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On the convergence of non-linear normal mode initialization methods

By STEFAN GOLLVIK[1], *Department of Meteorology[2], University of Stockholm (MISU), Arrhenius Laboratory, S-106 91 Stockholm, Sweden*

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ABSTRACT

Non-linear, normal mode initialization schemes are studied in a low order β-plane, shallow water model. In this model, Machenhauer's initialization scheme may diverge both due to linear advective terms and non-linear effects. Using more general methods to find non-linearily balanced states, convergence may be obtained and several possible balanced states can be found for a given meteorological field. Also, the effect of linearizing the model around a non-zero mean state and applying Machenhauer's method is investigated. Convergence is then more rapid and time integrations show an increased smoothness of the time evolution.

1. Introduction

During recent years, a powerful initialization method, non-linear normal mode initialization has been developed (Machenhauer, 1977; Baer and Tribbia, 1977). In this method, the eigenmodes of a certain equilibrium state (normally a state of rest) are utilized to describe states of the atmosphere, far from this equilibrium state. The idea is that the meteorologically significant evolution can be described by the dominant meteorological modes, and the fast gravity waves may be regarded as noise. Leith (1980) introduced the concept of the slow manifold in the phase space spanned by the eigenvectors for describing atmospheric states which evolve slowly in time. The normal mode approach is then to manipulate the amplitudes of the eigenmodes so that the initial state lies on the slow manifold. In a linear model, this initial state is simply given by zero amplitudes of the gravity modes (linear normal mode initialization). In a non-linear model this is not so, since the non-linear terms regenerate the gravity amplitudes, which then appear as noise. The non-linear normal mode initialization instead puts restrictions on the time derivatives of the gravity amplitudes initially. In Machenhauer's method, these tendencies are zero, which means that a system of non-linear equations has to be solved. He proposed an iterative procedure to arrive at this solution, in which only the gravity amplitudes are changed every iteration, and the non-linear terms are computed by the model itself. His proposal has proven to be effective and this method has been successfully implemented in global weather prediction models (Temperton and Williamson, 1979).

Since the equations are non-linear, it is not clear whether or not the solution obtained by Machenhauer's method is unique. Tribbia (1981) showed that several solutions to the normal mode initialization can exist in an f-plane model in cylindrical coordinates in which there are no variations in the azimuthal direction. He truncated his model to contain only one scale of motion. Because of the f-plane approximation, the behaviour of the model is very special, namely that the meteorological mode is stationary and the slow manifold coincides with a steady state gradient wind balance. Due to the stationarity of the meteorological mode, the linear transformation from physical space to eigenspace is not complete, and the normal mode initialization concept is not well-defined. In the present paper, we will consider a model in which

[1] Present affiliation: Swedish Meteorological and Hydrological Institute, P.O. Box 923, S-601 19 Norrköping, Sweden.

[2] Contribution No. 498.

the β-effect is included, which will make the meteorological modes non-stationary. The model is, however, severely truncated; only three wave components are present. Furthermore, there is no explicit latitudinal dependence; the β-effect is only included as a parameter. Here the gravity modes are affected by non-linear terms which are quadratic in the meteorological modes. Multiple solutions to the non-linear normal mode initialization procedure are shown to exist.

When utilizing the non-linear normal model initialization in a baroclinic model, the vertical dependence is decoupled in a series of shallow water systems with different equivalent depths. Temperton and Williamson (1979) pointed out that the iterative procedure failed to converge for the smaller equivalent depths and when strong physical forcing is included. Ballish (1981) used a very schematic model to study the convergence properties of the Machenhauer and the Baer-Tribbia schemes. His model was linear, and the dependence on the gravity modes was simulated by a linear advection term, which he showed had a critical value for convergence. Here we will study the convergence properties and their relation to linear advection and small depths of the fluid. Since this model is so easy to handle, we can examine its properties for many different parameter values. The results, however, must be interpreted with caution and can only give an indication of possible mechanisms in more general high resolution models. Whether or not the results are generalizable can only be answered by performing similar experiments with more complex models.

2. Model description

2.1. *The governing equations*

The simplest system that contains gravity waves is the shallow water system. Following Wiin-Nielsen (1979), we will apply the mid-latitude β-plane approximation with no latitudinal dependence in the variables. Thus the vorticity, divergence and continuity equations have the following form:

$$\frac{\partial \zeta}{\partial t} + U_g \frac{\partial \zeta}{\partial x} + f_0 \delta + \beta_0 v = -\left(u \frac{\partial \zeta}{\partial x} + \zeta \delta\right), \quad (2.1)$$

$$\frac{\partial \delta}{\partial t} + U_g \frac{\partial \delta}{\partial x} + \beta_0 u - f_0 \zeta + \frac{\partial^2 \phi}{\partial x^2} = -\left(u \frac{\partial \delta}{\partial x} + \delta^2\right), \quad (2.2)$$

$$\frac{\partial \phi}{\partial t} + U_g \frac{\partial \phi}{\partial x} + \Phi_0 \delta - f_0 U_g v = -\frac{\partial}{\partial x}(u\phi). \quad (2.3)$$

Here $\Phi_0 = gH$, where H is the mean depth of the fluid. All non-linear terms are on the right-hand side. Because of the disregarded y-dependence, u is purely divergent and v, the meridional wind, is purely rotational, i.e. $u = \partial\chi/\partial x$ and $v = \partial\psi/\partial x$ where χ is the velocity potential and ψ is the stream function, and $\zeta = \partial^2 \psi/\partial x^2$ and $\delta = \partial^2 \chi/\partial x^2$. The above equations are made non-dimensional by the following scaling:

$$x = Lx',$$

$$\psi = UL\psi',$$

$$\phi = Uf_0 L\phi',$$

$$t = LU^{-1} t',$$

$$\chi = U^2 L^2 f_0 (gH)^{-1} \chi',$$

where L is the length scale and U is the typical velocity amplitude.

With the primes dropped the resulting equations are:

$$\frac{\partial \zeta}{\partial t} + A \frac{\partial \zeta}{\partial x} + \kappa\delta + C \frac{\partial \psi}{\partial x} = -R_0 \kappa \left(\frac{\partial \chi}{\partial x}\frac{\partial \zeta}{\partial x} + \zeta\delta\right), \quad (2.4)$$

$$\frac{\partial \delta}{\partial t} + A \frac{\partial \delta}{\partial x} - B\zeta + C \frac{\partial \chi}{\partial x} + B \frac{\partial^2 \phi}{\partial x^2} = -R_0 \kappa \left(\frac{\partial \chi}{\partial x}\frac{\partial \delta}{\partial x} + \delta^2\right), \quad (2.5)$$

$$\frac{\partial \phi}{\partial t} + A \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x}\right) + \delta = -R_0 \kappa \left(\frac{\partial \chi}{\partial x}\frac{\partial \phi}{\partial x} + \phi\delta\right), \quad (2.6)$$

$$A = U_g U^{-1}, \quad R_0 = U(f_0 L)^{-1}, \quad C = L^2 \beta_0 U^{-1},$$
$$\kappa = f_0^2 L^2 (gH)^{-1}, \quad B = \kappa^{-1} R_0^{-2}.$$

In this study, f_0 and $\beta_0 = \mathrm{d}f/\mathrm{d}y$ are taken as their values at latitude 20°, and $L = 6 \cdot 10^6$ m. $U = 30$ m s^{-1} which gives a Rossby-number $R_0 \approx 0.1$. By this scaling, the non-linear terms are about one order of magnitude smaller than the linear terms, for a fluid depth, H, of 8 km.

2.2. *Transformation to normal mode form*

Assuming solutions of the form e^{ikx}, the eigenvectors of the linear part of the system (2.4)–(2.6) can be written

$$\hat{Y}_{k,j} = \begin{pmatrix}\hat{\zeta}\\ i\hat{\delta}\\ \hat{\phi}\end{pmatrix}_{k,j} = \begin{pmatrix}\kappa B k^2\\ -\left(\sigma_{k,j}+\dfrac{C}{k}-Ak\right)Bk_2\\ \left(\sigma_{k,j}+\dfrac{C}{k}-Ak\right)^2-\kappa B\end{pmatrix} = \begin{pmatrix}\hat{E}_{k,j}\\ \hat{G}_{k,j}\\ \hat{P}_{k,j}\end{pmatrix} \tag{2.7}$$

where k is the non-dimensional) wave number and $\sigma_{k,j}$ ($j = 1, 2, 3$) is the eigenvalue satisfying the following equation:

$$\sigma_k^3 + \sigma_k^2\left(\frac{2C}{k} - 3Ak\right) + \sigma_k\left(\frac{C^2}{k^2} + 3A^2k^2 - 4AC - \kappa B - Bk^2\right) + k\left(Ak - \frac{C}{k}\right)\left[Bk - A\left(Ak - \frac{C}{k}\right)\right] = 0. \tag{2.8}$$

Fig. 1 shows the frequencies as functions of κ for two different wavenumbers, assuming no zonal mean wind ($U_g = 0$). Note that all frequencies are non-zero. The Rossby wave has been given the index $j = 1$, while $j = 2, 3$ stand for the two gravity-inertia waves. The eigenvectors have been renormalized ($Y_{k,j} = (E_{k,j}, G_{k,j}, P_{k,j})^T$), so that the kinetic energy of an eigenvector, when integrated around a latitude circle is equal to unity.

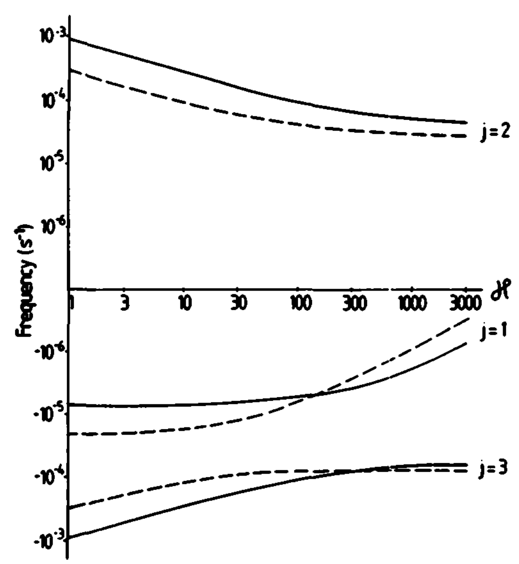

Fig. 1. The three frequencies as functions of κ ($=f_0^2 L_2 (gH)^{-1}$). The meteorological mode has index $j = 1$, and $j = 2, 3$ stands for the two gravity modes. The dashed lines show wavenumber 6 and the solid lines wavenumber 18.

All the variables are now projected onto the eigenvectors according to:

$$\begin{pmatrix}\zeta\\ i\delta\\ \phi\end{pmatrix} = \sum_k \sum_j D_{k,j}(t)\, Y_{k,j}\, e^{ikx} \tag{2.9}$$

Here $D_{k,j}(t)$ are complex amplitudes for the corresponding eigenvectors. By using this expression and utilizing the orthogonality properties of the trigonometric functions, it is possible to write the system (2.4)–(2.6) as follows:

$$\frac{d}{dt} D_{m,j} + i\sigma_{m,j} D_{m,j} = iR_0\, \kappa \sum_k \sum_l \sum_{l'} D_{k,l}\, D_{m-k,l'}\, W_{m,j,k,l,l'}, \tag{2.10}$$

where

$$W_{m,j,k,l,l'} = \frac{m}{k}\, G_{k,l}\,(s_{m,j,1} E_{m-k,l'} + s_{m,j,2} G_{m-k,l'} + s_{m,j,3} P_{m-k,l'}). \tag{2.11}$$

$s_{m,j,i}$ is the element with row index j and column index i in the matrix R_m^{-1}, if R_m is the matrix containing the eigenvectors for wavenumber m.

$W_{m,j,k,l,l'}$ are the interaction coefficients and eq. (2.10) is the complete non-linear system written in normal mode form. Since the β-effect is included, the non-linear terms in the equations for the gravity modes contain interactions between meteorological modes.

2.3. *Non-linear normal mode initialization*

The Machenhauer initialization procedure applied to the present model means that the following non-linear equation has to be solved, for all wavenumbers m and for the mode indices $j = 2, 3$:

$$i\sigma_{m,j} D_{m,j} = iR_0\, \kappa \sum_k \sum_l \sum_{ll} D_{k,l}\, D_{m-k,l'}\, W_{m,j,k,l,l'} \tag{2.12}$$

or in vector notation

$$i\Lambda_G\, \boldsymbol{G} = \mathrm{NL}(\boldsymbol{M}, \boldsymbol{G}) \tag{2.13}$$

Here Λ_G is a diagonal matrix containing all the eigenvalues of the gravity modes; G contains all the gravity amplitudes, and $NL(M, G)$ is a vector of all non-linear terms, which are functions of the meteorological amplitudes, M as well as of G. The equation system can be solved iteratively as

$$G^{\nu+1} = (i\Lambda_G)^{-1}\, NL(M, G^{\nu}). \tag{2.14}$$

This is done successively for all gravity amplitudes, starting with zero initial value. When the fluid is deep (corresponding to external modes in a baroclinic model), this procedure converges within a few iterations. For shallower fluids (smaller equivalent depths), and hence smaller eigenvalues, the method eventually diverges. The convergence properties of this method will be studied in the present paper.

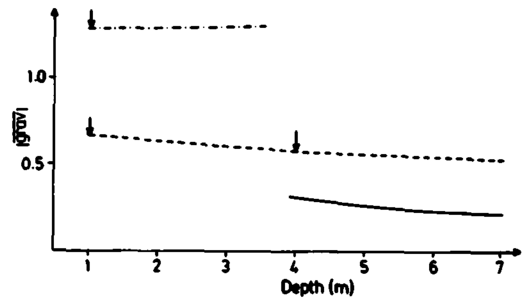

Fig. 2. The mean absolute value of the gravity amplitudes as a function of the depth of the fluid (see text). The solid line shows the solution obtained by Machenhauer's iteration scheme. The two other lines show other solutions obtained by the Newton–Raphson method. In the points marked with arrows, Machenhauer's scheme diverges, even if the initial guess is the solution obtained by the Newton–Raphson method.

3. Initialization experiments

3.1. *The convergence problem for small depths*

It was pointed out by Temperton and Williamson (1979) that the scheme proposed by Machenhauer did not converge for small equivalent depths. Now we will do experiments with this model to test the iteration scheme (2.14) for small values of the depth. We will then see that the scheme diverges at a critical depth. To see whether that has to do with this iteration scheme, another numerical method for solving (2.13) will also be utilized.

Here the model has been truncated to only three wave numbers ($k = 6, 12, 18$). The meterological modes are assumed to have the same phase, and their amplitudes are set to unity. The constant geostrophic mean wind (U_g) is here set to zero.

To display a solution of (2.13), we have utilized the mean absolute value of the gravity modes (i.e. $|\overline{\text{grav}}| = \sum_k (1/N)(|D_{k,2}| + |D_{k,3}|)$; $N = 6$). The solid line in Fig. 2 shows $|\overline{\text{grav}}|$ as a function of depth. This solution is obtained by Machenhauer's method, starting with a zero initial guess of the gravity amplitudes, and it is here called Machenhauer's solution. At a certain critical depth (3.9 m), the method does not converge.

3.2. *Multiple solutions for shallow fluids*

In this section, we are going to see if other methods of solving non-linear equation systems can yield solutions when Machenhauer's method diverges. We shall then find that additional solutions exist, even if they cannot be found by Machenhauer's method.

In the present model, we have only six complex gravity-amplitudes to determine, when solving the non-linear equation system (2.13). This makes it possible to utilize more refined numerical methods, and here we will use a Newton–Raphson (NR) type of solution to non-linear equation systems.

For the depths where (2.14) converges, the NR method converges to the same solution, if the initial guess is a zero amplitude of the gravity-modes.

For still shallower fluids, even the NR method ceases to yield a root with values in the vicinity of this solution. Instead it converges to another root (with zero initial guess) with a higher mean absolute value (see Fig. 2, dashed line). Convergence to this root also takes place if the initial guess is Machenhauer's solution for a depth just above the critical depth.

This suggests that a solution with smaller gravity amplitudes does not exist for small equivalent depths. By choosing another initial guess, the NR method converges to yet another solution (the dash-dotted line in Fig. 2). For one of the roots (dashed line, Fig. 2), the depth was gradually increased (the initial guess being the solution of the depth just below the actual depth) and it was found that this root exists for depths less than 48 m. Here $|\overline{\text{grav}}|$ is 0.5, while the corresponding value for the root to which Machenhauer's scheme converges is 0.05. Machenhauer's iteration scheme was also tried with the solution given by the NR method as initial guess at the points marked with arrows in Fig. 2, and yet it diverged, which shows that these solutions cannot be obtained by this procedure. These results are in good agreement with those of Thaning (1984), who found multiple solutions in an equatorial β-plane model, with latitudinal dependence, truncated to one meteorological mode and two gravity modes. He also found that Machenhauer's scheme did not converge to all possible solutions of (2.13). However, the roots with large amplitudes of the gravity modes are of no use for initialization purposes, since a field initialized with these modes would be very ageostrophic and unrealistic. This is indeed the case for the fluid depths, where the ordinary scheme converges to solutions with small amplitudes (e.g. for a fluid depth of 48 m, where there is a factor of 10 between the mean absolute values of the roots), while for shallower fluids, where the difference between the roots is smaller, this is not so obvious.

3.3. *Experiments with linear advection*

To study the convergence properties of Machenhauer's scheme, Ballish (1981) used the following linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = i\omega y + N\,\mathrm{e}^{i\gamma t} - \bar{U}iky, \tag{3.1}$$

where y is an amplitude of a gravity mode. The terms on the right-hand side are: the linear term corresponding to a fast oscillation with frequency ω, the slowly varying meteorological evolution (frequency $\gamma \ll \omega$) and the advection term. The last term is the simplest possible term that is a function of the amplitude y, and stands for a simulation of the non-linear terms containing y. The iterative technique of (2.14) applied to (3.1), starting with the initial guess $y_0 = 0$, yields after n iterations

$$y_n = iN[1 - (\bar{U}k/\omega)^n]\,[\omega(1 - \bar{U}k/\omega)]^{-1}$$

Consequently, for convergence $\bar{U}k/\omega$ must be less than 1.

To study how these results are modified by a non-linear system, an experiment was performed with a geostrophic mean wind advection treated in the following way. The eigenvectors are those of a system at rest, and even if the geostrophic advection terms are linear, they are treated together with the non-linear terms. This means that eq. (2.13) changes to

$$i\Lambda_G\,\boldsymbol{G} = \mathrm{NL}(\boldsymbol{M}, \boldsymbol{G}) + \boldsymbol{F}\cdot\boldsymbol{G}. \tag{3.2}$$

Here $\boldsymbol{F}\cdot\boldsymbol{G}$ are the linear advection terms, and $\boldsymbol{F}$ is a function of U_g.

For a depth of 8.5 km, this linear advection was increased until Machenhauer's initialization scheme failed to converge. It was found that we enter the region of no convergence, just below the point where the ratio $U_g\,k/\omega$ first reaches the value of 1. With regard to the close agreement with Ballish's result, it is reasonable to conclude that the non-linearity does not affect the convergence properties much in the case of deep fluids. For a shallower fluid, the non-linear terms become more and more important for the convergence properties. For a depth of 100 m, the lack of convergence already appears when $U_g\,k/\omega$ is about 0.75, because of the relatively greater non-linear terms.

The convergence problem can thus be looked upon as composed of two different mechanisms. One is the non-linear effect in the case of small fluid depths, and another one is the effect of strong linear advection on the system, in which case the non-linear terms are negligible. When the advection is linear, an alternative way to solve (3.2) iteratively is to utilize the form:

$$\boldsymbol{A}\cdot\boldsymbol{G} = \mathrm{NL}(\boldsymbol{M}, \boldsymbol{G}) \tag{3.3}$$

where

$$\boldsymbol{A} = i\Lambda_G - \boldsymbol{F}.$$

Alternatively, the linear advection can be included in the eigenvectors. In that case, the gravity modes have frequencies $\sigma' \approx U_g\,k + \sigma$, where σ is the frequency without linear advection included. The problems will thus appear when $\sigma' \approx 0$, which may occur for the westward moving gravity modes. In the following section, we will investigate this approach.

3.4. *Linearization around different basic states*

When non-linear normal mode initialization is used operationally, the linearization is done around a basic state at rest. This is because of the complexity of a real forecast model. This implies that the actual state of the atmosphere is far away from this basic state, and it would be more realistic to include a zonal wind profile in the basic state. In the simple model used here with only x-dependence, the eigenfunctions are always trigonometric functions even if additional linear terms are included. In the experiments of this section we will treat the geostrophic mean wind in two different ways. Either U_g in eqs. (2.1)–(2.3) is chosen to be non-zero, that is, the mean wind is contained within the eigenvectors of the system (alternative 1), and the non-linear normal mode initialization is performed by solving (2.13), or we have the other alternative (alternative 2), to solve eq. (3.2), where the eigenvectors are those of a system at rest.

To compare the two methods, we have regarded the physical field as that given by unity in the meteorological mode amplitudes, where the eigenvectors are those of a system not at rest (alternative 1). Eq. (2.13) was solved with the iterative technique of (2.14). This was repeated for higher values of the advecting wind, until the scheme diverged. This value of U_g is called critical forcing wind.

The critical forcing wind for convergence as a function of depth is shown in Fig. 3 (solid line), for depths greater than 80 m. When using the other method (alternative 2), the same physical field was projected on the new eigenvectors, and non-zero values therefore appear in the gravity amplitudes. The gravity amplitudes were then set to zero, and the system (3.2) was solved with the linear advection acting as a forcing. The critical wind of this method is also depicted in Fig. 3 (dashed line).

One can see from Fig. 3 that there is only a minor effect of including the mean wind in the eigenvectors, with regard to the critical forcing for convergence. There is, however, a big difference in the numbers of iterations needed for a given accuracy. The iterations are carried on until the relative changes of the gravity amplitudes between two consecutive iterations are less than 1%. The typical number of iterations is 3–4, when the linear advection is included in the eigenvectors. In the case with the linear advection acting as a forcing, the number of iterations is about 20. When the critical forcing wind is smaller (for shallower fluids), the difference between the two methods is of course less pronounced, as can be seen from Fig. 3.

From these experiments, it is concluded that only minor improvements, concerning the convergence properties, can be achieved by selecting a more realistic basic state for the linearization. The major difference is a more rapid convergence when the linear advection is included in the eigenvectors.

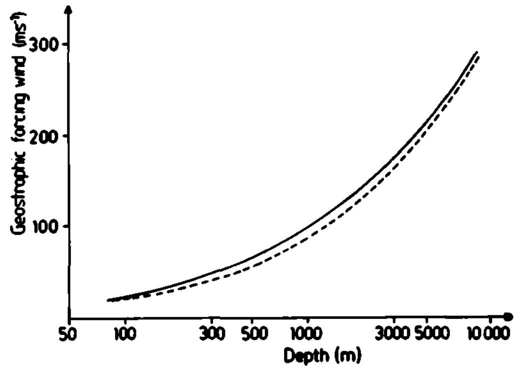

Fig. 3. The critical forcing wind for convergence as a function of the depth of the fluid. Solid line: the forcing is included in the eigenvectors. Dashed line: basic state at rest.

3.5. *Time integrations*

In this section, we will perform time integrations with the model given by eqs. (2.4)–(2.6), and using the same truncation and parameter values as before. The initial state will be defined, using the procedures given in subsection 3.4. We will then see how the two different initialization alternatives affect the amount of gravity noise in the time integrations.

As before, the physical field was chosen to be unity in the meteorological amplitudes, when the linearization was performed around a geostrophic mean wind assumed to be 50 m s^{-1}. The depth was 8.5 km and the other parameters were the same as before. Three different time integrations were performed, in each case for 0.2 time units, which corresponds to about 11 h.

(1) Linear normal mode initialization with the forcing wind included in the eigenvectors.

(2) Non-linear normal mode initialization with the forcing wind included in the eigenvectors.

(3) Non-linear normal mode initialization, where the linearization is performed around a basic state at rest.

Since the eigenvectors in (1) and (2) are different from those of (3), we have chosen to plot a physical variable (geopotential) as a function of time, instead of the individual amplitudes. Since the major part of the height variation is given by the meteorological evolution, the difference between the cases becomes fairly small with this simple model. To graphically enhance the differences between the forecasts, the difference between the forecast height and the height that is given by the linear meteorological forecast with the eigenvectors of (1) and (2), is chosen as the variable to be plotted.

The height-difference as a function of time is depicted in Fig. 4. The reason why the two curves in Fig. 4, (1) and (2), are so similar is that the gravity modes produced when solving (2.13) have very small amplitudes. Curve (3) on the other hand, shows more fluctuations. It is concluded from these experiments that in order to suppress gravity oscillations, the linearization should be performed in such a way that large linear terms are included in the eigenvectors.

To see what happens in a stronger non-linear system, the experiment was repeated, but now for a depth of 425 m. Here, the forcing wind of 50 m s^{-1} is near the critical value for convergence (see Fig. 3). The three different integrations shown in Fig. 5 were carried out to one time unit (55 h) to exhibit fluctuations, since the gravity mode periods are longer in this case.

All three curves show considerable fluctuations, and it is difficult to judge which of the three

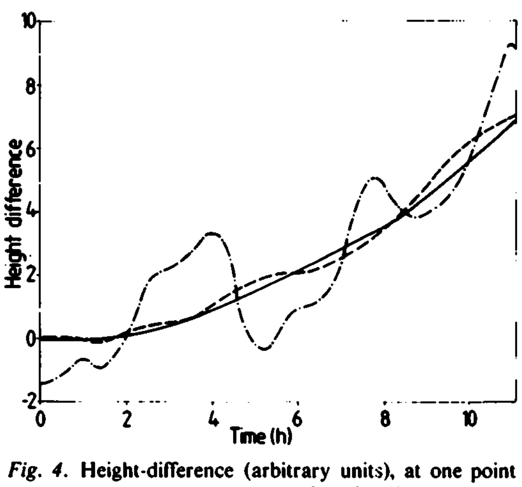

Fig. 4. Height-difference (arbitrary units), at one point as a function of time. The depth of the fluid is 8.5 km and the forcing wind is 50 m s^{-1}. Dashed line (1): linear normal mode initialization, with the forcing included in the eigenvectors. Solid line (2): the same but non-linear normal mode initialization. Dash-dotted line (3): non-linear normal mode initialization, with a basic state of rest.

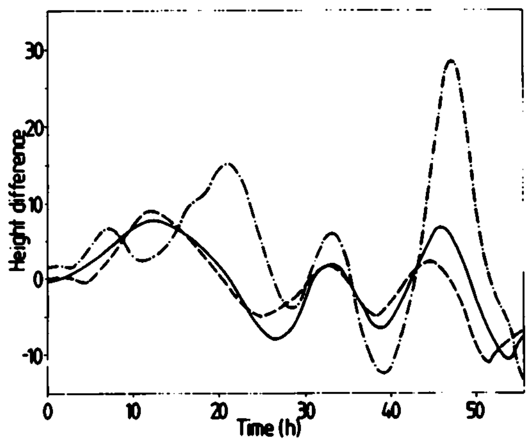

Fig. 5. As for Fig. 4, but for a fluid depth of 425 m.

initialization alternatives is preferable. This is in accordance with the theory developed by Baer and Tribbia (1977) that the non-linear terms should be at least one order of magnitude smaller than the linear terms. For the depth of 425 m, this is not the case since $R_0\kappa \approx 2$.

4. Conclusions

We have shown that multiple solutions to Machenhauer's non-linear normal mode initialization exist in a β-plane model without any latitudonal variations, and with three non-stationary meteorological modes. Tribbia (1981) also found multiple solutions in an f-plane model, but his results are not directly applicable to this model. This shows that the condition of vanishing time derivatives of the gravity modes can give rise to more than one solution, even if the meteorological modes are non-stationary. Our results are similar to those of Thaning (1984), who also found more than one solution, but with a latitudonally dependent model. Thaning (1984) restricted his analysis to a system with only one meteorological mode. Here, we have shown that these results can be extended to a system with more degrees of freedom. Convergence experiments indicate that the iteration scheme proposed by Machenhauer (1977) can only arrive at one of the solutions, namely the solution with small gravity wave amplitudes, a result that Thaning (1984) showed analytically for a three-mode system. The experiments suggest that Machenhauer's method converges, as long as the small amplitude solution exists. The other solutions, found through more general iteration procedures, have larger amplitudes in the gravity modes, and are therefore not of interest for initialization purposes.

The effect of linear advection upon the condition for convergence has been investigated. It is shown that for deep fluids, the strength of the linear advection term is critical for the convergence properties, while the non-linear terms dominate for shallower fluids. It is also shown that the time evolution from an initial state, which has been initialized with non-linear normal mode initialization is smoother if the linear advection terms are incorporated within the eigenvectors of the system, than if they act together with the non-linear terms in the iterative procedure. This is the case only for deep fluids, while for shallower fluids no improvements have been evident.

We have discussed some principal problems connected with non-linear normal mode initialization, and some possible solutions to these questions. Due to the simplicity of our model, the results are not directly applicable to a real complex weather prediction model, based on the primitive equations. However, we do believe that our results may serve as guidelines when the normal mode approach is used for initialization purposes.

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