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The development characteristics of quasi-geostrophic baroclinic disturbances

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ABSTRACT

An overall qualitative analysis is made for the development characteristics of quasi-geostrophic pure baroclinic disturbances by using both integral properties and the WKB method. The results obtained by these two methods are consistent, but the WKB method yields more conclusions.

The sufficient condition for stability and the necessary condition for instability are investigated. A local individual disturbance can be unstable only if it is located in the layer where the instability criteria is satisfied locally. For an arbitrarily given initial condition, the instability might be first realized locally, and then spread into adjacent levels.

More attention is paid to the problems of development. A disturbance is called developing (decaying) if its energy increases (decreases). In a vertical shear zonal flow, all characteristics of a disturbance, such as total wave length, vertical wave length, orientation of axis (trough-ridge line), amplitude and energy usually change with time during the evolutional process. The development of a disturbance depends only on the orientation of its axis and its location related to the basic current. A developing (decaying) disturbance enlarges (shortens) its total wave length and vertical wave length, and gradually turns its axis more vertically (more tilted). If the meridional gradient of zonal mean potential vorticity is everywhere positive during the period of growth (decay), the maximum amplitude of the disturbance moves toward (out from) the jet and is accompanied by a vertical energy flux toward (out from) the jet. In the unstable case, the energy flux is more complicated.

It is also pointed out that wave action might not exist in the unstable case, but the abovementioned conclusions are all still valid.

1. Introduction

The development of disturbances and their interaction with the basic current are fundamental problems in atmospheric dynamics and the theory of general circulation. Starting with Charney (1947), Eady (1949) and Kuo (1949), a great deal of investigation has been devoted to the development problem, which is usually studied as a problem of instability, using the normal mode method (see for example, Charney (1947), Kuo (1949, 1952), Lin (1955), Pedlosky (1964)).

There are many achievements in the field, such as the conclusions of the necessary conditions for instability, the proper structures of developing, decaying and neutral waves and their contributions to maintenance of the general circulation. These achievements have made great contributions to dynamic meteorology and moved it into the state of modern science. However, the investigation of these problems has not ended yet, and still attracts much attention of current investigators due to its importance and complexity.

Some questions still remain. For example, whether all kinds of the evolution or development of disturbances can be investigated through the concept of instability. In fact, observational study and the experience of weather forecasting tell us that there are many cases where strong development does not satisfy the barotropic or simple baroclinic instatibility conditions but which can be successfully predicted even by a very simple but non-linear barotropic or baroclinic model. It seems

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that in such cases the energy transformation and dispersion rather than the instability take place. The other question is connected with the limitation of the normal mode method. On the one hand, this method does not yield a complete set of eigenfunctions (normal modes) in some cases, as has been pointed out by Burger (1966) for the pure baroclinic two-dimensional problem. Therefore, some kinds of solution might be missed by this method. On the other hand, the normal modes for the instability problem usually do not form an orthogonal set even if they are complete. Consequently, they are not convenient for representing the energy and other squared properties of the disturbances. Therefore, it is desirable to investigate a more general problem, i.e., the initial value problem.

An individual disturbance such as a trough or ridge superimposed on a zonal current or ultra-long wave can be local, so that it is natural to represent it as an individual wave-packet and to investigate its evolution by the WKB method. By so doing, the general initial value problem is simplified. It has been shown by Lu and Zeng (1981) that the WKB method can be applied to a more general profile of basic current even with an ultra-long wave as the basic current. Besides, some new conclusions have been drawn, such as that the developing (decaying) disturbance enlarges (shortens) its wave length.

In this work we will combine the application of the WKB method with the analysis of general integral properties to make an overall qualitative analysis of the baroclinic development and instability. This yields many useful conclusions on the relationship between the development and the structure of a disturbance and the evolution of the structure and properties, although the corresponding analytical solution will not be obtained.

2. The model and its integral properties

The inviscid, adiabatic, quasi-geostrophic potential vorticity equation written in p-coordinates is as follows (see, for example, Haltiner and Williams, 1979, Chapter 3),

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) q = 0, \tag{2.1}$$

$$q = \beta_0 y + \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial \zeta}\left(\frac{f_0^2 \zeta^2}{C^2}\frac{\partial}{\partial \zeta}\right)\right]\psi, \tag{2.2}$$

where ψ is the quasi-geostrophic stream function, $\beta_0 = (\mathrm{d}f/\mathrm{d}y)_0$ and f_0 are constants, $\zeta = P/P_s$, P_s is the surface pressure and is also taken as a constant, and

$$C^2 = \alpha R\tilde{T}, \qquad \alpha = \frac{R}{g}\left(\frac{g}{C_p} + \frac{\mathrm{d}\tilde{T}}{\mathrm{d}Z}\right) = \frac{R\tilde{T}}{g^2}N^2, \tag{2.3}$$

R is the gas constant, $\tilde{T}$ is the reference "standard" vertical temperature profile, N is the Brunt-Vaisala frequency and α is a dimensionless parameter. The other symbols are those commonly used. $P_s\zeta^2/C^2$ is usually denoted by σ, the parameter of stratification, but here we prefer to use our notation, because σ is used by us as the local frequency, and $C = NR\tilde{T}/g$ defined by (2.3) determines a very useful characteristic velocity, whose standard (averaged) value, C_0, is slightly less than 100 m s^{-1}.

Introducing a characteristic time scale t_0, horizontal scale l_0, characteristic value of stream function, ψ_0, and $u_0 = \psi_0/l_0$, and denoting non-dimensional quantities by asterisks, we have

$$\left(\frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*}\right) q^* = 0, \tag{2.1$'$}$$

$$q^* = \beta^* y^* + \left[\frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial}{\partial \zeta^*}\left(\frac{\zeta^{*2}}{C^{*2}}\frac{\partial}{\partial \zeta^*}\right)\right]\psi^*, \tag{2.2$'$}$$

where $C^{*2} = (C^2/C_0^2)(l_r/l_0)^2$, $\beta^* = \beta_0(a/f_0)(l_0/a)R_0^{-1}$, and $\zeta^* = \zeta$; a is the earth's radius, R_0 is the Rossby Number, $u_0/f_0 l_0$, and $l_r = C_0/f_0$ is the internal Rossby deformation radius, whose standard value is equal to 10^6 m at middle latitude.

We call a disturbance ultra-long wave, long wave or short wave according to $O(l_0/l_r) > O(1)$, $= O(1)$ or $< O(1)$, respectively. Under the earth's actual atmospheric conditions, these definitions agree with those used in synoptic meteorology. For the long wave, we have $O(l_r/l_0) = O(1)$ and $O(l_0/aR_0) = O(1)$; hence $O(C^{*2}) = O(1)$ and $O(\beta^*) = O(1)$, while for the short wave we always have $O(C^{*2}) > O(1)$ and $O(\beta^*) < O(1)$ if $O(R_0^{-1})$ is not too small.

Hereafter we will omit the asterisk for all the non-dimensional quantities.

The bottom boundary is taken as $\zeta = 1$, and the corresponding boundary condition comprises the vanishing of the vertical velocity, i.e.,

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\frac{\partial \psi}{\partial \zeta} + \alpha_s \frac{\partial \psi}{\partial t} = 0, \quad (\zeta = 1), \tag{2.4}$$

where α_s is the value of α at surface $\zeta = 1$, and usually $\alpha_s \approx 0.1$. On the other hand, the energy included in a vertical atmospheric column is always considered to be finite, i.e.,

$$\frac{1}{2}\int_0^1 \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\zeta}{C}\frac{\partial \psi}{\partial \zeta}\right)^2\right] d\zeta < \infty. \tag{2.5}$$

This formulation itself includes the top boundary condition if the top boundary is taken as $\zeta \to 0$, (see, for example, Zeng, 1979).

As for the two-dimensional pure baroclinic problem, the solution ψ is considered to be independent of y. Hence, the zonal mean wind components $\bar{u}(y,\zeta) = -\partial\bar{\psi}/\partial y$ and $\bar{v} = 0$ are invariant with time in accordance with (2.1)–(2.5), and the disturbance $\psi' = \psi - \bar{\psi}$ automatically satisfies a linear equation and the corresponding boundary conditions

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + Bv' = 0, \tag{2.6}$$

$$q' \equiv \left[\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial \zeta}\left(\frac{\zeta^2}{C^2}\frac{\partial}{\partial \zeta}\right)\right]\psi', \tag{2.7}$$

$$\left(\frac{\partial}{\partial t} + \bar{u}_s\frac{\partial}{\partial x}\right)(T'_s + \alpha_s \psi'_s) + S\frac{\partial \psi'_s}{\partial x} = 0, \quad (\zeta = 1) \tag{2.8}$$

$$\frac{1}{2}\int_0^1 \left[\left(\frac{\partial \psi'}{\partial x}\right)^2 + \left(\frac{\zeta}{C}\frac{\partial \psi'}{\partial \zeta}\right)^2\right] d\zeta < \infty, \tag{2.9}$$

where the subscript s denotes quantities at $\zeta = 1$, $T' \equiv -\zeta\partial\psi'/\partial\zeta$, and B and S are the mean meridional gradient of potential vorticity and modified meridional gradient of surface temperature, respectively, i.e.

$$B \equiv \frac{\partial \bar{q}}{\partial y} = \beta - \frac{\partial}{\partial \zeta}\left(\frac{\zeta^2}{C^2}\frac{\partial \bar{u}}{\partial \zeta}\right),$$

$$S \equiv \frac{\partial \bar{T}_s}{\partial y} + \alpha_s \bar{u}_s = \frac{\partial}{\partial y}(\bar{T}_s - \alpha_s \bar{\psi}_s). \tag{2.10}$$

From (2.6)–(2.9) one can easily obtain the energy integral and the modified enstrophy integral

$$\frac{\partial E'}{\partial t} \equiv \frac{\partial}{\partial t}\left\{\int_{-l}^{+l}\int_0^1 \frac{1}{2}\left[\left(\frac{\partial \psi'}{\partial x}\right)^2 + \left(\frac{\zeta}{C}\frac{\partial \psi'}{\partial \zeta}\right)^2\right] d\zeta\, dx + \int_{-l}^{+l} \frac{\alpha_s}{2C_s^2}(\psi'_s)^2\, dx\right\}$$
$$= -\int_{-l}^{+l}\int_0^1 v'\,T'\frac{\partial \bar{T}}{C^2 \partial y}\, d\zeta\, dx, \tag{2.11}$$

$$\int_{-l}^{+l}\int_0^1 \frac{1}{2B}\frac{\partial q'^2}{\partial t}\, d\zeta dx = \int_{-l}^{+l} C_s^{-2} v'_s T'_s\, dx, \tag{2.12}$$

where the assumption has been made that ψ' is periodic along x with period $2l$ or has finite total energy as $l \to \infty$. E' is the total eddy energy consisting of kinetic, $(\partial\psi'/\partial x)^2/2$, available potential, $(\zeta\partial\psi'/\partial\zeta)^2/2$, and surface potential, $\alpha_s(\psi'_s/C_s)^2/2$.

From (2.12) it is concluded that in contrast to the barotropic case, a baroclinic disturbance does not conserve its modified enstrophy because of the meridional heat flux at the surface $\zeta = 1$. This makes the instability problem more complicated.

Besides, one can also obtain another integral

$$\int_{-l}^{+l}\int_0^1 \frac{\bar{u}}{B}\frac{\partial}{\partial t}\left(\frac{q'^2}{2}\right) d\zeta\, dx$$
$$= -\int_{-l}^{+l}\int_0^1 v'\,T'\frac{\partial \bar{T}}{C^2\,\partial y}\, d\zeta\, dx + \int_{-l}^{+l} \bar{u}_s C_s^{-2} v'_s T'_s\, dx, \tag{2.13}$$

and the surface energy integral

$$\frac{\partial}{\partial t}\int_{-l}^{+l} \frac{1}{2}(T'_s - \alpha_s \psi'_s)^2\, dx = -\int_{-l}^{+l} S v'_s T'_s\, dx. \tag{2.14}$$

Combining (2.11), (2.12), (2.13) and (2.14), we finally obtain the following three invariant integrals

$$\int_{-l}^{+l}\int_0^1 \left(\frac{C_s^2 S}{B}\right)\frac{\partial q'^2}{\partial t}\, d\zeta\, dx + \int_{-l}^{+l} \frac{\partial}{\partial t}(T'_s - \alpha_s \psi'_s)^2\, dx = 0, \tag{2.15}$$

$$SC_s^2\left[\frac{\partial E'}{\partial t}+\int_{-l}^{+l}\int_0^1\left(\frac{-\bar{u}}{2B}\right)\frac{\partial q'^2}{\partial t}\,d\zeta\,dx\right]+\int_{-l}^{+l}\left(\frac{-\bar{u}_s}{2}\right)\frac{\partial}{\partial t}(T'_s-\alpha_s\psi'_s)^2\,dx=0, \tag{2.16}$$

$$\frac{\partial E'}{\partial t}+\int_{-l}^{+l}\int_0^1\frac{-(\bar{u}-\bar{u}_s)}{2B}\frac{\partial q'^2}{\partial t}\,d\zeta\,dx=0. \tag{2.17}$$

Consequently, if the basic current does not vary with time, and $B \neq 0$, we have three invariants

$$\mathscr{E}_1\equiv\int_{-l}^{+l}dx\left[\int_0^1\frac{C_s^2S}{B}q'^2\,d\zeta+(T'_s-\alpha_s\psi'_s)^2\right]=\text{const.}, \tag{2.18}$$

$$\mathscr{E}_2\equiv\int_{-l}^{+l}dx\left[\int_0^1 C_s^2S\left\{\left(\frac{\partial\psi'}{\partial x}\right)^2+\left(\frac{\zeta}{C}\frac{\partial\psi'}{\partial\zeta}\right)^2+\frac{-\bar{u}}{B}q'^2\right\}d\zeta+S\alpha_s\psi_s'^2+(-\bar{u}_s)(T'_s-\alpha_s\psi'_s)^2\right]=\text{const.}, \tag{2.19}$$

$$\mathscr{E}_3\equiv E'+Q'=\text{const.}, \tag{2.20}$$

where

$$Q'\equiv\int_{-l}^{+l}\int_0^1\frac{-(\bar{u}-\bar{u}_s)}{2B}q'^2\,d\zeta\,dx. \tag{2.21}$$

It is clear, that the modified enstrophy $q'^2/2B$ or weighted modified enstrophy $(\bar{u}_r-\bar{u}_r)q'^2/2B$ is more important than the enstrophy itself.

Two invariants, similar to $\mathscr{E}_1$ and $\mathscr{E}_2$, have also been obtained by Blumen (1978) but only for $S \neq 0$. The third integral (2.17) and the invariant (2.20) are first obtained in our paper, and are more general, especially in that they can be used even if $S = 0$.

It must be pointed out that if $B = 0$ at some points, the integrals $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ all might not exist; hence they cannot be used for instability analysis. However, we can prove, that the integrals (2.15)–(2.17) are still valid. In fact, if q' and $v'T'/C^2$ are continuous functions, but $B(\zeta_c)=0$ (we assume that there is only one singular point for simplicity), we have, for example, the following integral

$$\int_{-l}^{+l}dx\left\{\left(\int_0^{\zeta_c-\delta}+\int_{\zeta_c+\delta}^1\right)\frac{1}{2B}\frac{\partial q'^2}{\partial t}\,d\zeta\right\}=\int_{-l}^{+l}\left[\frac{v'_sT'_s}{C_s^2}+\left(\frac{v'T'}{C^2}\right)\Big|_{\zeta_c+\delta}^{\zeta_c-\delta}\right]dx,$$

where $\delta > 0$ is small. Letting $\delta \to 0$, we come to (2.12). It is obvious that the right-hand side of (2.12) is bounded because of the continuity of $v'_sT'_s/C_s^2$.

Among (2.15)–(2.17), there are only two independent integrals. The same is true for $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$. In fact, if $S \neq 0$, (2.16) can easily be obtained by a linear combination of (2.15) and (2.17), while if $S = 0$, (2.15) and (2.16) degenerate into the same invariant integral—conservation of surface energy

$$\frac{\partial}{\partial t}\int_{-l}^{+l}\frac{1}{2}(T'_s-\alpha_s\psi')^2\,dx=0. \tag{2.22}$$

Besides, if $T'_s \propto \psi'_s$ is taken instead of (2.8), i.e., the bottom boundary is an isentropic surface, (2.17) is still valid, but (2.15) is replaced by

$$\int_{-l}^{+l}\int_0^1\frac{1}{2B}\frac{\partial q'^2}{\partial t}\,d\zeta\,dx=0. \tag{2.23}$$

The linear combination of (2.17) and (2.23) yields

$$\frac{\partial E'}{\partial t}+\int_{-l}^{+l}\int_0^1\frac{-(\bar{u}-\bar{u}_r)}{2B}\frac{\partial q'^2}{\partial t}\,d\zeta\,dx=0. \tag{2.24}$$

This integral differs from (2.17) only in that $\bar{u}_s$ is replaced by an arbitrary reference wind speed $\bar{u}_r$. It must be pointed out, that if $S \neq 0$, the linear combination of (2.15) and (2.16) yields a similar formula but with a boundary term:

$$\frac{\partial E'}{\partial t}+\int_{-l}^{+l}\int_0^1\frac{-(\bar{u}-\bar{u}_r)}{2B}\frac{\partial q'^2}{\partial t}\,d\zeta\,dx+\int_{-l}^{+l}\frac{-(\bar{u}_s-\bar{u}_r)}{2S}\frac{\partial}{\partial t}(T'_s-\alpha_s\psi'_s)^2\,dx=0. \tag{2.25}$$

3. Instability

We take the following definition of (linear) instability.

A disturbance ψ' is called stable, if its energy E' is always bounded; otherwise, it is called unstable.

From (2.17) we obtain that a sufficient condition for stability is

$$\frac{-(\bar{u}-\bar{u}_s)}{\partial\bar{q}/\partial y}\geqslant 0 \tag{3.1}$$

everywhere, and that the necessary condition for instability is

$$\frac{-(\bar{u}-\bar{u}_s)}{\partial\bar{q}/\partial y}<0 \qquad (3.2)$$

in some interval $0 \leqslant \zeta_1 < \zeta < \zeta_2 \leqslant 1$.

These criteria are valid for all cases, independent of whether $S = 0$ or whether the bottom boundary condition is chosen as (2.8) or as $T'_s \propto \psi'_s$. The wind speed $\bar{u}_s$ at the bottom boundary comes into these criteria as a reference speed. As an example of an application, we refer to Charney's case (1947), where $\bar{u}$ is a linear increasing function of height; hence the necessary condition for instability, (3.2), is always satisfied, and some typical unstable baroclinic waves indeed exist.

However, if $S \neq 0$, then from (2.25) we obtain additional, and more complicated conditions. They are stated as follws.

(a) If there exists at least one constant $\bar{u}_r$ such that

$$\frac{-(\bar{u}-\bar{u}_r)}{\partial\bar{q}/\partial y}\geqslant 0 \qquad (3.3)$$

everywhere, and

$$\frac{-(\bar{u}_s-\bar{u}_r)}{S}\geqslant 0, \qquad (3.4)$$

all disturbances are stable;

(b) The necessary condition for instability is that both inequalities (3.3) and (3.4) cannot be satisfied simultaneously for any constant $\bar{u}_r$; for example, there exists an interval $\Delta\zeta$, depending on $\bar{u}_r$, in which

$$\frac{-(\bar{u}-\bar{u}_r)}{\partial\bar{q}/\partial y}<0, \qquad (3.5)$$

or

$$-(\bar{u}_s-\bar{u}_r)/S<0. \qquad (3.6)$$

If the bottom boundary is taken as an isentropic surface, we get the same conditions (3.3) and (3.5) but without (3.4) and (3.6).

Condition (b), or (3.5) + (3.6), is the generalized condition for instability of an internal jet (Charney and Stern, 1962; Pedlosky, 1964; Blumen, 1968, 1978).

When condition (3.1) is satisfied, condition (a) is satisifed also. However, when condition (a) is satisfied, (3.1) may not be satisfied. The reverse is true when (3.2) and condition (b) are considered. In fact, condition (a) becomes (3.1) by taking $\bar{u}_r = \bar{u}_s$, while, if $\bar{u} > 0$ and is bounded everywhere, and if $\partial\bar{q}/\partial y > 0$, then (3.3) is satisfied by taking $\bar{u}_r = \bar{u}_{max}$, so that the internal jet is stable, even though (3.1) is not satisfied.

Usually S is negative, and it is difficult to satisfy (3.3) and (3.4) simultaneously. The bottom boundary condition very much influences the evolution of low-level distrubances and makes them unstable, except in some special cases.

4. Development

According to the definition of stability, a significant change of energy could still occur even if the distrubance is stable. In fact, according to (2.20), $E' + Q'$ is an invariant, but E' and Q' can both change with time. In order to investigate the gross properties of disturbances during their evolution, another useful definition can be taken as follows:

$$\begin{array}{ll}\partial E'/\partial t>0 & \text{developing;}\\ \partial E'/\partial t<0 & \text{decaying.}\end{array} \qquad (4.1)$$

Of course, an unstable disturbance must be developing, but the reversal is not true. Probably, the definition (4.1) is too wide and is connected only with the tendency of development. But, on the other hand, it is more convenient for analysis and clearly shows the relationship between development characteristics and the structure of a disturbance.

From (2.11), one can conclude, that the disturbance is developing (decaying), if the heat flux, $C^{-2}v'T'\,\mathrm{d}\bar{T}/\mathrm{d}y$, integrated over the whole atmosphere, is negative (positive). In other words, when the meridional heat flux is down-gradient, the disturbance develops; when the meridional heat flux is up-gradient, the disturbance decays. Suppose that the basic current is jet-like westerlies, and for simplicity there is a single jet at ζ^*, then the favorable axis for a developing disturbance is tilted westward below the jet, $\zeta > \zeta^*$, and eastward above the jet, $\zeta < \zeta^*$; and the reverse is true for the decaying disturbance. These are the common characteristics, no matter whether the instability conditions is satisfied or not.

The difference between the unstable disturbance and the stable, but developing one, is that the latter's energy is always bounded, probably reaches its maximum at some moment, and then decays.

If $B \neq 0$, $\mathscr{E}_3$ is an invariant, and if (3.1) is also satisfied, then the total weighted modified enstrophy Q' decreases (increases) as the total energy E' increases (decreases), and vice versa. Because $-(\bar{u} - \bar{u}_s)/B > 0$, we can determine a weighted average (non-dimensional) scale l_w as follows:

$$\left(\frac{l_w}{l_\beta}\right)^2 = \frac{E'}{Q'}, \tag{4.2}$$

where

$$l_\beta \equiv \left(\int_0^1 \frac{-(\bar{u} - \bar{u}_s)}{\partial \bar{q}/\partial y} \, d\zeta\right)^{1/2} \tag{4.3}$$

is a constant (non-dimensional length) for a given profile $\bar{u}$, and is the generalized stationary Rossby wave length in a baroclinic atmosphere. Thus, the weighted average scale l_w increases (decreases) during the period of growth (decay) due to the increase (decrease) of E'/Q'. This conclusion is valid under the conditions mentioned above. However, if the disturbance is located at a high level, and the boundary term in (2.25) is small, then (3.1) can be replaced by (3.3). Thus this conclusion is still approximately valid for $\bar{u}_s$ replaced by $\bar{u}_r$ in Q' and l_β. This is much more general and useful for qualitative analysis. The increase of l_w can be achieved either by a spreading of the disturbance or by a moving of the disturbance from a region with larger $-(\bar{u} - \bar{u}_r)/B$ to a region with smaller $-(\bar{u} - \bar{u}_r)/B$. As can be seen in the following sections, the former always takes place, being accompanied by the latter in some cases. Besides, in the unstable case, $B = 0$ at some points and $\mathscr{E}_3$ might no longer exist; however, the mean scale of the developing disturbance still increases.

5. Evolution of an individual disturbance

Hereafter, we investigate the evolutional process of an individual upper-level disturbance.

It is convenient to introduce a new vertical coordinate ξ and a new variable $\hat{\psi}$ as follows

$$\xi = \int_\zeta^1 \sqrt{\gamma(\zeta')} \, d\zeta', \qquad \gamma \equiv C^2/\zeta^2, \tag{5.1}$$

$$\psi' = W(\xi) \, \hat{\psi}(x,\xi,t), \tag{5.2}$$

where

$$W(\xi) = \left(\frac{\gamma}{\gamma_s}\right)^{1/4} = \left(\frac{C}{\zeta C_s}\right)^{1/2}. \tag{5.3}$$

$\xi = \log p_s/p$, and $W = \sqrt{P_s/P}$ if C is a constant. By using (5.1)–(5.3), the quasi-geostrophic potential vorticity equation (2.6) is transformed into the following

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\hat{q} + B\frac{\partial \hat{\psi}}{\partial x} = 0, \tag{5.4}$$

where

$$\hat{q} \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2} - H(\xi)\right)\hat{\psi}, \tag{5.5}$$

$$H(\xi) \equiv W\frac{d^2}{d\xi^2}\left(\frac{1}{W}\right) = \frac{1}{4}\left(\frac{d \ln \sqrt{\gamma}}{d\xi}\right)^2 - \frac{1}{2}\frac{d^2}{d\xi^2}\ln\sqrt{\gamma}. \tag{5.6}$$

Note that $1/H$ is the non-dimensional scaling height, and $H = \frac{1}{4}$ if C = constant. The boundary conditions should also be transformed; for example, from (2.9) we have

$$\frac{1}{\sqrt{\gamma_s}} \int_0^\infty \left[\left(\frac{\partial \hat{\psi}}{\partial x}\right)^2 + \left(\frac{\partial \hat{\psi}}{\partial \xi} + \frac{\hat{\psi}}{2}\frac{d \ln \sqrt{\gamma}}{d\xi}\right)^2\right] \times d\xi < \infty. \tag{5.7}$$

Because eq. (5.4) and the corresponding boundary conditions are linear, one can consider the disturbances in the whole atmosphere to be a superposition of several individual disturbances, every one of which can be represented as a wave-packet. Thus, we will naturally use the WKB method to investigate the evolution of an individual disturbance.

Here, we restrict ourselves to an investigation of the upper-level disturbances far away from the bottom boundary. Hence, the influence of the bottom boundary condition can be neglected. This is the case for the upper tropospheric and stratospheric disturbances. However, this assumption can be approximately applicable to the middle-low tropospheric disturbances, except those at very low level, which are significantly affected by the bottom boundary condition. Besides, we suppose, that $H > 0$, and that the basic current $\bar{u}$ and functions B and H are all slowly varying functions of ξ.

Because the WKB method is valid for the relatively short waves, we introduce a small parameter ε, a slow time T and enlarged spatial coordinates X and Z as follows

$$X = \varepsilon x, \qquad Z = \varepsilon \xi, \qquad T = \varepsilon t, \tag{5.8}$$

and represent the disturbance in the form of a wave packet

$$\hat{\psi} = \Psi(X,Z,T)\, e^{i\theta(X,Z,T)/\varepsilon}, \tag{5.9}$$

$$\Psi(X,Z,T) = \Psi_0(X,Z,T) + \varepsilon \Psi_1(X,Z,T) + \dots, \tag{5.10}$$

where Ψ_j ($j = 0, 1, 2, \dots$) vanish outside some interval in Z, and θ, Ψ_j and their derivatives of all orders with respect to the arguments (X,Z,T) are assumed to be of O(1).

Rewriting (5.4) in the following form

$$\left(\frac{\partial}{\partial T} + \bar{u}\frac{\partial}{\partial X}\right)\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2} - \varepsilon^{-2} H\right) \times \hat{\psi} + \varepsilon^{-2} B \frac{\partial \hat{\psi}}{\partial X} = 0, \tag{5.11}$$

substituting (5.9) and (5.10) into (5.11) and denoting the local and instantaneous frequency and wave numbers by σ, m and k, respectively, i.e.,

$$\frac{\partial \theta}{\partial T} = -\sigma, \qquad \frac{\partial \theta}{\partial X} = m, \qquad \frac{\partial \theta}{\partial Z} = k, \tag{5.12}$$

the terms of O(ε^{-3}) yield

$$(\sigma - mu)\, \nu^2 + Bm = 0; \tag{5.13}$$

the terms of O(ε^{-2}) yield

$$\left(\frac{\partial}{\partial T} + \bar{u}\frac{\partial}{\partial X}\right)\nu^2 \Psi_0 - (\sigma - m\bar{u}) \times \left[2\left(m\frac{\partial}{\partial X} + k\frac{\partial}{\partial Z}\right)\Psi_0 + \Psi_0\left(\frac{\partial m}{\partial X} + \frac{\partial k}{\partial Z}\right)\right] - B\frac{\partial \Psi_0}{\partial X} = 0, \tag{5.14}$$

where

$$\nu^2 \equiv m^2 + k^2 + H, \tag{5.15}$$

and the terms of higher order yield the equations for Ψ_j ($j = 1, 2, \dots$).

Eq. (5.13) is the dispersion relation, which gives the local and instantaneous group velocity $\vec{C}_g = \vec{i}C_{gx} + \vec{K}C_{gz}$. We have

$$\left(\frac{\partial \sigma}{\partial m} - \bar{u}\right)\nu^2 + B = -2m(\sigma - m\bar{u}), \tag{5.16}$$

$$\frac{\partial \sigma}{\partial k}\nu^2 = -2k(\sigma - m\bar{u}); \tag{5.17}$$

thus

$$C_{gx} = \frac{\partial \sigma}{\partial m} = \bar{u} - \frac{B}{\nu^4}(\nu^2 - 2m^2),$$

$$C_{gz} = \frac{\partial \sigma}{\partial k} = \frac{2mk}{\nu^4} B. \tag{5.18}$$

σ is considered to be a function of m, k, $\bar{u}$, B and H during differentiation in (5.17).

The evolution of Ψ_0 is described by (5.14), in which there are also the unknowns m, k, and σ; we thus need more equations to close the set of equations.

6. The variation of wavelength and the ray

From the very definitions (5.12) and $\vec{C}_g$, and by using formulas (5.13) we obtain

$$D_g \sigma/DT = -m\left(\frac{\partial \bar{u}}{\partial T} - \frac{1}{\nu^2}\frac{\partial B}{\partial T}\right), \tag{6.1}$$

$$D_g m/DT = 0, \tag{6.2}$$

$$D_g k/DT = -m\left(\frac{\partial \bar{u}}{\partial Z} + \frac{B}{\nu^4}\frac{\partial H}{\partial Z} - \frac{1}{\nu^2}\frac{\partial B}{\partial Z}\right), \tag{6.3}$$

where

$$D_g/DT \equiv \frac{\partial}{\partial T} + \vec{C}_g \cdot \nabla, \tag{6.4}$$

$$\nabla \equiv \vec{i}\frac{\partial}{\partial X} + k\frac{\partial}{\partial Z}. \tag{6.5}$$

Eqs. (5.13) and (5.14), together with (6.2) and (6.3), constitute the required closed set of equations which describe the evolution of the disturbance to the first approximation. Rather than solve this set of equations numerically, we now make an overall qualitative analysis of them.

According to (6.2), the individual derivative of m following the wave packet, or along a ray, $\mathrm{D_g}m/\mathrm{D}T$, is zero, due to the independence of $\bar{u}$, B and H on X; thus, the variation of m is caused by the initial spatial distribution of m, advected at a velocity $\vec{C}_g$. On the other hand, the vertical wave number k changes with time due to the dependence of $\bar{u}$, B and H on Z. Moreover, from (6.2) and (6.3) we obtain

$$\mathrm{D_g}(m^2+k^2)/\mathrm{D}T = \mathrm{D_g}\,k^2/\mathrm{D}T$$

$$= -2km\left[\frac{\partial\bar{u}}{\partial Z} + \frac{B}{\nu^4}\frac{\partial H}{\partial Z} - \frac{1}{\nu^2}\frac{\partial B}{\partial Z}\right], \tag{6.6}$$

$$\mathrm{D_g}(k/m)/\mathrm{D}T = -\left(\frac{\partial\bar{u}}{\partial Z} + \frac{B}{\nu^4}\frac{\partial H}{\partial Z} - \frac{1}{\nu^2}\frac{\partial B}{\partial Z}\right). \tag{6.7}$$

Since the local wave length is $2\pi/\sqrt{m^2+n^2}$ along the ray, and the angle made by the vertical (Z-axis) and the axis of disturbance (trough or ridge line, which is perpendicular to the ray) is given by $\tan^{-1}k/m$, eqs. (6.6) and (6.7) thus describe the time variation of local wave length and the orientation of the disturbance.

According to (2.10), we have

$$\frac{\partial B}{\partial Z} = -\varepsilon^2\frac{\partial}{\partial Z}\left(\frac{\partial^2\bar{u}}{\partial Z^2} - \frac{\partial\ln\sqrt{\gamma}}{\partial Z}\frac{\partial\bar{u}}{\partial Z}\right).$$

Thus, $\partial B/\partial Z$ is usually small, and could be of O(1) only if the higher order derivatives of $\bar{u}$ were much larger than O(1), i.e., in the case of an extremely narrow jet. Besides, $\partial H/\partial Z$ is always small outside the vicinity of the tropopause, and even $\partial H/\partial Z = 0$ if $C^2 =$ constant. Therefore, outside the jet axis, where $\partial\bar{u}/\partial Z \neq 0$, the first term on the right-hand side of (6.6) and (6.7) is dominant, if the jet is not too narrow. This indicates that the change of vertical wave number, the total wave length and the slope of the axis of the disturbance are primarily caused by the vertical shear of zonal wind, $\partial\bar{u}/\partial Z$. Without loss of generality, we can take $m > 0$; hence $k > 0$ corresponds to a trough line tilted westward, and $k < 0$ corresponds to that tilted eastward. According to (6.6) and (6.7) we come to the conclusion, that below the jet, where $\partial\bar{u}/\partial Z > 0$, both the vertical wave length, λ_Z, and the total wave length, λ, increase with time, and the trough line tends to be vertical, if $k > 0$, whereas, λ_Z and λ decrease, and the trough line tends to be more tilted, if $k < 0$. Above the jet, where $\partial\bar{u}/\partial Z < 0$, the opposite holds.

In the general case, including when $\partial H/\partial Z$ and $\partial B/\partial Z$ are not small, by using (5.16) we can transform (6.6) into a more compact form

$$B\mathrm{D_g}(\nu^2/B)/\mathrm{D}T = -2km\frac{\partial\bar{u}}{\partial Z}. \tag{6.8}$$

7. The variation of amplitude, energy and other integral properties

The original form of (5.14) is not convenient for analysis. We need further transformation. Substituting (5.17) into (5.14) yields

$$\nu^2\mathrm{D_g}\,\Psi_0/\mathrm{D}T + \Psi_0\left(\frac{\partial\nu^2}{\partial T} + \bar{u}\frac{\partial\nu^2}{\partial X}\right) - (\sigma - m\bar{u}) \times\left(\frac{\partial m}{\partial X} + \frac{\partial k}{\partial Z}\right)\Psi_0 = 0. \tag{7.1}$$

Differentiating (5.17) with respect to X and Z yields

$$\nabla\cdot(\nu^2\vec{C}_g) = \bar{u}\frac{\partial\nu^2}{\partial X} - 2(\sigma - m\bar{u})\left(\frac{\partial m}{\partial X} + \frac{\partial k}{\partial Z}\right) -2\left(m\frac{\partial}{\partial X} + k\frac{\partial}{\partial Z}\right)(\sigma - m\bar{u}). \tag{7.2}$$

According to the very definition of σ, m and k in (5.12), we have

$$-2\left(m\frac{\partial}{\partial X} + k\frac{\partial}{\partial Z}\right)\sigma = \frac{\partial}{\partial T}(m^2+k^2) = \frac{\partial\nu^2}{\partial T},$$

$$2\left(m\frac{\partial}{\partial X} + k\frac{\partial}{\partial Z}\right)m\bar{u} = \bar{u}\frac{\partial\nu^2}{\partial X} + 2km\frac{\partial\bar{u}}{\partial Z}.$$

Substituting these equations into (7.2) yields

$$\nabla\cdot(\nu^2\vec{C}_g) = 2\left(\frac{\partial}{\partial T} + \bar{u}\frac{\partial}{\partial X}\right)\nu^2 - 2(\sigma - m\bar{u}) \times\left(\frac{\partial m}{\partial X} + \frac{\partial k}{\partial Z}\right) + 2km\frac{\partial\bar{u}}{\partial Z} - \frac{\partial\nu^2}{\partial T}.$$

Then, substituting this formula into (7.1) yields the useful formula

$$\nu^2 \mathrm{D_g}\, \Psi_0/\mathrm{D}T + \frac{\Psi_0}{2} \mathrm{D_g}\, \nu^2/\mathrm{D}T + \frac{\nu^2 \Psi_0}{2} \nabla \cdot \vec{C}_g - \Psi_0 km \frac{\partial \bar{u}}{\partial Z} = 0. \tag{7.3}$$

Introducing

$$\Psi_0 = |\Psi_0|\, e^{i\alpha(X,Z,T)}, \tag{7.4}$$

the imaginary and real parts of (7.3) yield

$$\mathrm{D_g}\, \alpha/\mathrm{D}T = 0, \tag{7.5}$$

$$\nu^2 \mathrm{D_g} |\Psi_0|/\mathrm{D}T + \frac{|\Psi_0|}{2} \mathrm{D_g}\, \nu^2/\mathrm{D}T + \frac{\nu^2 |\Psi_0|}{2} \nabla \cdot \vec{C}_g - |\Psi_0|\, km \frac{\partial \bar{u}}{\partial Z} = 0, \tag{7.6}$$

respectively.

These equations indicate that the phase of the envelope, α, propagates purely along $\vec{C}_g$, whereas, the intensity of the envelope $|\Psi_0|$ undergoes some additional changes (development) besides its propagation along $\vec{C}_g$.

Multiplying (7.6) by $|\Psi_0|$, and then integrating over the whole wave packet denoted by S, we finally obtain the following equation:

$$\frac{\partial}{\partial T} \int\int_S \frac{1}{2} \nu^2 |\Psi_0|^2\, \mathrm{d}X\mathrm{d}Z = \int\int_S \frac{|\Psi_0|^2}{2} \times 2mk \frac{\partial \bar{u}}{\partial Z} \mathrm{d}X\mathrm{d}Z. \tag{7.7}$$

Substituting (6.8) into (7.7) yields

$$\frac{\partial}{\partial T} \int\int_S \frac{1}{2} \nu^2 |\Psi_0|^2\, \mathrm{d}X\mathrm{d}Z = -\int\int_S \frac{B|\Psi_0|^2}{2} \times \mathrm{D_g}(\nu^2/B)/\mathrm{D}T\mathrm{d}X\mathrm{d}Z. \tag{7.8}$$

Multiplying (7.6) by $\nu^2 |\Psi_0|$, using (6.8) and after some elementary calculations we obtain

$$\frac{1}{B} \frac{\partial}{\partial T} \left(\frac{\nu^4 |\Psi_0|^2}{2} \right) + \nabla \cdot \left(\frac{\nu^4 |\Psi_0|^2}{2B} \vec{C}_g \right) = 0. \tag{7.9}$$

Consequently, we have an invariant integral

$$\int\int_S \frac{1}{B} \frac{\partial}{\partial T} (\nu^4 |\Psi_0|^2)\, \mathrm{d}X\mathrm{d}Z = 0. \tag{7.10}$$

Besides, multiplying (7.9) by $\bar{u}$, we obtain

$$\frac{\bar{u}}{B} \frac{\partial}{\partial T} (\nu^4 |\Psi_0|^2) + \nabla \cdot \left(\frac{\bar{u}}{B} \nu^4 |\Psi_0|^2 \vec{C}_g \right) = -|\Psi_0|^2\, 2km \frac{\partial \bar{u}}{\partial Z}.$$

Consequently,

$$\int\int_S \left[\frac{-\bar{u}}{B} \frac{\partial}{\partial T} (\nu^4 |\Psi_0|^2) + \frac{\partial}{\partial T} (\nu^2 |\Psi_0|^2) \right] \mathrm{d}X\mathrm{d}Z = 0. \tag{7.11}$$

Adding (7.11) to (7.10) multiplied by an arbitrary constant $\bar{u}_r$ gives the other invariant integral

$$\int\int_S \left[\frac{\partial}{\partial T} (\nu^2 |\Psi_0|^2) + \frac{-(\bar{u} - \bar{u}_r)}{B} \frac{\partial}{\partial T} (\nu^4 |\Psi_0|^2) \right] \times \mathrm{d}X\mathrm{d}Z = 0. \tag{7.12}$$

It is not difficult to prove that the intergrals (7.8)–(7.12) are still valid if $B = 0$ at some points, provided Ψ_0 and ν^2 are bounded and continuous functions. In fact, for example, if there is one singular point Z_c, i.e., $B(Z_c) = 0$, integrating (7.9) over the whole area but without a small area $Z_c - \delta \leqslant Z \leqslant Z_c + \delta$ yields

$$\int\int_{S-S_\delta} \frac{1}{B} \frac{\partial}{\partial T} (\nu^4 |\Psi_0|^2)\, \mathrm{d}X\mathrm{d}Z = -\int_{(X)} \mathrm{d}X \left[-\frac{\nu^4 |\Psi_0|^2}{B} C_{gZ} \right] \Bigg|_{Z_c-\delta}^{Z_c+\delta},$$

where S_δ denotes the excluded area, and $\delta > 0$. Let $\delta \to 0$; the right-hand side tends to zero due to the continuity of $\nu^4 |\Psi_0|^2 C_{gZ} B^{-1} = -2km|\Psi_0|^2$. Thus (7.10) is valid.

$\nu^2 |\Psi_0|^2$ and $\nu^4 |\Psi_0|^2$ are just the analogues of energy and enstrophy densities, respectively, and (7.7), (7.10) and (7.12) are the analogues of (2.11), (2.12), and (2.25), respectively. In other words, (7.7), (7.10) and (7.12) are the energy equation, the invariant integral of modified enstrophy and the invariant integral of weighted modified enstrophy-energy for the whole wave packet, respectively. Now all the surface integrals with the bottom boundary condition automatically disappear due to the vanishing of Ψ_0 outside the wave packet. Thus, we come to the similar but simpler conclusions;

now they are valid for the individual disturbance, so that they are more useful. For example, from (7.12) we obtain the same necessary condition for instability, but without (3.6).

It must be pointed out that an individual disturbance can be unstable only if the instability condition is satisfied locally in the vertical interval occupied by it. Thus, the instability is first realized locally in the vertical, then spreads out into more levels and devolves into the structure of an unstable normal mode. For example, supposing $\bar{u}$ to be a linear increasing function of Z with $B > 0$ (the Charney case), the sufficient conditions (3.3) and (3.4) for stability cannot be satisfied in the whole atmosphere. However, (3.3) is satisfied in every finite vertical interval $0 < Z_1 < Z < Z_2 < \infty$; thus, every individual upper-level disturbance, free from the influence of bottom boundary, is stable. Therefore, the instability can first be realized only near the bottom boundary. However, in cases of a jet-like profile of zonal current $\bar{u}$ with changing sign of B and satisfying (3.5) within the atmosphere, the instability can first be realized there, and the amplitude of unstable normal modes will be pronounced in the unstable layer and adjacent area.

Now, for the development of an individual disturbance, we define as follows:

$$\frac{\partial}{\partial T}\int\int_{s}\frac{1}{2}\nu^2|\Psi_0|^2\,\mathrm{d}X\mathrm{d}Z > 0, \qquad \text{developing,}$$

$$\frac{\partial}{\partial T}\int\int_{s}\frac{1}{2}\nu^2|\Psi_0|^2\,\mathrm{d}X\mathrm{d}Z < 0, \qquad \text{decaying.} \tag{7.13}$$

The same conclusions about the dependence of development on the structure of the disturbance and its relative location on the basic current are more easily obtained from (7.7), because the slope of the trough-ridge (phase) line is directly represented by m and k. The same conclusions about l_w defined by (4.2), but with $\bar{u}_r$ replacing $\bar{u}_s$ in Q' and l_β, are obtained from (7.12). Moreover, if $B = \beta + O(\varepsilon)$ or if $\partial B/\partial Z$ is small, from (7.8) or from (7.7) together with (6.6), we conclude that the developing (decaying) disturbance is accompanied by a negative (positive) mean $D_g\nu^2/DT$, i.e., by an increase (decrease) of mean vertical scale λ_Z as well as mean wavelength λ, where the word "mean" means "averaged over the whole wave packet". This improvement in the conclusion is not only more useful, but also more general. It is valid for both stable and unstable cases provided $\partial B/\partial Z$ is small.

Note, that according to (7.7), the change of total energy of the disturbances is proportional to $mk\partial\bar{u}/\partial Z$, so that shorter disturbances must be more rapidly modified by the mean flow due to their large $|mk|$.

8. Concluding remarks

The problem of development and instability in the (pure) baroclinic two-dimensional quasi-geostrophic model can be investigated by both the integral properties and the WKB method. The results obtained by these two methods are in agreement with each other.

The concept of "development" is more convenient for analysis. A disturbance is called developing (decaying) if its energy is increasing (decreasing). A developing disturbance located in the layer where the instability criteria is satisfied might grow continuously, and its energy then becomes unbounded. The instability seems to first grow locally in the region where the instability criteria is satisfied, and then to spread into more levels. If the sufficient condition for stability is satisfied, the energy of a developing disturbance is always bounded, and in most cases it decays after its growth, if there is not an adjoining layer of instability.

In a vertical shear zonal flow, all characteristics of disturbance such as total wave length, vertical wavelength, ray direction and amplitude change with time during the evolutional process.

The development of an individual upper-level disturbance depends only on its structure and relative location in the basic current. For a single jet-like westerly current, the trough line of developing (decaying) disturbance is tilted westward (eastward) below the jet, $Z < Z^*$, the reverse being true above the jet, $Z > Z^*$. The developing (decaying) disturbance enlarges (shortens) its vertical scale and total wave length, and gradually turns its axis so as to be less (more) tilted, provided $\partial B/\partial Z$ is small. This indicates, that a baroclinic developing disturbance will gradually become more barotropic.

If $\partial\bar{q}/\partial y > 0$ (stable case), by using the conservation of $\mathscr{E}_3$, one can determine a supplementary weighted mean scale $l_w = l_\beta(E'/Q')^{1/2}$ and investigate its variation. During the period of growth (decay), l_w increases (decreases), and the maximum amplitude of the disturbances moves toward (out from) the jet.

All of these results are obtained using the WKB method, but only some of them (the criterion for instability, the dependence of the development on the structure and the relative location of disturbances, and the variation of l_w) can be obtained from an analysis of integral properties.

The WKB method even yields more and detailed results in comparison with the equation of conservation of wave action. In fact, the wave action, $\nu^2|\Psi_0|^2/2(\sigma - m\bar{u})$, can be expressed by $-\nu^4|\Psi_0|^2/2mB$, since $\sigma - m\bar{u} = -mB/\nu^2$. Thus (7.9) is just the equation of conservation of wave action, and the conservation of total modified enstrophy, (7.10), is just the conservation of total wave action. However, as we have mentioned, (7.10) is still valid even if $B = 0$ at some levels, while the total wave action

$$\int \int_s \frac{\nu^4|\Psi_0|^2}{2mB} \, dX dZ$$

might not exist. Secondly, eq. (7.9) alone is unable to yield the integrals (7.7) and (7.12), or the formulas (6.6) and (6.7), and hence cannot yield all our conclusions.

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10. Appendix

The transport properties of the disturbance and the change of the basic current

In this pure barclinic model, the meridional transport of momentum $\overline{u'v'}$ automatically vanishes. Moreover, the meridional transport of potential vorticity is also equal to zero to an accuracy of $O(\varepsilon)$. In fact, we have

$$v' = -W|\Psi_0| \, m \sin\left(\frac{\theta}{\varepsilon} + \alpha\right) + O(\varepsilon),$$

$$q' = W|\Psi_0| \, \nu^2 \cos\left(\frac{\theta}{\varepsilon} + \alpha\right) + O(\varepsilon),$$

and

$$\int_{x_0-\Delta x/2}^{x_0+\Delta x/2} [-W^2|\Psi_0|^2 m\nu^2] \sin\left(\frac{\theta}{\varepsilon} + \alpha\right) \times \cos\left(\frac{\theta}{\varepsilon} + \alpha\right) dx = O(\varepsilon)\Delta x,$$

where $\Delta x = 2\pi/m(x_0, \zeta)$ and x_0 is an arbitrarily given point. Note, that the vertically integrated $\overline{v'q'}$ might be zero exactly, if $\overline{v_s' T_s'} = 0$ (see Held, 1975). However, the meridional heat transport exists, and we have

$$\frac{1}{C}\overline{v'T'} \equiv \frac{1}{2l}\int_{-l}^{+l} \frac{1}{C} v'T' \, dx = \frac{1}{4L\varepsilon}\left\{\int_{-L}^{+L} W^2|\Psi_0|^2 \, km \, dX + O(\varepsilon)\right\}, \tag{A1}$$

where $L = \varepsilon l$. The relationship between the heat transport and the development for an individual disturbance is the same as mentioned in Section 4. This indicates that in both stable and unstable cases, a developing disturbance causes down-gradient meridional heat transport, while a decaying disturbance causes up-gradient heat transport.

Now we turn to the vertical transports and the interaction between disturbances and the basic current. If the ageostrophic motion is taken into account, the basic current changes slowly with time due to the influence of disturbances, and we have the following non-dimensional equations

$$\frac{\partial \bar{u}}{\partial t} = R_0\left(\bar{v} - \bar{w}\frac{\partial \bar{u}}{\partial \zeta} + R_0\frac{\partial \overline{w' u_a'}}{\partial \zeta}\right), \tag{A2}$$

$$R_0^3 \frac{\partial \bar{v}}{\partial t} = -\bar{u} - \frac{\partial \bar{\phi}}{\partial y} - R_0^2 \left(\frac{\partial \overline{v' w'}}{\partial \zeta} + R_0^2 \bar{w} \frac{\partial \bar{v}}{\partial \zeta} \right), \tag{A3}$$

$$\frac{\partial \bar{T}}{\partial t} = R_0 \left(\frac{C^2}{\zeta} \bar{w} - \frac{\partial \overline{T' w'}}{\partial \zeta} - R_0 \bar{v} \frac{\partial \bar{T}}{\partial y} \right), \tag{A4}$$

$$\bar{w} = \zeta \bar{w}_s = k' C_s^{-2} \frac{\alpha_s}{R_0} \zeta \frac{\partial \bar{\phi}_s}{\partial t}, \tag{A5}$$

where the characteristic scales are the same as before except that for $\bar{v}$ and $\bar{w}$, the dimensional mean meridional circulation is $u_0 R_0^2 \bar{v}$, and $\mathrm{d}\bar{\zeta}/\mathrm{d}t = f R_0^2 \bar{w}$; $k' = 0$ refers to taking $\bar{w}_s = 0$ and $k' = 1$ refers to the vanishing of mean vertical velocity at the bottom surface. u_a' and $w' = (fR_0^2)^{-1}(\mathrm{d}\zeta/\mathrm{d}t)'$ are the ageostrophic motions of the disturbance which are determined by the following quasi-geostrophic equations

$$u_a' = -\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) v', \tag{A6}$$

$$w' = \frac{\zeta}{C^2} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T' + \frac{\zeta}{C^2} v' \frac{\partial \bar{T}}{\partial y}. \tag{A7}$$

The terms of $O(R_0^3)$ in (A3) can be neglected, and hence we have

$$\frac{\partial \bar{u}}{\partial t} = -R_0^2 \frac{\partial^2 \overline{v' w'}}{\partial \zeta \partial t}, \tag{A8}$$

$$\frac{\partial \bar{T}}{\partial t} = -R_0 \frac{\partial \overline{T' w'}}{\partial \zeta} + O(R_0^3), \tag{A9}$$

$$\bar{v} = -R_0 \frac{\partial}{\partial \zeta} \left[\overline{w' u_a'} + \frac{\partial \overline{v' w'}}{\partial t} \right], \tag{A10}$$

$$\bar{w} = \bar{w}_s = \frac{\partial \bar{\phi}_s}{\partial t} = 0, \tag{A11}$$

$$\frac{\partial \bar{\phi}}{\partial y} = -\left(\bar{u} + R_0^2 \frac{\partial \overline{v' w'}}{\partial \zeta} \right)_{t=0}. \tag{A12}$$

By using (A7) and the formulae for T' and v', we obtain

$$\overline{T' w'} = \frac{\varepsilon}{4L\sqrt{\gamma}} \int_{-L}^{+L} W^2 |\Psi_0|^2 mk \frac{\partial \bar{u}}{\partial Z} \,\mathrm{d}X + O(\varepsilon),$$

$$\overline{v' w'} = \frac{B}{8L\sqrt{\gamma}} \frac{\mathrm{d} \ln \sqrt{\gamma}}{\mathrm{d}\xi} \int_{-L}^{+L} \frac{m^2}{v^2} W^2 |\Psi_0|^2 \,\mathrm{d}X + O(\varepsilon),$$

so that $\partial \bar{T}/\partial T$ is of $O(R_0)$, and $\partial \bar{u}/\partial T$ is of $O(R_0^2)$ in accordance with (A8)–(A9). These indicate that the influence of pure baroclinic disturbances on the basic flow is very small.

As for the vertical energy flux, we have

$$-\overline{\psi' w'} = \frac{1}{4L\sqrt{\gamma}} \int_{-L}^{+L} W^2 |\Psi_0|^2 \frac{Bmk}{v^2} \,\mathrm{d}X + O(\varepsilon). \tag{A13}$$

Because $C_{gz} = 2Bmk/v^4$, the vertical energy flux and the vertical group velocity have the same sign. This is in agreement with the envelope $|\Psi_0|$ and the energy density $v^2 |\Psi_0|^2$ propagating along the group velocity. Supposing $B > 0$, we have $C_{gz} > 0$ if $mk > 0$, and $C_{gz} < 0$ if $mk < 0$. Thus, the envelope and energy of a stable developing (decaying) disturbance propagate toward (out from) the jet region. The same conclusion is true for vertical energy flux.

If $B(Z_c) = 0$, the energy of a developing disturbance can also propagate toward Z_c from both sides.

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