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SHORT CONTRIBUTION

# A note on a geophysical fluid dynamics variational principle

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ABSTRACT

A variational principle for an inviscid stratified fluid on a beta plane is developed for horizontal flows. For the special case of solenoidal flow on an $f$ plane, it is shown that a certain velocity gradient invariant can be used to develop simple solutions to the equations of motion. The solution forms are critically dependent on relative magnitudes of the squared local vorticity and the squared total deformation.

## 1. Introduction

Because of the emphasis on energy and conservation principles, a variational formulation can provide physical insight into dynamical processes not readily apparent in the equations of motion. This approach is not widely used in hydrodynamics; thus its potential has not yet been realized in oceanography and meteorology. The purpose here is to investigate the use of some older results of Petterssen (1953) and Okubo (1970) in conjunction with a geophysical fluid dynamical variational principle. The results, although preliminary, suggest possible applications to quasi-geostrophic flows.

The basis for the analysis is a hydrodynamic variational principle for rotating, stratified and inviscid fluids. Most workers attribute the principle to Herivel (1955) for application to irrotational flows and to Lin (1963) for the generalization to include vorticity. This was done by constraining the Lagrangian so that continuity was satisfied and the material or Lagrangian parcel position vector was conserved following the parcel motion. It is not widely recognized, but Sasaki (1955), in a study of numerical weather prediction, had previously developed a more general version of the principle which included a rotating coordinate system. Stephens (1965, 1967) also used the principle in studies of atmospheric flows. More recently, Katz and Lynden-Bell (1982) and Henyey (1983) have used this principle in studying oceanographic problems.

The studies cited above all start with some version of the Sasaki, Herivel–Lin lagrangian. The differences in the analyses arise from the "Lin" or "parcel identity" constraints. Stephens (1965) and Katz and Lynden-Bell (1982), emphasized thermodynamic constraints, while Henyey (1983) utilized lagrangian displacement vectors in a study of internal waves.

Here, the use of geophysical fluid dynamics (GFD) flow invariants as the identity constraint is investigated. In particular, the potential vorticity (Ertel, 1942; Ertel and Rossby, 1949) and deformation rate invariants for geostrophic flow (Petterssen, 1953) are investigated as candidate constraints. A special case of these equations produces a solution first obtained by Okubo (1970) in a study of local diffusion. The use of the potential vorticity and deformation rate invariants as identity constraints and the consequent extension of Okubo's model to GFD scales are the principal new results.

## 2. Formulation

Since variational principles are not yet widely used in meteorology and oceanography, a review of the formulation is appropriate. With the usual subscript summation convention, the Sasaki lagrangian is

$$L = \tfrac{1}{2}\rho(v_k v_k + 2\Omega_{kj} r_j v_k) - \rho(e + U) - \phi\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}\rho v_j\right) + \rho\lambda_\alpha \frac{\mathrm{d}X_\alpha}{\mathrm{d}t}. \qquad (1)$$

Here $\rho$ is density, $\phi$ and $\lambda_\alpha$ are undetermined multipliers, $e$ is the specific internal energy density, $\Omega_{ij}$ is the bivector spin of the earth, $r_j$ is the position vector relative to the earth's center, $U$ is the potential energy density associated with the earth's gravity field and the centripetal kinetic energy density associated with the earth's rotation, and $v_k$ is the velocity relative to the rotating earth. The components of $\Omega_{ij}$ are

$$\Omega_{ij} = \Omega\begin{bmatrix} 0 & -\sin\theta & \cos\theta \\ \sin\theta & 0 & 0 \\ -\cos\theta & 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -f & g \\ f & 0 & 0 \\ -g & 0 & 0 \end{bmatrix}. \qquad (2)$$

In eq. (2), $\Omega$ is the angular velocity of the earth and $\theta$ is the latitude. Finally, $X_\alpha$ are the parcel properties which are conserved following the fluid motion. In Lin (1963), $X_\alpha$ was taken as the initial or lagrangian coordinate signature for an arbitrary parcel. However, as shown by Sasaki (1955) and others, the $X_\alpha$ do not necessarily refer to the flow geometry; thus the number of constraints may differ from the geometric flow dimension. Katz and Lynden-Bell (1982) and Henyey (1983) have succinctly discussed this point and have noted the connection of this issue with that of gauge invariances. Thus Greek subscripts of unspecified range are used for this term while Latin subscripts are reserved for geometric components. Obviously

$$\frac{\mathrm{d}}{\mathrm{d}t}X_\alpha = 0,$$

where $\mathrm{d}/\mathrm{d}t$ is the material derivative.

The terms in the first two parentheses in eq. (1) represent the absolute kinetic and total potential energy densities, respectively. The former is composed of the kinetic energy density as seen by one moving with the earth and the gyroscopic energy density which arises from the rotation of the earth. The total potential energy density is composed of an internal thermodynamic component and a gravitational component. The terms in the last two parentheses are the conservation of mass and the parcel identity constraints. Note that these constraints are non-holonomic. This means that the undetermined multipliers will be functions of the independent variables (Courant and Hilbert, 1966).

The general variational principle requires the functional

$$I = \int_{t_1}^{t_2}\int_V L(v_j, \rho, \phi, \lambda_\alpha, X_\alpha)\mathrm{d}V\mathrm{d}t, \qquad (3)$$

where $\mathrm{d}V$ is an arbitrary volume element, to be an extremum. Necessary conditions for this are that the variations with respect to the arguments of $L$ vanish. Two minor restrictions to these variations are noted. First the variations must vanish at the boundary of $V$. There are well-established techniques for handling inhomogeneous boundary variations. As they are problem specific and not essential to the principal results here, they are not considered. The other restriction is that the independent variables, time $t$ and eulerian position $x_i$, are not varied.

The variations with respect to $\phi$, $\lambda_\alpha$ and $X_\alpha$ yield

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_j}(\rho v_j) = 0, \qquad (4a)$$

$$\frac{\mathrm{d}X_\alpha}{\mathrm{d}t} = 0, \qquad (4b)$$

$$\frac{\mathrm{d}\lambda_\alpha}{\mathrm{d}t} = 0. \qquad (5)$$

It is seen from eq. (4b) that the variations with respect to the undetermined multipliers reproduce the original constraints, and from (5) that $\lambda_\alpha$ is conserved following the motion.

Variations with respect to $v_i$ and $\rho$ yield

$$v_i + \Omega_{ij} r_j + \frac{\partial}{\partial x_i}\phi + \lambda_\alpha \frac{\partial}{\partial x_i}X_\alpha = 0, \qquad (6)$$

$$\tfrac{1}{2}(v_k v_k + 2\Omega_{ij} r_j v_i) - U - h + \frac{\mathrm{d}\phi}{\mathrm{d}t} = 0. \qquad (7)$$

Here $h = \partial(\rho e)/\partial\rho$ is the enthalpy. Note that the number of equations contained in eqs. (4)–(7) exactly equals the number of dependent variables and the undetermined multipliers.

The variational equations for the velocity, eq. (6), suggest an interpretation of $\phi$, $\lambda_\alpha$ and $X_\alpha$ as Monge potentials. The importance of $X_\alpha$ in generating vorticity is apparent, for if $X_\alpha = 0$, eq. (6) admits only solutions with zero absolute vorticity. Eq. (7) is a bernoullian equation or an expression for the total enthalpy.

The equations of motion are obtained by taking the material derivative of eq. (6) and then subtracting the gradient of eq. (7). One obtains with the use of eq. (6), the thermodynamic relation between enthalpy and pressure $p$ for an isentropic process, and the beta plane assumption

$$\frac{\mathrm{d}}{\mathrm{d}t} v_1 - f v_2 + g v_3 = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} - \frac{\partial U}{\partial x_1},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} v_2 + f v_1 = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} - \frac{\partial U}{\partial x_2},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} v_3 - g v_1 = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} - \frac{\partial U}{\partial x_3}. \tag{8}$$

These are the nonlinear equations of motion for a compressible inviscid fluid on a rotating earth. These equations along with eq. (4a) comprise the formulation of many GFD studies. An important special case which is studied below occurs when the horizontal component of Corolis, $g$, along with the vertical acceleration in the third equation in (8), are neglected. Then the right-hand side of the first two equations can be replaced by the gradient of the geopotential $\Psi$. Another special case is geostrophy. This is obtained by neglecting the $v_k v_k$ term in (1).

## 3. Applications

Since the Monge potentials are eliminated from the equations of motion, there is no restriction on their interpretation. For example Sasaki (1955) provided several interpretations of these potentials. These included $\lambda$ as entropy, and $\phi$ as the geostrophic stream function. Stephens (1967) took $X_1$ as entropy and $\lambda_1$ as $\int T\mathrm{d}t$ ($T$ is absolute temperature). He also showed that if $X_\alpha$ was the material coordinate position vector, then $\lambda_\alpha$ was the initial velocity vector. A similar interpretation was made by Henyey (1983) in a study of internal waves.

Dutton (1976) has developed a general form of eq. (6) which includes most of the above interpretations as special cases. In that approach, eq. (7) is used to express $\phi$ in terms of the enthalpy, geopotential and kinetic energy per unit mass. Also there are four $\lambda_\alpha$ and $X_\alpha$ which encompass the Stephens (1967) and Henyey (1983) interpretations as well as some earlier results of Ertel (1952).

For GFD flows, the potential vorticity $\Pi$ is a logical candidate invariant. The general form for $\Pi$ is (Ertel, 1942; and Rossby, 1949)

$$\Pi = \varepsilon_{ijk}\left(\frac{\partial v_k}{\partial x_j} + 2\Omega_{ij}\right)\frac{\partial S}{\partial x_i}\Big/\rho. \tag{9}$$

Here $S = S(\rho, p)$ is a constant of motion and $\varepsilon_{ijk}$ is the alternating tensor. Katz and Lynden-Bell (1982) suggest taking $\Pi$ for $X$ in eq. (6); however, no specific interpretation for $\lambda$ was given. Note, however, that if the absolute vorticity is conserved and the fluid is incompressible, then eq. (9) is of the form $\lambda\ \partial X/\partial x_i$. In this case $\partial X/\partial x_i$ can be interpreted as the moment arm for the induced flow of a vortex.

Another interesting result arises if the flow is restricted to be horizontal and solenoidal. Starting with eq. (8), Petterssen (1953) derived, for this situation

$$\frac{\mathrm{d}}{\mathrm{d}t}(A - \beta x_1) = fB - \frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2}, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(B - \beta x_2) = -fA - 2\frac{\partial^2\Psi}{\partial x_1\,\partial x_2}, \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(C + \beta x_2) = 0. \tag{12}$$

Here

$A = \dfrac{\partial v_1}{\partial x_1} - \dfrac{\partial v_2}{\partial x_2}$, the normal deformation rate,

$B = \dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial v_2}{\partial x_1}$, the shear deformation rate,

$C = \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}$, the relative vorticity,

$\beta = \dfrac{\partial f}{\partial x_2}$, the so-called beta parameter.

Eq. (10) and (11) are the equations for the normal and shear deformation rates; (12) is a special case of Ertel potential vorticity theorem. Kirwan (1975) has provided additional interpretations of these equations.

Petterssen (1953) showed that for geostrophic flow on an $f$ plane ($\beta = 0$), eqs. (10)–(12) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}(C^2 - A^2 - B^2) = 0. \tag{13}$$

From eq. (13), it is seen that $M^2 = C^2 - A^2 - B^2$ can be used for $\lambda$ in (6). Since the motion is geostrophic and solenoidal, $v_l$ is neglected in (6) and $\phi$ is taken as a stream function. Then eq. (6) becomes

$$v_2 + M^2 \frac{\partial X}{\partial x_1} + \Omega_{12} r_2 + \Omega_{13} r_3 = 0,$$

$$-v_1 + M^2 \frac{\partial X}{\partial x_2} + \Omega_{21} r_1 = 0. \tag{14}$$

Following Sasaki (1957), $X$ may be interpreted as a geometric invariant of a particular streamline. For horizontal motion scales of the order of $10^2$ km or less, the streamline may be approximated by a conic; i.e.,

$$X = D_{11} x_1^2 + D_{12} x_1 x_2 + D_{22} x_2^2.$$

The parameters $D_{ij}$ may be normalized so that

$$4(4D_{11} D_{22} - D_{12}^2) = 1/M^2. \tag{15}$$

This is recognized as the characteristic equation for conic sections.

Eqs. (14) are readily integrated to yield parametic equations for the streamline followed by particular parcel of fluid. The result is

$$x_1 = Ut \times x_1^0 \cos \tfrac{1}{2} Mt - \frac{1}{M}(D_{12} x_1^0 + 2E_{22} x_2^0 \sin \tfrac{1}{2} Mt,$$

$$x_2 = Vt + x_2^0 \cos \tfrac{1}{2} Mt + \frac{1}{M}(D_{12} x_2^0 + 2E_{11} x_1^0 \sin \tfrac{1}{2} Mt. \tag{16}$$

Here $x_i^0$ refer to the starting positions and $t$ is now the streamline parameter. The constant terms in eq. (14) are identified with $U$ and $V$ in eq. (16).

Eq. (16) are the same form as that obtained by Okubo (1970) for the condition that $A$, $B$, and $C$ were constant in space and time and the horizontal divergence was zero. Through the use of Petterssen's geostrophic velocity gradient invariants and the variational formulation, that solution has been extended to just $M^2$ constant following the motion. The price for this is three additional path invariants.

The functional form of eq. (16) is critically dependent upon the sign of $M^2$. According to eq. (15), if $M^2 > 0$ then the streamlines form ellipses. In the special case where $A = B = 0$, the ellipses degenerate into circles. Then circularly symmetric vortices such as the Rankine or Bessel function models follow from the appropriate specification of the spatial dependency of $C$.

From (15) it is seen that negative and zero values of $M^2$ yield hyperbolic and parabolic streamlines, respectively. In this case, $M$ is imaginary or zero. It is interesting that the solution eq. (16) accommodates these possibilities as well. For imaginary $M$, the trigonometric functions revert to their hyperbolic counterparts, while for $M = 0$ the limiting condition for the trigonometric functions yields solutions which grow linearly in time.

These three classes of solution for horizontally sheared motions are analogous to the three cases of static vertical stability in a stratified ocean. The elliptic solutions ($M^2 > 0$) which produce periodic paths and swirl velocities correspond to stable stratification or positive values of the square of the Vässäilä–Brunt frequency, $N$. The cases of $M^2 = 0$ and $M^2 < 0$ produce solutions for horizontal motions which are the same as for $N^2 = 0$, $N^2 < 0$, or the neutral and unstable stratification cases.

Because both investigators recognized the pivitol rôle of $M$ in horizontal motions, the name Petterssen–Okubo or PO frequency is suggested.

The PO frequency can be related to the kinematical vorticity number $K$ of Truesdell (1954) defined by

$$K = \sqrt{C^2/(A^2 + B^2)}.$$

Clearly

$$M^2 = (A^2 + B^2)(K^2 - 1).$$

Notice that $M^2 > 0$ requires that $K^2 > 1$. Fortak (1963) has provided an elegant and general interpretation of the rôle of $K$ in the stability of geophysical fluid scale flows.

## 4. Discussion

It has been shown that the variational approach has provided a dynamical basis for a means of extending the local kinematic model of Okubo to GFD scales. The key to this was the use of Petterssen's deformation rate invariants in the identity constraint. The consequent analysis suggests that there is a fundamental frequency which separates the streamlines into periodic and non-periodic classes. The periodic class allows non-circular vortices.

The variational approach as outlined above also provides a basis for combining lagrangian and eulerian observations. The model for this is eq. (6). If the flow invariants are observed, then these equations can be used to specify the velocity. The alternative is to observe the velocity or the path (eq. (14)) and then deduce the invariants.

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