assumed constant)
c_p = specific heat at constant pressure (assumed constant)
$\kappa = R/c_p \cong 2/7$
Φ = geopotential
T = temperature
J = diabatic heating rate per unit mass
a = radius of earth

$$\nabla_2\cdot\mathbf{A} \equiv \frac{\partial A_\lambda}{a\cos\phi\,\partial\lambda} + \frac{\partial(A_\phi\cos\phi)}{a\cos\phi\,\partial\phi}$$

for any vector $\mathbf{A} = (A_\lambda, A_\phi, A_Z)$

$$\nabla_3\cdot\mathbf{A} \equiv \nabla_2\cdot\mathbf{A} + \frac{\partial A_Z}{\partial Z}$$

Eqs. (1)–(5) obey the approximate angular momentum principle suggested by Phillips (1966), with angular momentum per unit mass $M \equiv a\cos\phi\,(\Omega a\cos\phi + u)$. Multiplying (1) by ρu and (2) ρv yields a kinetic energy equation:

$$\frac{\partial}{\partial t}(\rho K) = -\nabla_3\cdot[\rho\mathbf{v}(K+\Phi)] + R\rho WT, \tag{6}$$

where $K \equiv \frac{1}{2}(u^2 + v^2)$ is the local kinetic energy per unit mass appropriate to a hydrostatic system. Multiplying (3) by ρc_p gives a total potential energy equation:

$$\frac{\partial}{\partial t}(\rho c_p T) = -\nabla_3\cdot(\rho\mathbf{v}c_pT) - R\rho WT + \rho J. \tag{7}$$

The total energy equation is simply the sum of (6) and (7):

$$\frac{\partial}{\partial t}\rho(K + c_pT) = -\nabla_3\cdot[\rho\mathbf{v}(K + \Phi + c_pT)] + \rho j. \tag{8}$$

Clearly, in the absence of boundary fluxes and with the assumption of no net heating,

$$\int J\rho\,\mathrm{d}V = 0, \tag{9}$$

where an element of pseudo-volume $\mathrm{d}V \equiv (a\cos\phi\,\mathrm{d}\lambda)\,(a\,\mathrm{d}\phi)\,(H_s\,\mathrm{d}Z)$ and $H_s \equiv RT_s/g$ is a constant

reference "scale height", the total energy of the atmosphere is conserved:

$$\frac{d}{dt}\int(K + c_pT)\rho\, dV = 0. \tag{10}$$

Lorenz (1955) showed that available potential energy (APE) is often a more useful concept than total potential energy for understanding the interaction between kinetic energy (due to the motion of the fluid) and the energy associated with the mass distribution within the fluid. This utility results from the fact that only a small fraction of the total potential energy is generally available for conversion to kinetic energy under an adiabatic redistribution of the mass field. The remainder of the potential energy is the energy contained within some reference state, which is often taken to be horizontally homogeneous and at rest; as discussed by Dutton and Johnson (1967), other reference states with dynamical constraints could also be selected.

In the theory of available potential energy, the reference state is generally defined in terms of the mass field relative to surfaces of constant potential temperature. Isentropic coordinates then provide a useful framework because air parcels are constrained to remain on their respective isentropic surface when adiabatic processes are assumed. However, for a variety of reasons, pressure is usually the vertical coordinate chosen for numerical models. A primary rationale for this choice is that the hydrostatic approximation (for which pressure is a logical vertical coordinate) is generally more applicable for many large-scale problems than is the assumption of adiabatic motion. Additionally, observations are reported and analyzed routinely in isobaric coordinates.

Lorenz showed that available potential energy can be approximated as a weighted integral of the temperature variance on pressure surfaces divided by the static stability. We therefore seek an energy conservation statement relating kinetic energy and a quantity involving temperature variance. Defining $\bar{F}$ to be the horizontal area average of any quantity F on a constant pressure surface over the entire domain and F' to be the local deviation therefrom ($F' \equiv F - \bar{F}$), the kinetic energy eq. (6) averaged over the area of a pressure surface can be written

$$\frac{\partial}{\partial t}(\rho\bar{K}) = -\frac{\partial}{\partial Z}[\rho\overline{W'(K + \Phi)}] + R\rho\overline{W'T'}. \tag{11}$$

In obtaining (11), we have assumed that the horizontally averaged divergence of any flux vanishes; i.e., $\overline{\nabla_2 \cdot \rho v F} = 0$ for any quantity F. This follows either from a domain in which pressure surfaces are closed surfaces, or from the absence of any fluxes across the horizontal boundaries of an open pressure surface. An implication from the continuity eq. (4) is that $\bar{W} = 0$ on any pressure surface, assuming that $pW \to 0$ as $Z \to \infty$; hence, the total mass above a pressure surface is invariant. For purposes of this discussion, boundary fluxes at the lower surface have been tacitly ignored.

Clearly $R\rho\overline{W'T'}$ represents the conversion from potential energy to kinetic energy. This term must be present (with opposite sign) in an equation for APE. Multiplying the total potential energy eq. (7) by T' and averaging over a pressure surface, we find

$$\frac{\partial}{\partial t}\left(\rho\frac{\overline{RT'^2}}{2\hat{S}}\right) = -\frac{\partial}{\partial Z}\left[\rho\overline{W'\left(\frac{RT'^2}{2\hat{S}}\right)}\right]$$

$$R\rho\overline{W'T'} + \frac{\kappa}{\hat{S}}\rho\overline{J'T'} - \rho\overline{W'\left(\frac{RT'^2}{2\hat{S}}\right)}\left(\frac{\partial \ln \hat{S}}{\partial Z} + 2\kappa\right)$$

$$-\rho\left(\frac{\overline{RT'^2}}{2\hat{S}}\right)\frac{\partial \ln \hat{S}}{\partial t} \tag{12}$$

Here S is a measure of static stability in the log p coordinate system which is proportional to the vertical gradient of potential temperature θ

$$S \equiv \frac{\partial T}{\partial Z} + \kappa T = e^{-\kappa Z}\frac{\partial \theta}{\partial Z} = \left(\frac{p}{p_0}\right)^{\kappa}\frac{\partial \theta}{\partial Z}. \tag{13}$$

Adding (11) and (12) with this notation results in:

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{K} + \frac{\overline{RT'^2}}{2\hat{S}}\right)\right] =$$

$$= -\frac{\partial}{\partial Z}\left[\rho\overline{W'\left(K + \phi + \frac{RT'^2}{2\hat{S}}\right)}\right] + \frac{k}{\hat{S}}\rho\overline{J'T'}$$

$$-\rho\overline{W'\frac{RT'^2}{2\hat{S}}}\left(\frac{\partial \ln \bar{S}}{\partial Z} + 2\kappa\right)$$

$$-\rho\left(\frac{\overline{RT'^2}}{2S}\right)\left(\frac{\partial \ln \hat{S}}{\partial t}\right) \tag{14}$$

The final two terms on the right side of (14) prevent $\int(\overline{RT'^2}/2\hat{S})\rho\,dV$ from being an exact expression for APE with a rigorous energy conservation principle.

3. Energetics of the approximate form of the primitive equations

Holton (1975) approximated the thermodynamic energy eq. (3) with the stated objective of obtaining an expression for available potential energy which conserves the sum of APE and KE exactly in the absence of friction, diabatic forcing, and boundary fluxes. The recent stratospheric primitive equation models have followed his strategy.

The first step is to define a temperature deviation T^* from a *specified* function of the vertical coordinate, $T_0(Z)$, which resembles the observed globally averaged vertical temperature profile $\hat{T}(Z, t)$:

$$T^* \equiv T - T_0(Z) \tag{15}$$

The arbitrariness in T_0 (and therefore also in available potential energy below) can be removed by defining it to be the initial global temperature $\hat{T}(Z, 0)$. The thermodynamic energy eq. (3) can then be re-written

$$\frac{DT^*}{Dt} + W\left(\frac{dT_0}{dZ} + \kappa T_0\right) + W(\kappa T^*) = \frac{J}{c_p} \tag{3a}$$

The crucial approximation is to neglect the $W\kappa T^*$ term, which is observed to be much smaller than the $W\kappa T_0$ term. Note that it may be more appropriate to compare κT^* with $S_0 \equiv dT_0/dZ + \kappa T_0$. Fig. 1 displays κT_0 and S_0 for the US Standard Atmosphere during January and July at 45° N, from the surface to the lower thermosphere. In regions where the temperature lapses with height (troposphere and mesosphere), S_0 can be significantly smaller than κT_0. For example, in the summer mesosphere, Holton's (1975) estimate of a 10% potential error from neglecting κT^* relative to κT_0 could actually be almost doubled by a comparison of κT^* with S_0.

One additional point should be mentioned regarding the interpretation of S_0. It is often stated that the square of the buoyancy frequency N is

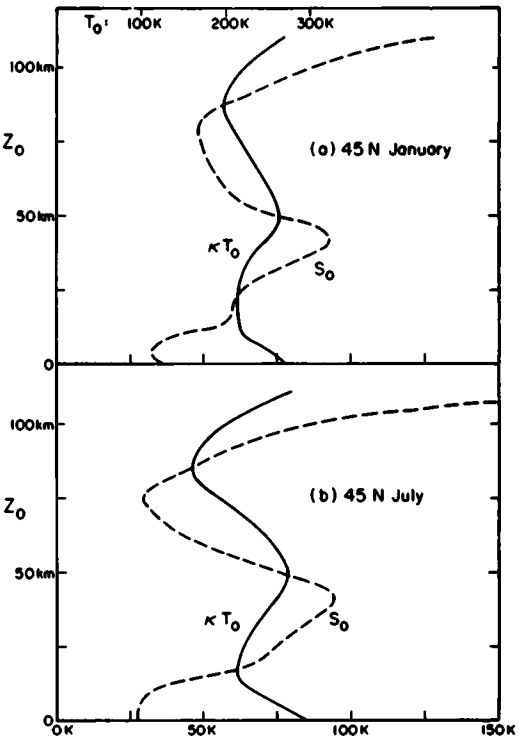

Fig. 1. κT_0 and stability S_0 for the US Standard Atmosphere at latitude 45° N during (a) January and (b) July. Scale for temperature T_0 is at the top.

directly proportional to S_0 (e.g., Holton, 1975; Holton and Wehrbein, 1980):

$$N^2 \equiv \frac{RS_0}{H_s^2} \tag{16}$$

where H_s is an arbitrary constant reference scale height. However, if we define geopotential surfaces Φ_0 corresponding hydrostatically to the temperature profile T_0, and further define geopotential height $z_0 \equiv \Phi_0 g$, then Z and z_0 are related hydrostatically by a variable scale height $H_0(Z)$:

$$\frac{dz_0}{dZ} = \frac{RT_0}{g} = H_0(Z). \tag{17}$$

Using the conventional definition of buoyancy frequency as g times the vertical derivative of log (potential temperature) with respect to *height*, and then identifying height with geopotential height z_0, we find that

$$N^2 = \frac{RS_0}{H_0(Z)^2}. \tag{18}$$

Since H_0 varies considerably in the stratosphere/mesosphere, the difference in the buoyancy frequency as defined by (16) *vis-à-vis* (18) is nearly a factor of two, according to Fig. 1.

In order to compare the energetics with and without the Holton approximation, the tracer δ is applied to the $W\kappa T^*$ term in (3a). With $\delta = 1$, no approximation; $\delta = 0$ is applicable for the Holton model with this term neglected. The total potential energy equation is easily found to be

$$\frac{\partial}{\partial t}(\rho c_p T) = \frac{\partial}{\partial t}(\rho c_p T^*) = -\nabla_3 \cdot (\rho \mathbf{v} c_p T^*)$$

$$- \rho W c_p S_0 - \delta \rho W c_p \kappa T^* + \rho J \qquad (7a)$$

Multiplying this equation by $(RT^*/c_p S_0)$ and averaging over a horizontal pressure surface yields an equation similar to (12):

$$\frac{\partial}{\partial t}\left(\rho \frac{\overline{RT^{*2}}}{2S_0}\right) = -\frac{\partial}{\partial Z}\left[\rho \overline{W'\left(\frac{RT^{*2}}{2S_0}\right)}\right]$$

$$- R\rho \overline{W'T^*} + \frac{\kappa}{S_0}\rho \overline{JT^*}$$

$$- \rho \overline{W'\left(\frac{RT^{*2}}{2S_0}\right)}\left(\frac{\mathrm{d} \ln S_0}{\mathrm{d}z} + 2\kappa\delta\right) \qquad (12a)$$

In this form, T^* (in 12a) replaces T' (in 12) and the specified stability $S_0(Z)$ replaces the globally averaged stability $\bar{S}(Z,t)$. Recalling that $\bar{W} = 0$, the conversion terms between kinetic and potential energy forms are equivalent: $\overline{W'T'} = \overline{W'T^*}$. However, the diabatic heating terms are *not* equivalent because the horizontal average of T^* does not vanish: $\overline{JT^*} = \overline{J'T'} + \bar{J}\,\bar{T}^*$. Adding (12a) to the kinetic energy eq. (11), we obtain

$$\frac{\partial}{\partial \tau}\left[\rho\left(\bar{K} + \frac{\overline{RT^{*2}}}{2S_0}\right)\right] = -\frac{\partial}{\partial Z}$$

$$\times\left[\rho \overline{W'\left(K + \Phi + \frac{RT^{*2}}{2S_0}\right)}\right] + \frac{\kappa}{S_0}\rho \overline{JT^*}$$

$$- \rho \overline{W'\left(\frac{RT^{*2}}{2S_0}\right)}\left(\frac{\mathrm{d} \ln S_0}{\mathrm{d}Z} + 2\kappa\delta\right). \qquad (14a)$$

Considering eqs. (14) and (14a), we see that neither $\int (\overline{RT^2}/2\bar{S})\rho \,\mathrm{d}V$ *nor* $\int (\overline{RT^{*2}}/2S_0)\rho \,\mathrm{d}V$ is an appropriate expression for available potential energy. If the heating vanishes and $W' = 0$ on the upper and lower boundaries (i.e., no mechanical forcing through pressure-work terms), the sum of kinetic energy and available potential energy in the entire fluid domain must be conserved. In the Holton approximation, the term involving the temporal change of stability is removed by using a prescribed, time-dependent stability. The flux term proportional to 2κ is dropped by neglecting the $W\kappa T^*$ term in (3a). The other term involving $\mathrm{d} \ln S_0/dZ$ remains.

This energy sum is conserved if $S_0(Z)$ is chosen to be independent of height. Fig. 1 shows that this is not a good approximation in the lowest 100 km of the earth's atmosphere, where S_0 varies by a factor of three to four. In this region, $\mathrm{d} \ln S_0/\mathrm{d}Z$ varies between +1 and −1, while $2\kappa \cong 0.6$. Hence the variable static stability term makes about an equal contribution as the neglected 2κ term to the energy budget. Additionally, if S_0 were prescribed to be a constant, $T^* = \bar{T} - T_0 + T'$ would be dominated by $(\bar{T} - T_0)$ rather than representing the more dynamically relevant temperature perturbation T'.

An even more significant difficulty arises with the approximate formulation ($\delta = 0$). *Total energy is not conserved.* Setting $\delta = 0$ in the total potential energy eq. (7a), and assuming that boundary fluxes vanish, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int c_p T \rho \,\mathrm{d}V = \int J\rho \,\mathrm{d}V. \qquad (19)$$

Here there is no conversion term from total potential energy to kinetic energy. Therefore, when added to the global kinetic energy budget,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int K\rho \,\mathrm{d}V = \int RWT\rho \,\mathrm{d}V, \qquad (20)$$

we find a spurious source of energy resulting from the unbalanced conversion term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (K + c_p T)\rho \,\mathrm{d}V = \int RWT\rho \,\mathrm{d}V + \int J\rho \,\mathrm{d}V. \qquad (21)$$

As a result, the Holton model contains no total energy invariants. Hence, in spite of the claim that "this approximation is necessary to obtain a set of energetically consistent equations" (Boyd, 1976), the exact opposite is actually true.

We may contrast this approximation to the primitive equations with a similar approximation by Lorenz (1960) to the quasi-geostrophic

equations. He defined APE in terms of the variance of temperature on a pressure surface; but the mean temperature consisted of the global average, not an arbitrarily specified function of pressure. The stability was an arbitrarily specified function of pressure, as in the Holton model. In the quasi-geostrophic model, the key approximation for the formulation of APE is the neglect of horizontal temperature advection by the divergent component of the wind. As a result, the offending final term in (12a) and (14a) is missing, as is the vertical flux divergence term. Hence APE can be defined in such a way that KE + APE is conserved. However, Lorenz pointed out that total energy is not conserved in this quasi-geostrophic formulation. Both approximate systems have this feature in common.

4. A zonally symmetric annual cycle with the approximate ($\delta = 0$) and exact ($\delta = 1$) thermodynamic equation

In Section 3, we found that the approximate primitive equations formulated by Holton do not contain any total energy invariants. This causes the possibility of error in any calculation using the approximate set.

The approximate set of equations have been used in studying stratospheric sudden warmings forced by stationary wave fluxes from below (Holton, 1976; Hsu, 1980; Lordi, et al., 1980; Dunkerton et al., 1981; Bridger and Stevens, 1982) and the annual cycle of the stratosphere/mesosphere (Holton and Wehrbein, 1980; Holton and Wehrbein, 1981). It is additionally conceivable that the approximate set could be used for instability studies. In this investigation, we have chosen to consider a simple dynamical model of the annual cycle to illustrate the effect of the approximation.

Consider the zonally averaged circulation on an f-plane at latitude 45° N. Following Holton and Wehrbein (1980), the eddies are ignored and a Rayleigh friction coefficient α is assumed. The zonal momentum equation is

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial y} - W\frac{\partial u}{\partial Z} + fv - \alpha u, \tag{22}$$

where y is meridional distance from latitude 45° N. In this section all dependent variables are assumed to be zonal averages. The geostrophic approximation is made in the v equation:

$$fu = -\frac{\partial \Phi}{\partial y}. \tag{23}$$

Because the continuity equation is two-dimensionally non-divergent,

$$\frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial Z}(\rho W) = 0, \tag{24}$$

a meridional streamfunction ψ can be defined:

$$\rho v = -\frac{\partial \psi}{\partial Z}, \quad \rho W = \frac{\partial \psi}{\partial y}. \tag{25}$$

The diabatic heating J/c_p in (3a) and (7a) consists of a prescribed annual cycle $Q = Q_1(y, z)\cos(2\pi t/1\text{ year})$ and Newtonian cooling $-\beta T^*$. Using the δ notation of Section 3, the governing equations can be combined so as to eliminate time derivatives through application of the thermal wind equation,

$$B \equiv -\frac{R}{\rho}e^{-\kappa z}\frac{\partial \theta}{\partial y} = \frac{f}{\rho}\frac{\partial u}{\partial Z}, \tag{26}$$

where B is the baroclinity. There results the standard elliptic balance equation (Eliassen, 1962) for the streamfunction:

$$\frac{\partial}{\partial y}\left(A\frac{\partial \psi}{\partial y} + B\frac{\partial \psi}{\partial Z}\right) + \frac{\partial}{\partial Z}\left(B\frac{\partial \psi}{\partial y} + C\frac{\partial \psi}{\partial Z}\right) = F, \tag{27}$$

where

$F = R\frac{\partial Q}{\partial y} - R\beta T^* - \frac{\partial}{\partial Z}(f\alpha u)$ is the forcing,

$A = \frac{R}{\rho}\left[e^{-\kappa z}\frac{\partial \theta}{\partial Z} + (\delta - 1)\kappa T^*\right]$ is the gravitational stability, and

$C = \frac{1}{\rho}f\left(f - \frac{\partial u}{\partial y}\right)$ is the inertial stability.

The model annual cycle is computed by integrating the thermodynamic eq. (7a) forward in time using a leapfrog time scheme and an Asselin filter. The variables are represented on a staggered $y - Z$ grid, and ψ is assumed to vanish on the boundaries

$y = y_1 = -a\pi/4$, $y = y_2 = a\pi/4$, $Z = Z_1$, and $Z = Z_2$. Once temperature T^* is obtained, the zonal wind u is calculated from the thermal wind relation (27), with the integrated zonal flow constrained by conservation of total angular momentum. Then the streamfunction ψ is calculated from the balance eq. (28). $T^* = u = \psi = 0$ are required as initial conditions.

a. *Experiment I: No net heating* ($\int Q\rho\, dV = 0$), *with dissipation* ($\alpha, \beta \neq 0$)

Experiment I simulates the annual cycle in the stratosphere/mesosphere. The lower boundary $Z_1 = 4.5$ scale heights (~35 km) and the upper boundary is $Z_2 = 9.5$ scale heights (~70 km). The basic temperature profile $T_0(Z)$ is the average of January and July Standard Atmospheres at 45° N. The heating rate $Q(y, Z) = (1.5\ \text{K day}^{-1}) \times \sin(2y) \times \sin[\pi(Z - Z_1)/(Z_2 - Z_1)]$, crudely representing a typical maximum 3 K day^{-1} pole-to-equator differential heating rate of Holton-Wehrbein's (1980) radiative calculations. The dissipation rates α, β of Holton and Wehrbein were reduced by a factor of four so that a time step of six days would be numerically stable. Fig. 2 and 3 display the energetics through slightly longer than one-half year for a calculation with $\delta = 1$ and a separate calculation with $\delta = 0$. Fig. 2 shows the time development of a domain-averaged kinetic energy [KE] and *change* in total potential energy [TPE]

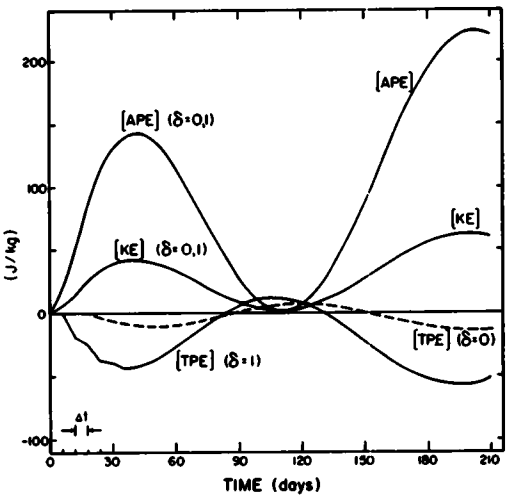

Fig. 2. Kinetic energy [KE], available potential energy [APE], and total potential energy [TPE] change from initial state for Experiment I, cases $\delta = 0, 1$.

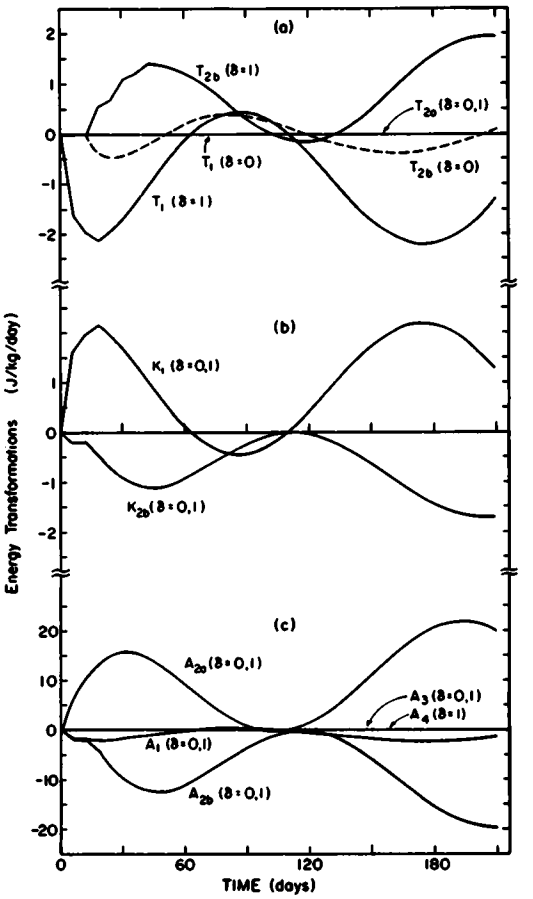

Fig. 3. Generation, conversion and dissipation terms in Experiment I for (a) [TPE] (b) [KE], and (c) [APE]. Subscript 1 refers to conversion between potential energy and kinetic energy; 2*a* refers to generation; 2*b* to dissipation; 3 to the stability gradient term in the [APE] budget; and 4 to the Holton-dropped term in the [APE] budget.

which is associated with the deviation temperature T^*. The background total potential energy of the global initial temperature distribution $T_0(Z)$ is $2.42 \cdot 10^5$ J kg^{-1}, approximately 1000 times larger than the variable part of the energy. The quantity $\int(\overline{RT^{*2}}/2S_0)\rho dV/\int\rho dV$ is calculated as a proxy available potential energy, even though we have seen in Section 3 that the identification is not rigorous. For convenience, this quantity is symbolized as [APE] and eqn. (12a) is referred to as the APE budget.

During the first 60 days, the circulation adjusts to the impulsively started annual cycle of diabatic heating. Initially a noticeable $2\Delta t$ oscillation in [TPE] is observed which is rapidly damped by the Asselin filter. After this initial period, the circulation adjusts to a steady annual cycle controlled by a quasi-balance among the generation, conversion, and dissipation terms, with a maximum in [KE] and [APE] at 200 days. The energies exhibit a semi-annual oscillation because of the quadratic form of [KE] and [APE].

Comparing the energetics for the $\delta = 0$ and $\delta = 1$ calculations, we find that [TPE] differs substantially, typically by a factor of 4–5. The phase of the [TPE] oscillation differs by about 10 days. Perhaps most surprising is the insensitivity of [KE], [APE], and indeed the circulation itself to the choice of a thermodynamic equation. The [KE] curves for $\delta = 0$, 1 are indistinguishable on the scale of Fig. 2: they differ only in the third to fourth significant digit. The same is true for [APE]. Local differences of mean zonal wind (u) and temperature deviation (T^*) are less than 1 m s^{-1} and 1 K, respectively—both on the order of 1 % of the calculated fields.

These energy results are illuminated by considering the generation, conversion, and dissipation terms affecting the time development of the various energy forms. Table 1 lists these terms and the corresponding labels. From (7a), three different mechanisms affect the [TPE] budget: $T_1 = -\delta R[W'T^*]$ is the conversion term which is present in the "exact" thermodynamic equation, $T_{2a} = c_p[Q]$ is the generation term; and $T_{2b} = -[\beta T^*]$ is the Newtonian dissipation term. Here $[f]$ is the domain average of any quantity f:

$$[f] \equiv \int f\rho \, dV / \int \rho \, dV. \tag{28}$$

Fig. 3a displays the relevant terms in the [TPE] budget for both $\delta = 0$, 1. In both calculations, the generation term T_{2a} vanishes identically because the specified heat source Q has a domain average of zero. With $\delta = 1$, the baroclinic conversion term T_1 generally acts against the dissipation term T_{2b}, with the smaller residual sum forcing changes in [TPE].

Table 1. *Energy terms*

Eq. (7a) *Total Potential Energy* [TPE]			
$\frac{d}{dt}[c_p T] =$	$-\delta R[W'T^*]$	T_1	conversion
	$+ c_p[Q]$	T_{2a}	heating
	$-[\beta T^*]$	T_{2b}	Newtonian cooling
Eq. (6) *Kinetic Energy* [KE]			
$\frac{d}{dt}\left[\frac{u^2+v^2}{2}\right] =$	$+ R[W'T^*]$	K_1	conversion
	$-[\alpha u^2]$	K_{2b}	Rayleigh friction
Eq. (12a) *"Available Potential Energy"* [APE]			
$\frac{d}{dt}\left[\frac{RT^{*2}}{2S_0}\right] =$	$-R[W'T^*]$	A_1	conversion
	$+\left[\frac{R}{S_0}QT^*\right]$	A_{2a}	heating
	$-\left[\frac{\kappa}{S_0}\beta T^{*2}\right]$	A_{2b}	Newtonian cooling
	$-\left[W'\frac{RT^{*2}}{2S_0}\right]\frac{d \ln S_0}{dz}$	A_3	stability gradient term
	$-\delta\left[W'\frac{RT^{*2}}{2S_0}\right]2\kappa$	A_4	Holton-dropped term

$\delta = 1$ Primitive equations.
$\delta = 0$ Holton model equations.

With $\delta = 0$, the conversion term is missing and only dissipation acts to alter [TPE]; the dissipation term is substantially reduced. Since β has been assumed to depend on Z only, the dissipation term can be interpreted as a weighted vertical average of the net temperature deviation on each pressure surface, with weighting function $\beta(Z)$. With a heating function identically anti-symmetric about the central latitude $y = 0$, and a forced meridional circulation with W' predominantly antisymmetric, it is understandable that the anti-symmetric component of T^* is much greater than the symmetric component. The net thermal dissipation reflects only the small symmetric component of T^* and can therefore differ substantially between $\delta = 0,1$ even though the total T^* fields are almost indistinguishable. Both the T_1 and T_{2b} terms exhibit substantial differences between the $\delta = 0,1$ models.

According to the dissipative analog to (6), two mechanisms can change the total kinetic energy [KE]: baroclinic conversion $K_1 = R[W'T^*]$ and dissipation $K_{2b} = -[\alpha u^2]$ by Rayleigh friction. Fig. 3b shows that these two terms generally oppose each other, with friction lagging the conversion by about 25 days. Because the circulations for $\delta = 0,1$ are virtually indistinguishable, these two terms are essentially identical on the scale of Fig. 3b, leading to the same curve for [KE] in Fig. 2.

According to (12a), the [APE] budget contains five terms: $A_1 = -R[W'T^*]$ is the baroclinic conversion term; $A_{2a} = [RQT^*/S_0]$ represents generation; $A_{2b} = -[\kappa\beta T^{*2}/S_0]$ is Newtonian dissipation; $A_3 = -[W'(RT^{*2}/2S_0)]\mathrm{d}\ln S_0/\mathrm{d}Z$ is the stability gradient term; and $A_4 = -\delta[W'(RT^{*2}/2S_0)]2\kappa$ is the term dropped in the Holton approximation. Fig. 3c shows the dominant terms of the budget—generation, conversion, and dissipation—which differ little in the $\delta = 0,1$ cases. Terms A_3, A_4 are both smaller than 0.01 in magnitude throughout the calculation, at least two orders of magnitude less than the prevailing terms. Thus for the diabatically forced, viscous annual cycle, the terms which cause [APE] as defined by Holton to be non-conservative are indeed negligible. For this particular case, available potential energy as defined by Holton is for all *practical* purposes an appropriately conservative quantity *a posteriori*. Further, this definition is equally valid with or without the approximation in the thermodynamic equation. Therefore the neglect of κWT* is unnecessary as well as undesirable.

b. *Experiment II: No net heating* ($\int Q\rho\, dV = 0$), *no dissipation* ($\alpha, \beta = 0$)

In Experiment II, α and β are set to zero, removing all dissipation, in order to test whether the energy results of Experiment I depend on the presence of dissipation. Additionally, the calculation is carried out for the troposphere and lower stratosphere, from $Z = 0$ to $Z = 3$ scale heights, with a smaller static stability S_0—thus reducing the credibility of neglecting $W\kappa T^*$ relative to WS_0. Fig. 4 shows that the primary conclusions are unchanged. [KE] and [APE] are again insensitive to the approximation: the $\delta = 0,1$ cases are indistinguishable in Fig. 4. [TPE] differs as expected: [KE] + [TPE] is conserved for $\delta = 1$ but not for $\delta = 0$. The $2\Delta t$ oscillation in [KE] + [TPE] is quite apparent initially in the $\delta = 1$ case, for which the total energy is an analytically conservative quantity. The Asselin filter (with coefficient 0.1) damps out the oscillation in time but also dissipates a small amount (~2%) of the energy of the annual cycle after one-half year. In the $d = 0$ case, as shown in Section 3, [TPE] is approximately zero, never exceeding 0.5 $\mathrm{Jkg^{-1}}$ in magnitude in this integration; thus [KE] + [TPE] is not a conservative quantity.

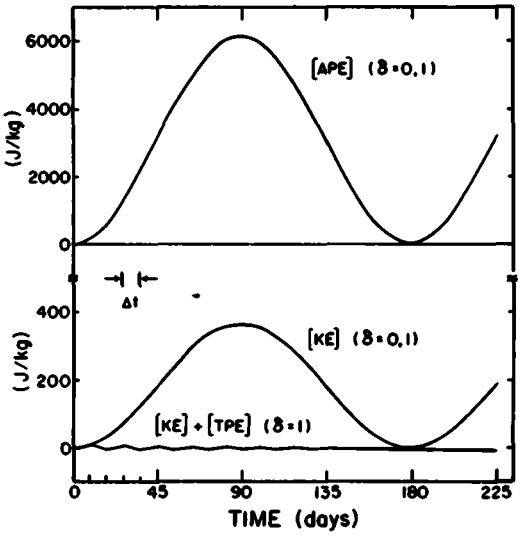

Fig. 4. Experiment II energies: [APE] and [KE] for cases $\delta = 0,1$; [KE] + [TPE] for case $\delta = 1$. [TPE] ($\delta = 0$) is smaller than 0·5 J $\mathrm{kg^{-1}}$ in magnitude throughout the integration; therefore the [KE] line is representative of [KE] + [TPE] for case $\delta = 0$. Time step is 9 days.

Nevertheless, this lack of energetic consistency again has surprisingly little effect on the circulation pattern itself, despite the fact that the neglect of $W\kappa T^*$ is completely unjustified *a posteriori*. In this example, S_0 is in the range of 25 K–75 K; because of the absence of dissipation, T^* is rather large, exceeding 80 K in magnitude. Even though the approximation $\kappa T^* \ll (S_0, \kappa T_0)$ is not justified *a posteriori*, the resulting circulations for $\delta = 0,1$ are still quite comparable. u and T^* differ by only 1–2% between the two cases.

It might be noted that [TPE] and [KE] have the same magnitude for $\delta = 1$, while [APE] is more than an order of magnitude larger. In this calculation, the traditional rationale for using available potential energy rather than total potential energy —that [APE] and [KE] have roughly equal magnitudes—is not valid. Total potential energy *deviation* [TPE] from the initial state appears to be a better form of potential energy for this purpose.

c. *Experiment III: Net heating* ($\int Q\rho\, dV \neq 0$), *no dissipation* ($a, b = 0$)

Experiment III was designed in order to test the comparison of $\delta = 0,1$ when the heating is unbalanced ($[Q] \neq 0$) and non-symmetric, causing a change in total energy of the model atmosphere. As in Experiment II, $Z_1 = 0$ and $Z_2 = 3$ are chosen, and there is no dissipation. We assume heating of the form

$Q = (3\text{K day}^{-1}) \times \sin(y - y_1) \times \sin[\pi(Z - Z_1)/(Z_2 - Z_1)] \times \cos(2\pi t/1 \text{ year})$, which is asymmetric in y.

Fig. 5 shows that [APE] and [TPE] are both two orders of magnitude larger than [KE]. We also see that the errors in [APE] and [TPE] caused by the approximation $\delta = 0$ are as large as the kinetic energy itself! Nevertheless, the qualitative character of the circulation is still insensitive to the δ approximation. This insensitivity is reflected in the relatively small difference of a few percent in the kinetic energy.

The source of the differences in [TPE] has already been identified as the baroclinic conversion term T_1. Fig. 6a documents the energy transformations in the [APE] budget for the $\delta = 1$ case. For this experiment the A_4 term which is dropped in the approximate set is just as large as the baroclinic conversion term A_1. The stability gradient term A_3 is an order of magnitude smaller. The differences in

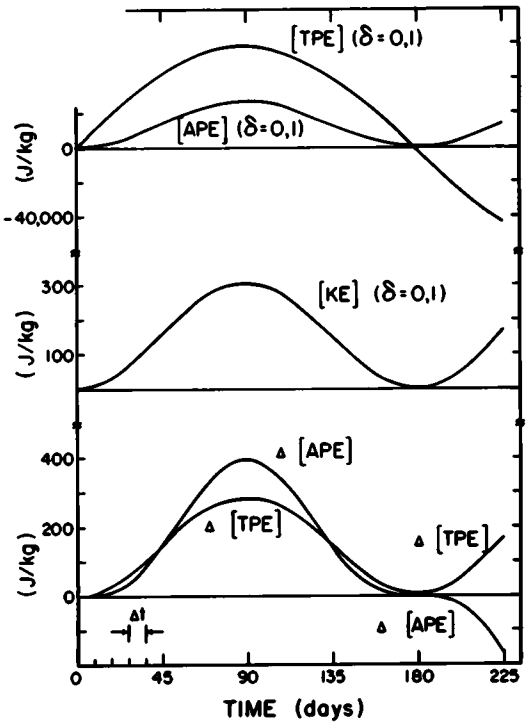

Fig. 5. Experiment III energies: [TPE], [APE], and [KE] for cases $\delta = 0,1$. The lower displays the differences [APE]($\delta = 0$) − [APE]($\delta = 1$) and [TPE]($\delta = 0$) − [TPE]($\delta = 1$).

energy transformations between the $\delta = 1$ case and the $\delta = 0$ case are shown in Fig. 6b. Two of the terms add together to cause most of the error: the generation term A_{2a} and the neglected term A_4. In this experiment A_4 is not negligible.

Fig. 7 displays the energy box diagrams for the two cases of experiment III, using total potential energy on the left and the approximate available potential energy on the right. Since total energy is increasing on day 45, the transformations do not sum to zero. The triple correlation terms (A_3, A_4) are indicated as arrows leaving the [APE] box to the right; recall that the existence of these terms causes [APE] to be only an approximation. The substantial contribution of A_4 in the $\delta = 1$ case suggests some doubt in the utility of the approximate [APE] definition. The lack of a connection between the [TPE] and [KE] boxes for $\delta = 0$ reflects the lack of conservation of total energy.

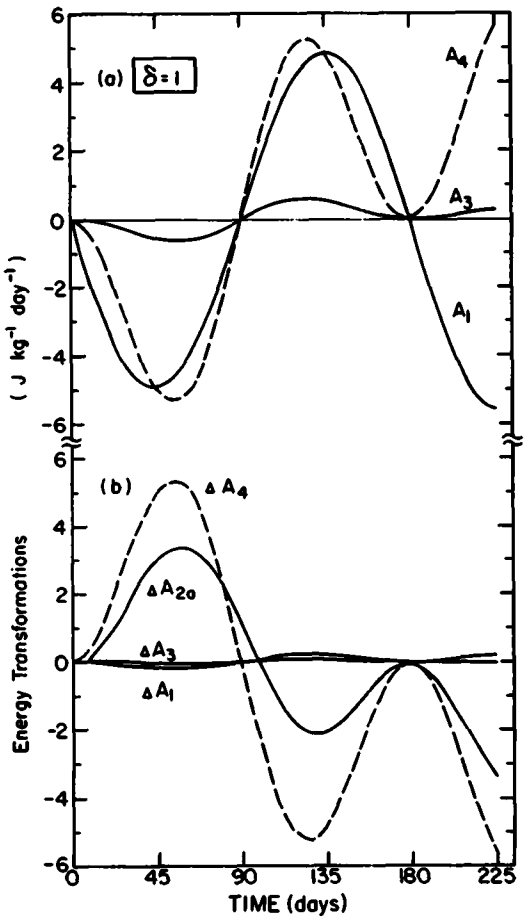

Fig. 6. Energy transfer terms in Experiment III for [APE] budget. (a) Energy transformations for $\delta = 1$ case. (b) Differences in generation and conversion terms (A_1, A_{2a}, A_3, A_4) between $\delta = 0$ and $\delta = 1$ case.

5. Summary and conclusion

In this paper we have analyzed the energetics of an approximate set of primitive equations introduced by Holton (1975). The sole approximation consists of dropping the component of adiabatic warming/cooling due to the temporally and horizontally variable part of the specific volume on a constant pressure surface, $W\kappa T^*$ in (3a). For generally observed basic temperature distributions, *there is no total energy invariant for the approximate set of primitive equations.* The second component of this study addressed the issue of an approximate expression for available potential energy which is suggested by the approximate set of equations. As defined by Holton, [APE] is proportional to the temperature variance on pressure surfaces, $\int (RT^{*2}(2S_0)\rho\, dV$. However, [KE] + [APE] is conserved only in the rather special case in which the static stability S_0 is set to a constant value. Since the concept of available potential energy as proposed by Lorenz (1955) satisfies appropriate energy conservation principles, the recent approximate expression for [APE] introduces questions regarding its validity and utility.

A simple dynamically-balanced f-plane analog to the annual cycle has been integrated in order to compare the approximate set with the "exact" primitive equations. As expected, [TPE] differs significantly between the two cases. [KE], [APE], and the circulation itself are quite similar, even when the critical approximation is not justified *a posteriori.*

The reason for the similarity between the circulations for the approximate and exact equations lies in the dominance of the retained terms over the neglected one. The similarity is greatest in Experiments I and II with anti-symmetric heating. With T^* and W predominantly anti-symmetric and v symmetric with respect to y, all terms in the thermodynamic eq. (3a) *except* for $W\kappa T^*$ are primarily anti-symmetric. The presence of $W\kappa T^*$ serves mainly to force a much smaller symmetric component in T^*. Only the symmetric component of T^* is associated with [TPE]; hence, the sensitivity of [TPE] to the approximation. Because [APE] as defined by Holton is quadratic in T^*, both symmetric and anti-symmetric components contribute.

Lorenz (1955) suggested that the terms involving the triple correlation $[W'T'^2]$ would be negligible in the budget for true available potential energy. Although the true available potential has not been computed in the calculations reported here, it has been established that triple correlations are much smaller than the baroclinic conversion term when the heating is balanced ($[Q] = 0$). Thus our results confirm Lorenz's suggestion, but also place a condition on its applicability.

In conclusion, we find that the approximate form of the thermodynamic energy equation eliminates the conservation principle for total energy. Although the approximation does not cause

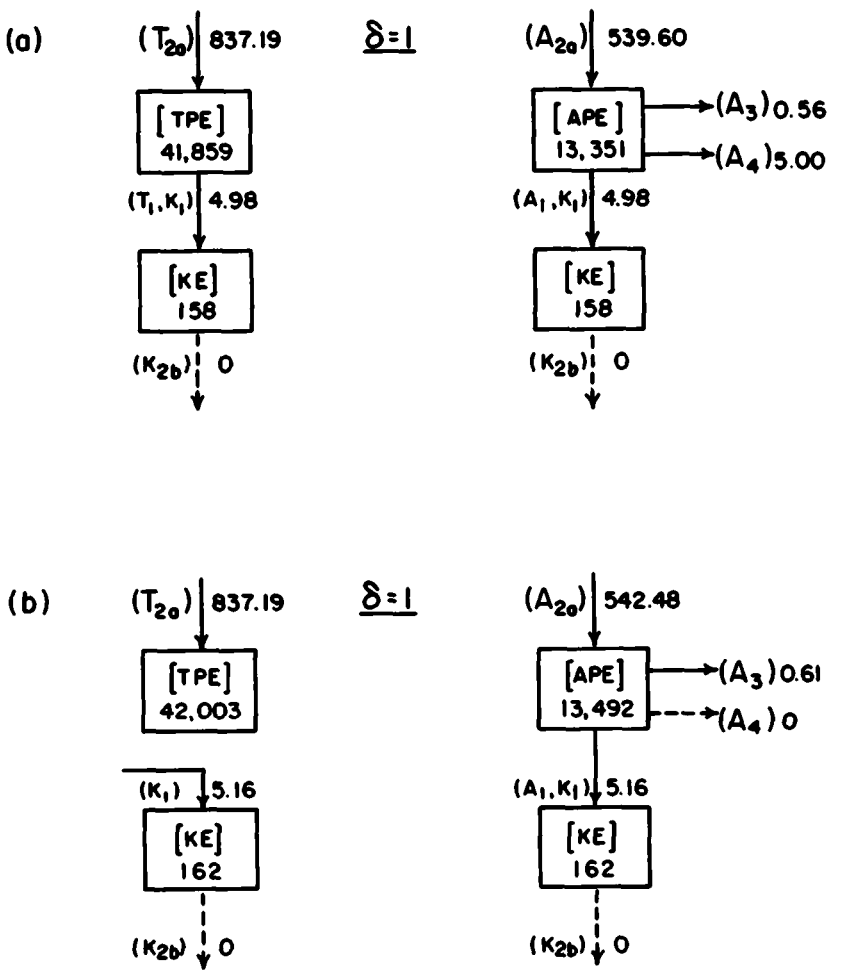

Fig. 7. Energy box diagrams at day 45 for (a) case $\delta = 1$ and (b) case $\delta = 0$, using [TPE] and [APE]. Energy units are J kg^{-1}; energy transfers have units J kg^{-1} day^{-1}.

substantial error in the annual cycle simulation, there appears to be no advantage to its application. Whether it would prove harmful to other types of atmospheric simulations, involving wave forcing, dynamic instabilities, or strong non-linearities, is still an open question. It seems prudent to avoid possible difficulties by using the standard set of primitive equations.

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REFERENCES

Boyd, J. 1976. The noninteraction of waves with the zonally averaged flow on a spherical earth and the interrelationships of eddy fluxes of energy, heat, and momentum. *J. Atmos. Sci. 33*, 2285–2291.

Bridger, A. F. C. and Stevens, D. E. 1982. Numerical modeling of the stratospheric sudden warming. Some sensitivity studies. *J. Atmos. Sci. 39*, 666–679.

Dunkerton, T., Hsu, C.-P. F. and McIntyre, M. E. 1981. Some Eulerian and Lagrangian diagnostics for a model stratospheric warming. *J. Atmos. Sci. 38*, 819–843.

Dutton, J. A. and Johnson, D. R. 1967. The theory of available potential energy and a variational approach to atmospheric energetics. *Advances in Geophysics 12*, 333–436. Academic Press, New York.

Eliassen, A. 1962. On the vertical circulation in frontal zones. *Geophys. Publ. 24*, No. 4, 147–160.

Holton, J. R. 1975. *The Dynamic Meteorology of the Stratosphere and Mesosphere.* American Meteorological Society, Boston, 218 pp.

Holton, J. R. 1976. A semi-spectral numerical model for wave-mean flow interactions in the stratosphere: applications to sudden stratospheric warmings. *J. Atmos. Sci. 33*, 1639–1649.

Holton, J. R. 1979. Equatorial wave-mean flow interaction: A numerical study of the role of latitudinal shear. *J. Atmos. Sci. 36*, 1030–1040.

Holton, J. R. and Wehrbein, W. M. 1980. A numerical model of the zonal mean circulation of the middle atmosphere. *Pageoph 118*, 284–306.

Holton, J. R. and Wehrbein, W. M. 1981. A further study of the annual cycle of the zonal mean circulation in the middle atmosphere. *J. Atmos. Sci. 38*, 1504–1509.

Hsu, C.-P. F. 1980. Air parcel motions during a numerically simulated sudden stratospheric warming. *J. Atmos. Sci. 37*, 2768–2792.

Lordi, N. J., Kasahara, A. and Kao, S. K. 1980. Numerical simulation of stratospheric sudden warmings with a primitive equation spectral model. *J. Atmos. Sci. 37*, 2746–2767.

Lorenz, E. N. 1955. Available potential energy and the maintenance of the general circulation. *Tellus 7*, 157–167.

Lorenz, E. N. 1960. Energy and numerical weather prediction. *Tellus 12*, 364–373.

Phillips, N. A. 1966. The equations of motion for a shallow rotating atmosphere and the "Traditional Approximation". *J. Atmos. Sci. 23*, 626–628.

Phillips, N. A. 1968. Reply to Veronis' comments. *J. Atmos. Sci. 25*, 1155–1157.

Veronis, G. 1968. Comment on Phillips' proposed simplification of the equations of motion for a shallow rotating atmosphere. *J. Atmos. Sci. 25*, 1154–1155.