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On quasi-geostrophic, finite-amplitude disturbances forced by topography and diabatic heating

By JACQUES DEROME, *Department of Meteorology, McGill University, Montreal, Canada, H3A 2K6*

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ABSTRACT

It is shown that, in the absence of dissipation, the tropographically-forced disturbances that Charney and Eliassen obtained as solutions to the linearized barotropic vorticity equation are actually *finite-amplitude* solutions. Similarly, in a baroclinic quasi-geostrophic model with no dissipation, topographically-forced disturbances obtained as solutions to the linearized potential vorticity equation are also shown to satisfy the non-linear version of the latter, and are therefore finite-amplitude solutions for arbitrary shapes of the topography, provided the mean zonal wind $\bar{u}$ is such that the index of refraction is a function of the vertical coordinate only. Finally, it is shown that when $\bar{u}$ satisfies the above condition, finite-amplitude thermally-forced disturbances of the kind discussed by Mitchell and Derome can coexist with the topographically-forced flow without interacting.

1. Introduction

Charney and Eliassen (1949), hereafter referred to as CE, were the first to investigate topographically-forced disturbances with a quasi-geostrophic model. Assuming a topographic height field of the form $h = \hat{h}(x) \cos \alpha(\phi - \phi_0)$, $\hat{h}(x)$ being the topographic height at $\phi_0 = 45\,^\circ$N, $\alpha =$ constant, they used an equivalent barotropic model with and without Ekman pumping to obtain the forced 500 mb height. The model was linearized about a constant mean zonal flow on the assumption that $\hat{h}(x)$ is sufficiently small. At least when dissipation was included, their linear model yielded results in reasonable agreement with the observed mean monthly 500 mb January height field at 45 °N.

It is a simple matter to show that in the absence of dissipation, the solution to CE's linear problem satisfies the condition $q = \zeta + f + f_0 h/H = -\lambda\psi$, where $\zeta + f$ is the absolute vorticity on a β plane, f_0, H and λ are constants and ψ is the stream function (zonal plus perturbation). This implies that the Jacobian $J(\psi,q) \equiv 0$ so that the steady-state, non-linear vorticity equation is satisfied. In short, when dissipation is neglected, CE's solution is a finite-amplitude solution.

In Section 2, we show that the above result also holds for an arbitrary $h(x,y)$ and a mean zonal wind $\bar{u}$ that is equal to a constant plus a simple sinusoidal structure in the meridional coordinate. In Section 3, we extend the results to a baroclinic quasi-geostrophic model and show that in the absence of dissipation, the linear and finite-amplitude topographically-forced disturbances will be identical when the mean zonal wind has both vertical and horizontal shear, provided the index of refraction is a function of the vertical coordinate only. It is also shown that in that case, the finite-amplitude response to thermal forcing can also be obtained by solving a linear problem, provided the diabatic heating satisfies a constraint related to its spatial structure. It is then easily demonstrated that the above topographically-forced and thermally-forced eddies can coexist without interacting between themselves or with the zonal flow.

Our focus in this paper is the presence or absence of non-linearities when the only perturbations in the flow are those forced by topography and/or diabatic heating, as was the case, for example, in the studies of Egger (1976) and Ashe (1979). The interactions between the free travelling disturbances and the forced ones discussed by Salmon et al. (1976), Frederiksen and Sawford (1981) and by Youngblut and Sasamori (1980), in particular, will not be considered. Atmospheric data studies such as that by Youngblut and Sasamori have demonstrated that transient disturbances do modify the structure of the forced quasi-stationary planetary waves. In interpreting the results of this paper, we must therefore keep in mind this limitation of our model equations. Our aim, it should be stressed, is primarily to show that under certain circumstances, rather complex solutions to the linearized potential vorticity equation are also finite-amplitude solutions, and to provide some information on the nature of these special circumstances.

The possibility of obtaining the structure of finite-amplitude forced disturbances by solving a linear problem has been exploited previously in the literature, for example, by Buzzi and Speranza (1979). Typically, the forcing is a localized mountain and the far upstream conditions are invoked to relate the potential vorticity to the stream function. Here, we will assume the flow to be periodic in the zonal direction and the structure of the mean zonal wind will be used to relate the potential vorticity to the stream function. A further, more fundamental difference is the fact that we will consider flows forced by both topography and diabatic heating.

2. The barotropic model

The inviscid quasi-geostrophic barotropic vorticity equation can be written in the form

$$(\nabla^2 - \mu^2)\, \partial\psi/\partial t + J(\psi, \nabla^2 \psi + \beta y + f_0 h/H) = 0, \tag{1}$$

where t is time, ∇^2 is the horizontal Laplacian, $\mu^2 = f_0^2/gH$, the Coriolis parameter has been written as $f_0 + \beta y$, g is gravity and the other symbols are as defined in Section 1. While CE obtained (1) using the equivalent barotropic model, the same equation applies to a quasi-geostrophic free-surface model.

If we let

$$\psi(x, y) = \bar{\psi}(y) + \psi'(x, y),$$

where the overbar represents a mean zonal value and a prime a deviation from it, not necessarily small, the steady-state version of (1) can be written as

$$\frac{\partial}{\partial x}\left[\nabla^2\psi' + \frac{f_0}{H}h' + \frac{1}{\bar{u}}\left(\beta - \frac{d^2\bar{u}}{dy^2}\right)\psi'\right] + \frac{1}{\bar{u}}J\left(\psi', \nabla^2\psi' + \frac{f_0}{H}h'\right) = 0, \tag{2}$$

where $\bar{u} = -\partial\bar{\psi}/\partial y$ and $\bar{h}$ has been assumed to be zero. (The contribution from $\bar{h}$ can easily be included by replacing β in (2) by $\beta + (f_0/H)\, d\bar{h}/dy$). Let us now suppose that $\bar{u}$ is such that

$$\frac{1}{\bar{u}}\left(\beta - \frac{d^2\bar{u}}{dy^2}\right) = \lambda = \text{constant}, \tag{3}$$

that is,

$$\bar{u} = \beta/\lambda + A \sin \lambda^{1/2}(y - y_0), \tag{4}$$

where A and y_0 are arbitrary constants. For example, with $\beta = 16 \times 10^{-12}\ \text{m}^{-1}\ \text{s}^{-1}$, if we choose $\beta/\lambda = u_0 = 15\ \text{m s}^{-1}$, then $\bar{u}$ will have a maximum value of $(15\ \text{m s}^{-1} + A)$ and a meridional wavelength $L_y = 2\pi\sqrt{u_0/\beta} = 6000$ km.

If $\bar{u}$ satisfies (3), the linearized version of (2) is

$$\nabla^2\psi' + \frac{f_0}{H}h' + \lambda\psi' = 0. \tag{5}$$

Note that this is precisely the problem that CE solved (in the absence of dissipation) with $A = 0$ in (4) and λ a positive constant. Now if ψ' satisfies (5), it is seen that $\nabla^2\psi' + (f_0/H)h'$ is a linear function of ψ' and hence the Jacobian in (2) vanishes identically. So if $\bar{u}$ satisfies (4), the solution to the linearized version of (2) is also a finite amplitude solution for an *arbitrary* field $h'(x,y)$.

It is worth pointing out that the topographically-forced stream function obtained from (5) is independent of A, the amplitude of the sinusoidal part of $\bar{u}$ in (4). This means that the forced inviscid solution obtained by CE applies not only to $\bar{u} =$ constant as they assumed, but also to a more realistic mean zonal wind profile as given by (4) with $A \neq 0$. Salmon et al. (1976) have also noted that the solution to the linear equation

$$\nabla^2\psi + \frac{f_0 h}{H} + \beta y + \lambda\psi = 0,$$

which is equivalent to (3) and (5), is an exact steady state solution to (1). We see that this solution will be resonant if h' and ψ' have a two-dimensional wave number $K = \lambda^{1/2}$ (see (5)). This is clearly due to the neglect of dissipation and time dependence. In fact, Salmon et al. (1976) and Frederiksen and Sawford (1981), in their studies of the statistical equilibrium of quasi-geostrophic flows over topography, have shown that the resonance phenomenon disappears when the interaction with transients is allowed.

In summary, we have seen that CE's inviscid, linear solution satisfies (1) and that it applies not only to $\bar{u}$ = constant as they assumed but also to $\bar{u}(y)$ given by (4).

3. The baroclinic model

We will now extend the comparison of Section 2 between the solutions of the non-linear and linearized barotropic vorticity equations to a baroclinic model. In addition we will include the effects of a thermal forcing term. The β-plane potential vorticity and thermodynamic equations (Phillips, 1963; Charney, 1973) can be written as

$$\frac{\partial q}{\partial t} + J(\psi, q) = G, \tag{6}$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial Z} + J(\psi, \psi_Z) + \frac{N^2}{f_0} w = \frac{\kappa Q}{f_0 H}, \tag{7}$$

where

$$q = \nabla^2 \psi + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial Z}\left(\frac{\rho_0}{N^2}\frac{\partial \psi}{\partial Z}\right) + \beta y,$$

$$G = \frac{f_0}{\rho_0}\frac{\kappa}{H}\frac{\partial}{\partial Z}\left(\frac{\rho_0}{N^2} Q\right),$$

$$Z = H \ln (p_s/p),$$

$$\rho_0 = \rho_s \exp(-Z/H),$$

and p is pressure, a subscript s represents a surface value, $w = \mathrm{d}Z/\mathrm{d}t$, $N^2 = g\,\mathrm{d}\ln\theta_0/\mathrm{d}Z$, θ_0 is the horizontally-averaged potential temperature, H is the constant scale height, Q is the diabatic heating rate per unit mass and κ is the ratio of the gas constant to the specific heat at constant pressure.

We will use the upper and lower boundary conditions

$$w(Z = Z_T) = 0,$$

$$w(Z = 0) = \boldsymbol{V}\cdot\nabla h = J(\psi, h),$$

where Z_T is the constant height of the upper boundary. Substituting the conditions on w into (7) we get

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial Z} + J(\psi, \psi_Z) = \frac{\kappa Q}{f_0 H} \quad \text{at} \quad Z = Z_T, \tag{8a}$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial Z} + J\left(\psi, \psi_Z + \frac{N^2}{f_0}\right) h = \frac{\kappa Q}{f_0 H} \quad \text{at} \quad Z = Z_T, \tag{8b}$$

As pointed out by Phillips (1963), it is consistent with quasi-geostrophic scaling to apply the lower boundary condition at $Z = 0$ rather than at $Z = h$. At this point, we can proceed along the same lines as for the barotropic model except for the fact that the mountain term now appears in the lower boundary condition.

Using

$$\psi = \bar{\psi} + \psi',$$

$$u = \bar{u} + u',$$

where an overbar represents a mean zonal average, we can write (6) and (8) as

$$\frac{\partial q}{\partial t} + \bar{u}\frac{\partial}{\partial x}\left(q' + \frac{\partial \bar{q}/\partial y}{\bar{u}}\psi'\right) + J(\psi', q') = G, \tag{9}$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial Z} + \bar{u}\frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial Z} - \frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial Z}\psi'\right) + J(\psi', \psi_Z') = \frac{\kappa Q}{f_0 H} \quad \text{at} \quad Z = Z_T, \tag{10a}$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial Z} + \bar{u}\frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial Z} - \frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial Z}\psi' + \frac{N^2}{f_0}h'\right) + J\left(\psi', \psi_Z' + \frac{N^2}{f_0}h'\right) = \frac{\kappa Q}{f_0 H} \quad \text{at} \quad Z = 0. \tag{10b}$$

Again, we have assumed that $\bar{h} = 0$, but the analysis can easily be modified for a non-constant $\bar{h}$ by replacing $\partial\bar{u}/\partial Z$ in (10b) by $\partial\bar{u}/\partial Z - (N^2/f_0)\mathrm{d}\bar{h}/\mathrm{d}y$.

Let us now suppose that $\bar{u}$ is such that

$$\frac{1}{\bar{u}}\frac{\partial \bar{q}}{\partial y} = \lambda(Z), \tag{11}$$

that is,

$$\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial Z}\left(\frac{\rho_0}{N^2}\frac{\partial \bar{u}}{\partial Z}\right) + \lambda(Z)\,\bar{u} = \beta, \tag{11)*$$

and

$$\frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial Z} = a \quad \text{at} \quad Z = Z_T, \tag{12a}$$

$$\frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial Z} = b \quad \text{at} \quad Z = 0, \tag{12b}$$

where a and b are constants.

In the absence of thermal forcing, the steady-state linearized version of (9) and (10) is then

$$q' = -\lambda(Z)\,\psi' \tag{13}$$

that is,

$$\nabla^2\psi' + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial Z}\left(\frac{\rho_0}{N^2}\frac{\partial \psi'}{\partial Z}\right) + \lambda(Z)\,\psi' = 0, \tag{13)*$$

$$\psi'_Z = a\psi' \quad \text{at} \quad Z = Z_T, \tag{14a}$$

$$\psi'_Z + (N^2/f_0)\,h' = b\psi' \quad \text{at} \quad Z = 0. \tag{14b}$$

But (13) and (14) imply, respectively, that

$$J(\psi', q') = 0, \tag{15}$$

$$J(\psi', \psi'_Z) = 0 \quad \text{at} \quad Z = Z_T, \tag{16a}$$

$$J(\psi', \psi'_Z + (N^2(f_0)\,h') = 0 \quad \text{at} \quad Z = 0. \tag{16b}$$

So, if $\bar{u}$ satisfies (11) and (12) and ψ' satisfies the linear (13)* and (14), then it also satisfies the non-linear (6) and (8) (in the absence of thermal forcing), as all the non-linear terms vanish identically, regardless of the structure of the topography.

Let us now reintroduce the thermal forcing mechanism. We will suppose, as before, that $\bar{u}$ satisfies (11) and (12) and that the diabatic heating field Q is such that it forces a $\psi_Q = \bar{\psi}_Q + \psi'_Q$ field which in turn is such that $\bar{\psi}_Q$ satisfies the homogeneous part of (11)* and (12) and ψ'_Q satisfies (13)*, (14a) and the homogeneous part of (14b). Then because of the linearity of the equations, the total flow (thermally and topographically forced) satisfies (11)–(14). This means that (15) and (16) are satisfied so that (9) and (10) reduce to

$$\frac{\partial q}{\partial t} = G,$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial Z}\right) = \frac{\kappa Q}{f_0 H} \quad \text{at} \quad Z = 0 \text{ and } Z_T.$$

We see that if Q is time-independent, we obtain a linear (resonant) growth in ψ, a result that was exploited by Mitchell and Derome (1983), hereafter referred to as MD. The important difference here is that the thermally-forced part of the flow coexists with the orographically forced disturbances, while MD considerd only thermal forcing. It is also important to note that Q can be a function of time; the restriction is on its spatial structure which, as mentioned above, must be such that ψ_Q satisfies the homogeneous parts of (11)–(14).

4. Observations

The condition (11) on $\bar{u}$ can be integrated once with respect to y to yield

$$\bar{q} = -\lambda(z)\,\bar{\psi} + \mu(z), \tag{17}$$

where $\mu(z)$ is the integration function. To see to what extent atmospheric data satisfy (17), we have written $\bar{q}$ in spherical coordinates and used p at the vertical coordinate to obtain

$$\bar{q}^* = \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\bar{\psi}^*}{\partial\phi}\right) + \frac{\partial}{\partial p_*}\left(\frac{1}{\sigma^*}\frac{\partial\psi^*}{\partial p_*}\right) + \sin\phi = -\lambda^*(z)\,\bar{\psi}^* + \mu^*(z), \tag{18}$$

where $\bar{q}^* = \bar{q}/2\Omega$, $\psi^* = \psi/2\Omega a^2$, $\sigma^* = \sigma p_s^2/f_0^2 a^2$, $p_s = 1000$ mb, $p_* = p/p_s$, $\sigma = -\alpha_0\,\mathrm{d}\ln\theta_0/\mathrm{d}p$, α_0 being the area-averaged specific volume, $\lambda^* = a^2\lambda$, $\mu^* = \mu/2\Omega$. The January 1979 FGGE data were used to obtain the mean monthly $\bar{u}(\phi, p)$ field from which $\bar{\psi}^*$ was computed by trapezoidal integration using data at intervals of $\Delta\phi = 1.875°$ and the average value of $\bar{\psi}^*$ was set to zero. The potential vorticity q^* was then computed using second-order finite differences in ϕ and p_*, using the static stability computed from the mean January temperatures averaged over the area north of 20°N. The results, plotted against $-\bar{\psi}^*$, appear in Fig. 1a, b

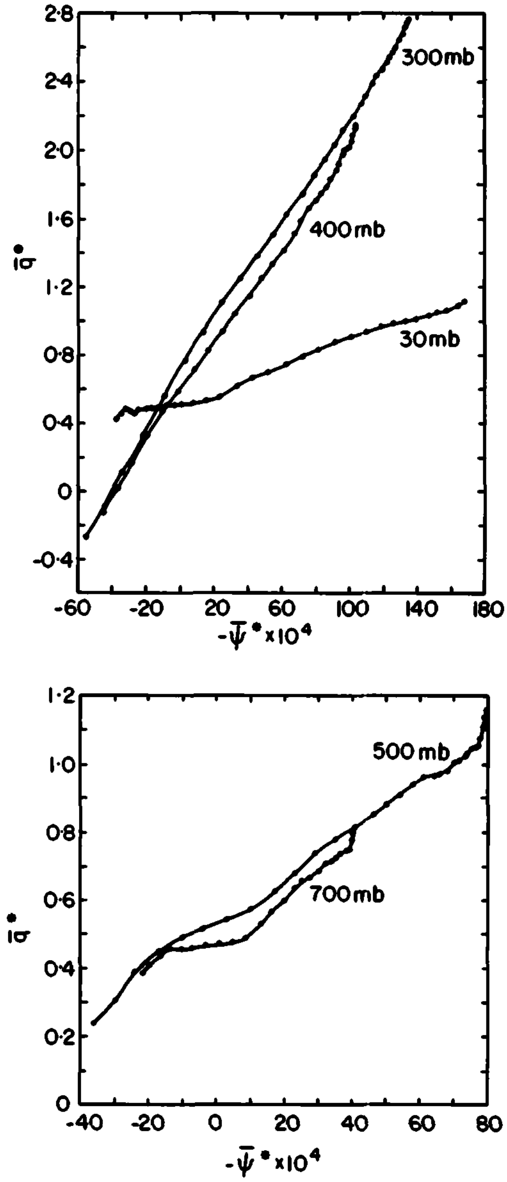

Fig. 1. The potential vorticity computed by (18) using mean January 1979 data versus the negative of the stream function, (a) for 30, 300 and 400 mb and (b) for 500 and 700 mb. In all cases, the latitude increases from 22.5 °N on the left to 82.5 °N on the right.

for $p = 30$, 100, 300, 400, 500 and 700 mb for the latitude band 22.5 °N to 82.5 °N. In all cases, the smallest value of $-\bar{\psi}^*$ corresponds to the southernmost point, while the largest value of $-\bar{\psi}^*$ applies to the northernmost point, reflecting the fact that $\bar{u}$

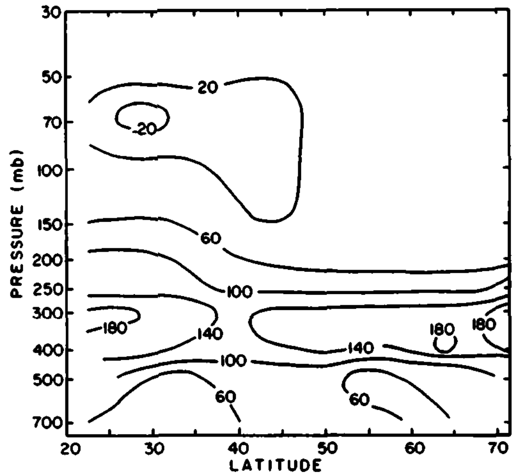

Fig. 2. The ratio $(\partial\bar{q}^*/\partial\phi)/\bar{u}^*$ as a function of pressure and latitude.

is positive everywhere. We see that at 300, 400 and to some extent 500 mb also, the potential vorticity is nearly linear in $\bar{\psi}$ as assumed in this study, while at other levels the relationship is more complex. The change in axes from Fig. 1a to Fig. 1b is such that the slopes, equal to λ^*, are comparable. We note that λ^* is appreciably larger at 300 and 400 mb than at the other levels.

Another way to present the results is to plot $\lambda^* = (\partial\bar{q}^*/\partial\phi)/\bar{u}^*$, with $\bar{u}^* = \bar{u}/2\Omega a$, as a function of latitude and the vertical coordinate, as did Matsuno (1970). While our results, shown in Fig. 2, differ appreciably from his, presumably in large part due to our different static stability profiles, they do agree on the fact that λ^* is more nearly independent of latitude in the upper troposphere than anywhere else.

While our choice of mean January conditions is arbitrary, our results are nevertheless of interest in that they show that the atmosphere is not far, on the average in January, from obeying a linear relationship between $\bar{q}$ and $\bar{\psi}$, at least in the upper troposphere. It might then be speculated that on occasion, the mean zonal wind, as it oscillates about its time-mean state, can come even closer to obeying the linear relationship than is indicated in Figs. 1 and 2. These would be times when the resonant thermal forcing mechanism discussed by MD would have the best chance to operate.

The structure of topographically-forced flows using observed mean January zonal conditions has been computed using two-level steady-state non-linear models by Egger (1976) and Ashe (1979). In both cases (Egger, p. 79; Ashe, p. 118) the topographically forced flow was found to be relatively insensitive to the non-linearities. This may be a reflection of the fact that for mean January conditions, $\partial\bar{q}/\partial y/\bar{u}$ is not a strong function of latitude over an appreciable part of the troposphere. When the effects of eddy diabatic heating are considered, on the other hand, as was done by Ashe (1979) and Youngblut and Sasamori (1980), there is no reason to expect the non-linearities to be negligible even in a steady state model, unless the heating field satisfies the conditions discussed in Section 3.

It should be pointed out that another, exactly equivalent method can be used to obtain the results of this paper. Rather than beginning by partitioning the flow into a mean zonal part and an eddy part and then looking for conditions on $\bar{u}$ to reduce the problem to a linear one, we could have proceeded as in MD and as in studies of modons (e.g. Flierl et al., 1980). The procedure is to look for flows satisfying $J(\psi, q) = 0$, that is, flows for which $q = H(\psi, Z)$, where H is some function of ψ and Z. If H is assumed to be linear in ψ, the results of the present paper follow. We have opted for a different presentation which, by formally representing the flow as a sum of a zonal part and an eddy part, parallels to some extent that of linear studies.

5. Concluding remarks

We have seen that provided some special conditions are satisfied, it is possible to obtain the structure of finite-amplitude disturbances by solving a linear problem. One condition that we imposed was that the flow be inviscid. When only topographic forcing is included, this is a logical consequence of our requirement that the flow be time-independent; no non-trivial steady-state is possible in a system having a dissipation mechanism but no energy source. When thermal forcing is included, it is possible to have a thermal dissipation mechanism (Newtonian thermal forcing-dissipation) as in MD. It was shown in that study that the flow can remain time-independent if the thermal forcing and dissipation balance everywhere; otherwise a time dependence can be driven by the imbalance between the two mechanisms, which involves no non-linear interactions provided the diabatic heating has the proper spatial structure.

No restriction was placed on the amplitude of the diabatic heating and topography nor, in fact, on the structure of the latter. For sufficiently large-amplitude disturbances, however, the flow can be expected to become unstable due to the large local shears. Naturally, the details of the stability properties would depend on the particular forcing fields used and so it is not possible here to make general statements regarding the stability of the forced flow.

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