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SHORT CONTRIBUTION

Note on simplified forms of the quasi-geostrophic potential vorticity equation

By H. L. KUO, *Department of Geophysical Sciences, The University of Chicago, Chicago, Illinois 60637, U.S.A.*

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ABSTRACT

Different approximations of the general quasi-geostrophic potential vorticity equation under high Richardson number condition are obtained for different types of large-scale disturbances in the atmosphere. It is found that the complete equation is needed for the long waves in middle latitudes, but that the barotropic and non-divergent approximation is adequate for ordinary long waves in low latitude, whereas for disturbances with ultra-long horizontal length-scale in both zonal and meridional directions, the potential vorticity can be approximated by the thermal portion alone. On the other hand, for the EW-elongated ultra-long waves in the tropics, the u_y term in the potential vorticity must be included.

1. Introduction

It is well-known that the large-scale flows in the atmosphere and the oceans, including the stationary ultra-long waves, are quasi-geostrophic and their general properties can be represented adequately by the quasi-geostrophic potential vorticity equation provided the influences of the diabatic heating and surface topography are taken into consideration. It is also known that for the ultra-long waves with their horizontal wavelengths of the order 10^7 m, the vorticity equation reduces to a balance between the transport of the planetary vorticity of earth's rotation and the divergence of the geostrophic wind (Burger, 1958) and hence it loses its prognostic ability, so that the development of the system is represented by the heat equation together with the hydrostatic and geostrophic relations. However, this conclusion is valid only when the horizontal wavelength in both x-and y-directions are in the ultra-long range, and it becomes invalid when the wavelength is ultra-long only in one direction, as exemplified by the elongated high pressure cells in the tropics. The significance of this type of disturbance can also be seen from the results given by many observational studies, such as those of Eliasen and Machenhauer (1965, 1969), Van Loon and Jenne (1972) and Van Loon et al. (1973). The purpose of this note is to derive appropriate approximations of the quasi-geostrophic potential vorticity equation for different types of the large-scale disturbance, including those with their wavelength in the zonal direction much longer than that in the meridional direction.

2. Theory

We shall first non-dimensionalize the governing equations by using L as the representative horizontal scale length, $H = RT/g$ as the vertical scale length, V_0 as the horizontal velocity scale, L/V_0 as the time scale, and express the departures of the geopotential and temperature from their mean values on the horizontal or isobaric surfaces in units of f_0LV_0 and f_0LV_0/R, and the diabatic heating rate Q aside from conduction in units of $f_0V_0^2c_p/R$, where f_0 is the representative mean value of the Coriolis parameter and R is the gas constant of air in mechanical units. We then find that the non-dimensionalized hydrostatic equation, the continuation equation, the heat equation, the ordinary vorticity equation and the potential vorticity equation in $Z = \ln/(P_{se}/P)$ vertical coordinates are given by

$$\frac{\partial \phi}{\partial Z} = T, \tag{1}$$

$$\nabla \cdot \mathbf{V} + e^{Z} \frac{\partial}{\partial Z}(e^{-Z}\dot{Z}) = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla - \frac{\partial}{\partial x_j} v_j \frac{\partial}{\partial x_j}\right)\frac{\partial \phi}{\partial Z} + \frac{R_1}{R_0} S\dot{Z} = Q, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla - \frac{\partial}{\partial x_j} v_j \frac{\partial}{\partial x_j}\right)\zeta + \hat{\beta}v = f\frac{R_1}{R_0} e^{Z}\frac{\partial}{\partial Z}(e^{-Z}\dot{Z}), \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla - \frac{\partial}{\partial x_j} v_j \frac{\partial}{\partial x_j}\right)q + \hat{\beta}v = f e^{Z}\frac{\partial}{\partial Z}\left(\frac{e^{-Z}}{S} Q\right). \tag{5}$$

Here ζ is the relative vorticity, q is the relative potential vorticity, v_j is the dimensionless eddy diffusion coefficient in units of $V_0 L$ and $V_0 H^2/L$, $\dot{Z} = dZ/dt$, P_{se} is the mean sea-level pressure, $f = f'/f_0$, R_1 is the ratio of the divergent part of the wind speed to the actual wind, R_0 is the Rossby number, S is the planetary stability factor and $\hat{\beta}$ is the dimensionless Rossby parameter, which are respectively given by

$$R_0 = \frac{V_0}{f_0 L}, \quad S = \frac{N^2 H^2}{f_0^2 L^2} = \frac{L_R^2}{L^2}, \quad \hat{\beta} = \frac{f_0 L^2}{a V_0} = \frac{L}{a R_0} \tag{6a, b, c}$$

where N is the Brunt-Väsälä frequency and $L_R = NH/f_0$ is the internal deformation radius.

According to the quasi-geostrophic approximation, we have

$$\mathbf{V} \doteq \mathbf{V}_g = \frac{1}{f}\mathbf{k} \times \nabla\phi, \quad \zeta \doteq \zeta_g = \frac{1}{f}\nabla^2\phi - R_0 \frac{\hat{\beta}}{f^2}\frac{\partial \phi}{\partial y}, \tag{7a, b}$$

$$q = \zeta + f e^{Z}\frac{\partial}{\partial Z}\left(\frac{e^{-Z}}{S}\frac{\partial \phi}{\partial Z}\right). \tag{7c}$$

Ordinarily, we take (5) as the governing equation for the large-scale flow field and use (3) with $\dot{Z} = 0$ or $\dot{Z} = Z_b = \mathbf{V}\cdot\nabla h$ at the surface as the lower boundary condition, where $h(x,y)$ is the surface topography. Here the heating term Q is included for the study of the influence of diabatic heating on the development of the large-scale flow field, while the eddy diffusion terms are included for completeness.

It should be mentioned that even though we have used the same length scale L in both x- and y-directions in the above, different scales are usually involved under many situations.

We shall now try to simplify the potential vorticity equation (5) further for disturbances of different scales and different latitudinal positions. Specifically, we shall designate $L \sim 10^6$ m as the length scale of ordinary long waves and $L \geq 10^7$ m as the scale of ultra-long waves, and use $f_0 \sim 10^{-4}$ s^{-1} for middle and higher latitude and $f_0 \sim (1-3) \times 10^{-5}$ s^{-1} for low latitude. In addition, we also have the ultra-long waves of the second kind characterized by $L_x \geq 10^7$ m, but with $L_y \leq 3 \times 10^6$ m, as exemplified by the elongated high pressure cells in the tropics. Further we shall take $V_0 \sim 10$ m s^{-1} as the representative velocity for all large-scale disturbances in the atmosphere. We then find that the values of the most significant parameters S, R_0, $\hat{\beta}$, $\nabla f/f_0$ and R_1 for the different kinds of disturbance are as given in Table 1.

Here the value of R_1 is inferred from the heat equation (3). From the values of $\Delta f/f_0$ given above, we conclude that f can be replaced by the constant value f_0 when not being differentiated for the long waves a_1 in middle latitude, while for all other cases f should not be approximated by f_0. Further, the values of R_1 given above show that the advecting horizontal wind $\mathbf{V}$ can be approximated by the non-divergent wind $\mathbf{V}_\psi$ for the long waves a_1 and a_2 and the ultra-long wave b_3, whereas for the ultra-long waves b_1 and b_2 the divergent part of $\mathbf{V}$ should also be included, such as given by the local geostrophic wind given by (7a). The appropriate simplifications of eq. (5) for the different classes of motion are given below in accordance with this analysis.

2.1. *Middle latitude long waves*

As has been mentioned above, for this case the advecting horizontal wind can be approximated by the non-divergent wind $\mathbf{V}_\psi$ which can be repre-

Table 1. *Characteristic values of the significant dynamic parameters for different classes of disturbance*

	Long waves		Ultra-long waves		
	a_1 Middle lat.	a_2 Low lat.	b_1 Middle lat.	b_2 Low lat.	b_3 Low lat. $L_y \leq 3 \times 10^6$ m $< L_x$
S	1	100	0.01	0.1	1
R_0	0.1	1	0.01	0.1	0.04
$\hat{\beta}$	1	1	100	10	1
$\Delta f/f_0$	0.1	1	1	10	1
R_1	0.1	0.01	1	1	0.04

sented more conveniently by the stream function $\psi = \phi'/f_0 = p'/f_0\rho_0$, so that $\mathbf{V}$, ζ and q are given by

$$\mathbf{V} = \mathbf{V}_\phi = \mathbf{k} \times \nabla\psi, \quad \zeta = \nabla^2\psi, \tag{8a, b}$$

$$q = \nabla^2\psi + f^2 e^Z \frac{\partial}{\partial Z}\left(\frac{e^{-Z}}{S}\frac{\partial\psi}{\partial Z}\right). \tag{8c}$$

2.2. *Long waves in low latitude*

According to the values of S and R_1 obtained above for this case, the wind is also non-divergent, but the baroclinic part of q represented by the second term in (8c) is negligibly small in comparison to the first term except when the vertical wave-length is small. This means that for this class of disturbance, eq. (5) reduces to the non-divergent vorticity equation, viz.,

$$\frac{\partial}{\partial t}\nabla^2\psi + \psi_x(\nabla^2\psi + f)_y - \psi_y\nabla^2\psi_x = \nu\nabla^4\psi. \tag{9}$$

This result indicates that at low latitude, the long waves at different levels are decoupled from each other, in conformity with the conclusion reached by Charney (1963). No lower boundary condition is needed for this case. This property is also revealed by the two-layer baroclinic wave solutions at low latitude (Kuo, 1978).

Since the geostrophic relation breaks down at the equator, the balance equation for low latitude should be used to obtain the geopotential height ϕ' from the stream function ψ, viz.,

$$\nabla^2\phi' = f\nabla^2\psi + 2(\psi_{xx}\psi_{yy} - \psi_{xy}^2) + \beta\psi_y. \tag{10}$$

Thus, even though the development of the motion field is carried out solely by the non-divergent vorticity equation (9), the change of the ϕ'-field can be obtained from this equation and consequently also the change of the temperature field. The heat equation (3) can then be used to obtain the vertical velocity $\dot{Z}$.

2.3. *Disturbances with ultra-long wavelengths in both* x- *and* y-*directions in middle latitude*

For the ultra-long waves in middle latitudes, we have $S \sim 0.01$ and $\hat{\beta} \sim 100$; hence the contribution of ζ to the potential vorticity q is neglegible and therefore the potential vorticity equation (5) reduces to the following

$$\left(\frac{\partial}{\partial t} + \frac{1}{f}\mathbf{k} \times \nabla\phi\cdot\nabla\right)\left[e^Z\frac{\partial}{\partial Z}\left(\frac{f e^{-Z}}{S}\frac{\partial\phi}{\partial Z}\right)\right] + \frac{1}{f}\hat{\beta}\frac{\partial\phi}{\partial x} = e^Z\frac{\partial}{\partial Z}\left(\frac{fQ}{S}e^{-Z}\right). \tag{11}$$

The boundary condition at the bottom for these ultra-long waves is still given by (3) but without the $\nu_h\nabla^2$ term. This equation also holds for the ultra-long waves at low latitude with $L_x \simeq L_y \geq 10^7$ m, since here both S and R_0 are one order of magnitude larger than in middle latitudes. It can also be seen that under this circumstance the vorticity equation (4) reduces to

$$\hat{\beta}v + \frac{1}{R_0}f\nabla\cdot\mathbf{V} = 0, \tag{12}$$

and hence it loses its prognostic ability, as has been discussed by Burger (1958). Notice also that this relation is satisfied exactly by the geostrophic wind $\mathbf{V}_g$. It is readily seen that elimination of $\nabla\cdot\mathbf{V}$ between (2), (3) and this relation yields (11).

2.4. *Ultra-long waves of the second kind*

Besides, there is another important class of quasi-geostrophic disturbances at low latitude with ultra-long length scale $L_x > 10^7$ m in WE direction, but moderately long length scale $L_y \sim 3 \times 10^6$ m in the NS direction, and with $f_0 \sim 3 \times 10^{-5}$ s^{-1}, as exemplified by the elongated high pressure cells in the tropics. For this case, the vorticity is mainly given by ψ_{yy} and hence the length scale L in S should be L_y, and the ratio $v/|V|$ is of the order $L_y/L_x \sim 0.1$, so that we have

$$S \sim 1, \quad \frac{\Delta f}{f_0} \sim 1, \quad R_0 \sim 0.04, \quad \hat{\beta} \sim 1.$$

Accoring to (1), the divergence ratio R_1 is also of the order of 0.04 and hence the velocity is nearly non-divergent, but the flow is now governed by the complete eq. (5) with $\mathbf{V}$ represented by the non-divergent wind $\mathbf{V}_\psi$, viz.,

$$\frac{\partial q}{\partial t} + \psi_x(q_y + \hat{\beta}) - \psi_y q_x = e^Z \frac{\partial}{\partial Z}\left(\frac{f e^{-Z}}{S} Q\right), \quad (13)$$

where q is given by

$$q = \psi_{yy} + e^Z \frac{\partial}{\partial Z}\left(\frac{f^2 e^{-Z}}{S} \frac{\partial \psi}{\partial Z}\right). \quad (13a)$$

Similarly, for disturbances with ultra-long wavelength in the y direction only, ζ can be approximated by ψ_{xx}.

Thus, these analyses show that, under the prevailing stable stratification of the atmosphere, the behaviors of the different types of large-scale disturbance, including the ultra-long standing waves, can be adequately represented by the different simplified versions of the potential vorticity equation. Specifically, it is found that the ordinary long waves at low latitude can be represented by the barotropic and non-divergent vorticity equation, and the disturbances with ultra-long wavelengths in both the x- and y-directions are governed by the thermal limit of the potential vorticity equation, while for the class of disturbances characterized by ultra-long wavelengths in the zonal direction and a shorter wavelength in the meridional direction, ζ can be approximated by $-u_y$.

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