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SHORT CONTRIBUTION

Some additional properties of the spectral multiplication matrix

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In a recent paper on spectral multiplication, Merilees (1980) (hereafter referred to as M80) succeeded in deriving some interesting symmetry properties of the multiplication matrix P_{ljk}. However, if one looks at either Tables 1 or 2 in M80, one is struck by the number of elements of equal value, which seem to occur in an orderly fashion but which cannot be explained with the given properties. The purpose of this short contribution is to show that there exist additional properties satisfied by the P_{ljk} elements which greatly simplify their computation.

First, we found that, in addition to the property $P_{ljk} = P_{lkj}$ reported in M80, there is a complete symmetry of the P_{ljk} under any permutation of its indices, i.e.,

$$P_{ljk} = P_{jkl} = P_{klj} = P_{lkj} = P_{kjl} = P_{jlk}. \tag{1}$$

Moreover, we have $P_{ljk} = P_{-l-j-k}$.

The derivation of these properties is rather straightforward. Using eq. (8) of M80 for the expression for P_{ljk}, we rewrite it in the following way:

$$P_{ljk} = \frac{1}{N^2}\frac{(-)^{j-k+1}}{\sin\theta} \times \left\{(-)^{l-j}\frac{\sin^2 M\theta'}{\sin\theta'} - (-)^{l-k}\frac{\sin^2 M\theta''}{\sin\theta''}\right\}, \tag{2}$$

where

$$\theta = \frac{\pi}{N}(j-k), \quad \theta' = \frac{\pi}{N}(l-j), \quad \theta'' = \frac{\pi}{N}(l-k),$$

$N = 2M + 1$,

N, an odd integer.

From eq. (2), one sees that $P_{ljk} = P_{-l-j-k}$. To prove (1), it is sufficient to show that $P_{ljk} = P_{jlk}$. In other words, one has to show that

$$\frac{1}{\sin\theta}\left\{(-)^{j+1}\frac{\sin^2 M\theta'}{\sin\theta'} + (-)^{k}\frac{\sin^2 M\theta''}{\sin\theta''}\right\} = \frac{1}{\sin\theta''}\left\{(-)^{l}\frac{\sin^2 M\theta'}{\sin\theta'} + (-)^{k}\frac{\sin^2 M\theta}{\sin\theta}\right\}$$

or that

$$(-)^k[\sin^2 M\theta'' - \sin^2 M\theta]\sin\theta' + \sin^2 M\theta'[(-)^{j+1}\sin\theta'' + (-)^{l+1}\sin\theta] = 0.$$

Now, if $N = 2M + 1$ and X is an integer, we have in general that

$$\sin^2\frac{M\pi X}{N} = \tfrac{1}{2} + (-)^{X+1}\tfrac{1}{2}\cos\frac{\pi X}{N}.$$

Using this equality and the fact that $\theta + \theta' = \theta''$, the rest of the proof follows.

Because the P_{ljk} depend only on the difference between the indices (M80), it seems more convenient to redefine P_{ijk} as $B_{j-kl-jl-k}$. From eq. (1), we are then able to derive the following additional symmetry properties:

$$\begin{aligned} B_{j-kl-jl-k} &= B_{k-jj-lk-l} = B_{l-jk-lk-j} \\ &= B_{k-lj-kj-l} = B_{k-jl-kl-j} \\ &= B_{l-kj-lj-k} = B_{j-lk-jk-l}. \end{aligned}$$

In addition, we have $B_{j-kl-jl-k} = (-)\,B_{k-jl-jl-k}$.

The previous properties do not exhaust all the apparent symmetries which can be observed in Tables 1 or 2 of M80. We have found additional relations, owing to the periodic nature of the P_{ljk}, in

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one, two or all of its indices. These relations may be written as follows:

$$B_{j-kl-jl-k} = B_{\alpha N+j-k\,\beta N+l-j\,\gamma N+l-k}, \tag{4}$$

where α, β and γ may independently take any integer value (zero, positive or negative), and where $N = 2M + 1$ as defined before. These properties are simple to prove, using e.g.,

$$\frac{(-)^{\alpha N+X}}{\sin\frac{\pi}{N}(\alpha N + X)} = \frac{(-)^X}{\sin\frac{\pi}{N}X},$$

$$\sin^2\frac{M\pi}{N}(\alpha N + X) = \sin^2\frac{M\pi X}{N}$$

which are always true for any integer X.

It is not always possible to rewrite the expression (4) in terms of P_{ljk}, and even when it is possible, the result is not unique, as one can see from the following example.

From $B_{j-kl-jl-k} = B_{j-k\pm N+l-j\pm N+l-k}$, we obtain, amongst many other possible choices that

$$P_{ljk} = P_{\pm N+ljk} = P_{l\pm N+j\pm N+k}, \quad \text{etc.}$$

For this reason, we prefer to keep the much more general expression (4).

The properties derived here are very useful in calculating and tabulating the values of P_{ljk}. Indeed, of the 45 values reported in Table 1 of M80, only 12 are independent, while of the 153 values given in Table 2, only 33 are independent. One can also check that P_{03-1} for $M = 8$ in Table 2 should be -0.0367 since, according to (3), $P_{03-1} = P_{0-3-4} = P_{0-1-4} = B_{134}$. Furthermore, one can show that, for instance, with use of eq. (4), for $M = 4$, $P_{02-4} = P_{03-4} = P_{02-3} = B_{76-5}$.

In conclusion, we have seen that the spectral multiplication matrix elements P_{ljk} are fully symmetric under the permutation of their indices, and have periodicity symmetry properties. Such properties could have been suspected *a priori* from another form of spectral multiplication, e.g., that involving the triple product of spherical harmonics. The resulting matrix element expressed in terms of a $3-j$ symbol (Thiebaux, 1971) is highly symmetric under interchange of its indices. These symmetries are a consequence of the properties of the physical space and can be used to advantage for simplification and error checking.

REFERENCES

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