Tellus (1983), *35A*, 1–16

On low-order non-linear stochastic-dynamic systems

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(Manuscript received January 12; in final form April 6, 1982)

ABSTRACT

A low-order system of equations derived from the advective equation with forcing and dissipation is investigated. A stochastic-dynamic treatment of these equations based on the closure assumption of the neglect of third and higher moments is carried out. For this system, the steady states, including those created by the closure assumption, are determined, and their linear stability properties are found. It turns out that 5 steady states exist in addition to the 3 related to the deterministic equations. Among the 8 steady states, 3 are stable, 2 of them are the steady states coming from the deterministic problem.

The asymptotic solutions in a stochastic-dynamic sense are determined without any closure assumption. It is shown that the 3 stable steady states for the system with closure form a subset of the continuum of asymptotic states for the system without closure. Several examples showing the differences between the asymptotic solutions of the 2 systems are calculated.

Numerical experiments based on backward integrations in time show that the system with closure also has a limit cycle characterized by a closed curve in the mean values and in each point a degenerate uncertainty ellipse consisting of a line segment.

1. Introduction

The state of the atmosphere cannot be known in full detail at any time. Each initial state used for purposes of predictions of future states has therefore a certain uncertainty. The initial error will normally increase with time for a while, but it is not necessarily true that the asymptotic state will be influenced by the initial uncertainty, because the asymptotic states may be determined by the forcing and the dissipation of the system.

So far it has not been possible to investigate the stochastic-dynamic problem in an asymptotic sense for high-order systems containing many components, but certain low-order systems can be analysed in detail in spite of their non-linear nature. Such a deterministic system was analysed by Källen and Wiin-Nielsen (1980). For certain values of the external forcing it was found that the system has 2 stable and 1 unstable steady states of which one of the stable steady states is surrounded by an unstable closed "limit cycle". For this system it is thus obvious that a state inside the "limit cycle" will asymptotically approach the steady state inside the closed curve while all other states will approach the other stable steady state asymptotically.

There are reasons to expect that the transition between atmospheric circulation types may follow similar mechanisms in such a way that the transition takes place when the external forcing, created either by heating or topographic effects, exceeds a critical value. If we assume that the initial state with its uncertainty is close to the unstable "limit cycle" it is intuitively obvious that the asymptotic state will have considerable uncertainty. It is thus of great interest to investigate the stochastic-dynamic problem for this case.

The purpose of this paper is to investigate the solution to this problem and to study the impact of a given closure approximation on the solutions. It is well-known that the formulation of the equations governing a stochastic-dynamic prediction problem from the deterministic equations leads to an infinite set of equations involving all the statistical moments. In practice it is therefore necessary to introduce a closure approximation which truncates the infinite set to a finite set. A number of such closure assumptions are in

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existence, but the one most commonly used in atmospheric prediction problems is the neglect of third and higher moments. This closure procedure reduces the moments to the means, the variances and the co-variances among the dependent variables of the problem.

In certain cases it is also possible to avoid the closure approximation because the expected value of a function can be obtained by a direct integration of the integral whose integrand is the product of the function and the probability density function. In this regard it is of great convenience to note that due to the conservation of probabilities it is possible to replace the integration over the total phase space at an arbitrary time with an integration over the initial phase space using only the initial values of the probability density function (Pitcher, 1974). This procedure will work only when analytical or numerical solutions of the deterministic equations are available in the phase space, and it follows that the procedure can be used in practice only when the phase-space is of very limited dimensions. If numerical integrations only are available, it is necessary to evaluate the expected value of a function by approximate integration formulae. For Gaussian probability density functions it is then possible to use the very accurate procedure employing the roots and weights of the Hermite polynomials.

The approach described above has been used earlier to investigate the solutions of the advective, non-linear equation without forcing and dissipation and treated as a low-order system (Wiin-Nielsen, 1979). To employ this procedure for more realistic systems it is necessary to add forcing and dissipation. In this paper we shall again consider a two-component version of the advective equation, but we shall include a Newtonian forcing and a linear dissipation. The deterministic problem has been considered in detail by Källén and Wiin-Nielsen (1980). In this paper it was shown that for certain values of the external forcing, the system has 3 steady states of which one is always unstable and another is always stable. The third steady state may be stable or unstable depending on the values of the external parameters. If it is unstable, the system has only one asymptotic state, i.e., the single stable steady state, but if it is stable, we have two possible asymptotic states. The investigation of the deterministic equations shows that in the latter case we may have an unstable "limit cycle" which was determined in the previous paper by numerical means.

In this paper we shall describe the results of an investigation of the properties of the stochastic-dynamic system corresponding to the deterministic problem considered earlier. We shall consider the problem without and with a closure approximation, and in this investigation we shall also be interested in the asymptotic solutions. The cases in which the deterministic problem possesses limit cycles will be of special interest.

2. The stochastic-dynamic problem

The deterministic equations are those employed by Källén and Wiin-Nielsen (loc. cit.). With a slight change of the scaling we may write them in the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy - x + x_E,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x^2 - y + y_E. \tag{2.1}$$

In spite of their apparent simplicity, analytic solutions are not readily available. On the other hand numerical solutions are easily created by an integration using some form of the Runge–Kutta method. It is pertinent to mention that 3 real solutions to the steady state problem exist if

$$y_E \geqslant 1 + \frac{3}{2^{2/3}} x_E^{2/3}, \tag{2.2}$$

and that the values of x_s where the subscript s denotes a steady state can be found from the equation

$$x_s^3 - (y_E - 1)\, x_s - x_E = 0. \tag{2.3}$$

By applying the usual procedure and neglecting third and higher order moments we obtain from (2.1) the stochastic-dynamic equations

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{x}\bar{y} - \bar{x} + x_E + \sigma(xy),$$

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}t} = -\bar{x}^2 - \bar{y} + y_E - \sigma(x^2),$$

$$\frac{d\sigma(x^2)}{dt} = 2(\bar{y}-1)\,\sigma(x^2) + 2\bar{x}\sigma(xy), \tag{2.4}$$

$$\frac{d\sigma(y^2)}{dt} = -4\bar{x}\sigma(xy) - 2\sigma(y^2),$$

$$\frac{d\sigma(xy)}{dt} = -2\bar{x}\sigma(x^2) + (\bar{y}-2)\,\sigma(xy) + \bar{x}\sigma(y^2).$$

As in the deterministic problem (2.1) there is no difficulty in obtaining numerical solutions to (2.4). We shall, however, first consider the steady states of (2.4). It is obvious that the steady states in the deterministic problem as found from (2.3) will also be steady states for the stochastic-dynamic problem as described by (2.4). We need only note that (2.4) reduces to (2.1) if $\sigma(x^2) = \sigma(y^2) = \sigma(xy) = 0$. Additional steady states are, however, introduced by the closure approximation. To find these we consider the set of 5 equations obtained when all the time-derivatives are equated to zero. Using the third and fourth of these equations we find

$$\sigma_s(x^2) = -\frac{\bar{x}_s}{\bar{y}_s - 1}\,\sigma_s(xy), \qquad \sigma_s(y^2) = -2\bar{x}_s\,\sigma_s(xy). \tag{2.5}$$

Inserting these expressions in the fifth equation we obtain:

$$[2\bar{x}_s^2 + (\bar{y}_s - 1)(\bar{y}_s - 2 - 2\bar{x}_s^2)]\,\sigma_s(xy) = 0. \tag{2.6}$$

Eq. (2.6) has the solution $\sigma_s(xy) = 0$ in which case (2.5) shows that $\sigma_s(x^2) = \sigma_s(y^2) = 0$. If $\sigma_s(xy) \neq 0$ we must require that the expression in the bracket vanishes. It is seen that a root in this polynomial is $\bar{y}_s = 2$, and we therefore find

$$(\bar{y}_s - 2)(\bar{y}_s - 1 - 2\bar{x}_s^2) = 0, \qquad \sigma_s(xy) \neq 0. \tag{2.7}$$

In the first case when $\bar{y}_s = 2$ we find from the first 2 steady state equations:

$$\bar{x}_s + x_E + \sigma_s(xy) = 0, \qquad -\bar{x}_s^2 - 2 + y_E + \bar{x}_s\,\sigma_s(xy) = 0. \tag{2.8}$$

The formal solution of (2.8) is

$$\bar{x}_s = -\tfrac{1}{4}x_E \pm \tfrac{1}{4}\sqrt{x_E^2 + 8(y_E - 2)}, \quad \bar{y}_s = 2, \quad \sigma_s(x^2) = \bar{x}_s(\bar{x}_s + x_E), \quad \sigma_s(y^2) = 2\bar{x}_s(\bar{x}_s + x_E), \quad \sigma_s(xy) = -(\bar{x}_s + x_E). \tag{2.9}$$

This stochastic-dynamic solution requires that

$$y_E \geqslant 2 - \tfrac{1}{8}x_E^2, \tag{2.10}$$

because otherwise $\bar{x}_s$ will not be real. It is also required that the inequality

$$\sigma_s(xy)^2 \leqslant \sigma(x^2)\,\sigma(y^2) \tag{2.11}$$

should be satisfied. Eq. (2.11) results in the inequalities

$$\bar{x}_s \geqslant 2^{-1/2}, \qquad \bar{x}_s \leqslant -2^{-1/2}. \tag{2.12}$$

Using (2.9) it may be shown by elementary calculations that the first inequality is satisfied for

$$y_E \geqslant 3 + \sqrt{\tfrac{1}{2}}x_E \tag{2.13}$$

while the second equality in (2.12) is satisfied for

$$x_E \geqslant 2\sqrt{2}, \qquad y_E \leqslant 3 - \sqrt{\tfrac{1}{2}}x_E \quad \text{and} \quad y_E \geqslant 3 - \sqrt{\tfrac{1}{2}}x_E. \tag{2.14}$$

Fig. 1 shows the various curves defined by (2.2), (2.10), (2.13) and (2.14). Note that the 2 straight lines through (0,3) are tangents to the remaining 2 curves at the points $(4\sqrt{2}, 7)$ and $(2\sqrt{2}, 1)$. For later use we note that all solutions to the steady state equations exist for points above the upper straight line in Fig. 1.

In the second case originating from (2.7) we have

$$\bar{y}_s = 1 + 2\bar{x}_s^2. \tag{2.15}$$

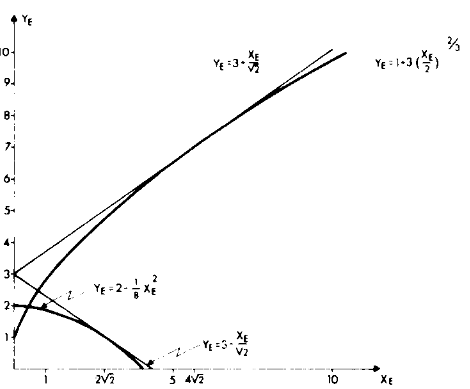

Fig. 1. The region above the upper straight line in the (x_E, y_E) plane is in the region in which all steady states for the system with closure exist.

Inserting (2.15) in the first 2 steady state equations we find that

$$
\begin{aligned}
&2\bar{x}_s^3 + x_E + \sigma_s(xy) = 0, \\
&(y_E - 1) - 3\bar{x}_s^2 + \frac{1}{2\bar{x}_s}\sigma_s(xy) = 0,
\end{aligned}
\tag{2.16}
$$

leading to

$$
8\bar{x}_s^3 - 2(y_E - 1)\,\bar{x}_s + x_E = 0. \tag{2.17}
$$

Replacing $\bar{x}_s$ by $-\frac{1}{2}z$ we can transform (2.17) into the eq. (2.3). We find therefore that the solutions to (2.3) also provide the solutions to (2.17) by multiplication by $-\frac{1}{2}$. If we denote the solutions to (2.3) by $\bar{x}_{1s}$, $\bar{x}_{2s}$ and $\bar{x}_{3s}$ we may write from (2.15) that

$$
\bar{y}_s = 1 + \tfrac{1}{2}\bar{x}_s^2, \tag{2.18}
$$

while the first equation in (2.16) leads to

$$
\sigma_s(xy) = \tfrac{1}{4}\bar{x}_s^3 - x_E. \tag{2.19}
$$

From (2.5) we find finally

$$
\sigma_s(x^2) = \bar{x}_s^{-1}\sigma_s(xy),
$$

$$
\sigma_s(y^2) = \bar{x}_s\,\sigma_s(xy),
$$

where $\bar{x}_s$ in (2.18), (2.19) and the above 2 equations represents any one of the 3 solutions of (2.3).

We note therefore that in this special case we have that

$$
\sigma_s(xy)^2 = \sigma_s(x^2)\,\sigma_s(y^2),
$$

indicating that the uncertainty ellipse has degenerated to a straight line.

The results of the investigation of the steady states are summarized in Table 1. In this table x_{1s}, x_{2s} and x_{3s} denote the solutions to eq. (2.3), while x_{4s} and x_{5s} are the solutions to (2.9).

As seen from Table 1 there are 3 steady states when the uncertainty ellipse degenerates to a straight line. A general description of the calculation of the parameters of the ellipse is given by Wiin-Nielsen (1979). It is pertinent to mention that the orientation of the ellipse is calculated from the formulae

$$
\cos 2\theta = \frac{\sigma(x^2) - \sigma(y^2)}{N}, \qquad \sin 2\theta = \frac{2\sigma(xy)}{N}, \tag{2.20}
$$

where

$$
N = [(\sigma(X^2) - \sigma(y^2))^2 + 4\sigma(xy)^2]^{1/2}. \tag{2.21}
$$

We note that (2.20) and (2.21) may be used even in the degenerate case in which

$$
\sigma(xy)^2 = \sigma(x^2)\,\sigma(y^2). \tag{2.22}
$$

However, the orientation may also be calculated directly from the definition of the uncertainty ellipse. The equation in the degenerate case is

$$
\left(\frac{x}{\sqrt{\sigma(x^2)}}\right)^2 \mp \frac{2}{\sqrt{\sigma(x^2)}\,\sqrt{\sigma(y^2)}}\, xy + \left(\frac{y}{\sqrt{\sigma(y^2)}}\right)^2 = 0, \tag{2.23}
$$

when the double sign in the second term depends on the sign of $\sigma(xy)$. The upper sign is used when $\sigma(xy)$ is positive, the lower when $\sigma(xy)$ is negative. Eq. (2.23) is a straight line with the equation

$$
y = \pm\sqrt{\frac{\sigma(y^2)}{\sigma(x^2)}}\,x. \tag{2.24}
$$

Table 1

$\bar{x}_s$	$\bar{y}_s$	$\sigma_s(x^2)$	$\sigma_s(\overline{y^2})$	$\sigma_s(xy)$	Comments
$\bar{x}_{1s}$	$y_E - \bar{x}_{1s}^2$	0	0	0	
$\bar{x}_{2s}$	$y_E - \bar{x}_{2s}^2$	0	0	0	Deterministic solutions
$\bar{x}_{3s}$	$y_E - \bar{x}_{3s}^2$	0	0	0	
$\bar{x}_{4s}$	2	$\bar{x}_{4ds}(x_E + \bar{x}_{4s})$	$2\bar{x}_{4s}(x_E + \bar{x}_{4s})$	$-(x_E + \bar{x}_{4s})$	Exist for
$\bar{x}_{5s}$	2	$\bar{x}_{5s}(x_E + x_{5s})$	$2\bar{x}_{5s}(x_E + \bar{x}_{5s})$	$-(x_E + x_{5s})$	$\bar{x}_s^2 \geqslant \frac{1}{2}$
$-\frac{1}{2}\bar{x}_{1s}$	$1 + \frac{1}{2}\bar{x}_{1s}^2$	$\bar{x}_{1s}^{-1}(\frac{1}{4}\bar{x}_{1s}^3 - x_E)$	$\bar{x}_{1s}(\frac{1}{4}\bar{x}_{1s}^3 - x_E)$	$\frac{1}{4}\bar{x}_{1s}^3 - x_E$	Uncertainty ellipse is
$-\frac{1}{2}\bar{x}_{2s}$	$1 + \frac{1}{2}\bar{x}_{2s}^2$	$\bar{x}_{2s}^{-1}(\frac{1}{4}\bar{x}_{2s}^3 - x_E)$	$\bar{x}_{2s}(\frac{1}{4}\bar{x}_{2s}^3 - x_E)$	$\frac{1}{4}\bar{x}_{2s}^3 - x_E$	straight line
$\frac{1}{2}\bar{x}_{3s}$	$1 + \frac{1}{2}\bar{x}_{3s}^2$	$\bar{x}_{3s}^{-1}(\frac{1}{4}\bar{x}_{3s}^3 - x_E)$	$\bar{x}_{3s}(\frac{1}{4}\bar{x}_{3s}^3 - x_E)$	$\frac{1}{4}\bar{x}_{3s}^3 - x_E$	

The result in (2.24) agrees with those displayed in (2.20) and (2.21). It is finally of interest to determine the limiting value of the half axis of the ellipse. The major half axis a is computed from the formula (Wiin-Nielsen, 1979)

$$a^2 = 4\mathscr{L}\frac{\sigma(x^2)\,\sigma(y^2) - \sigma(xy)^2}{\sigma(x^2) + \sigma(y^2) - N} \tag{2.25}$$

where $\mathscr{L}$ is a constant so that $1 - \exp(-\mathscr{L})$ is the probability that a point is inside the ellipse.

Using l'Hopital's rule we find that the limiting value of a is

$$a_* = [2\mathscr{L}(\sigma(x^2) + \sigma(y^2))]^{1/2}. \tag{2.26}$$

For later use we note that the position of the last steady state in Table 1 is located in the mid-point of the connecting line between the first 2 steady states in the table. The slope of the connecting line is $\bar{x}_{3s}$, while the slope of the degenerate uncertainty ellipse is $-\bar{x}_{3s}$.

3. Stability of the steady states in the system with closure

The non-linear equations governing this sytem are given in (2.4). The steady states can be calculated from Table 1. It is straightforward to formulate the stability problem. Let an arbitrary steady state be denoted by $\bar{x}_s$, $\bar{y}_s$, $\sigma_s(x^2)$, $\sigma_s(y^2)$ and $\sigma_s(xy)$. Using the classical procedure of linear perturbation analysis we may convert the system (2.4) into a set of linear equations. We are seeking solutions to these equations of the form $\exp(\nu t)$ where ν is the frequency. A positive value of the real part of ν indicates instability, a negative value stability, and a zero value indicates a neutral solution.

The general eigenvalue problem leads to the following equation:

$$\begin{vmatrix} \bar{y}_s - 1 - \nu & \bar{x}_s & 1 & 0 & 0 \\ -2\bar{x}_s & -1-\nu & 0 & -1 & 0 \\ \sigma_s(y^2) - 2\sigma_s(x^2) & \sigma_s(xy) & \bar{y}_s - 2 - \nu & -2\bar{x}_s & \bar{x}_s \\ 2\sigma_s(xy) & 2\sigma_s(x^2) & 2\bar{x}_s & 2\bar{y}_s - 2 - \nu & 0 \\ -4\sigma_s(xy) & 0 & -4\bar{x}_s & 0 & -2-\nu \end{vmatrix} = 0 \tag{3.1}$$

It is seen from (3.1) that the determinant in general will result in a frequency equation of the fifth degree. In the remaining part of this section we shall limit ourselves to an example when $x_E = 3$, $y_E = 20$. The reason for this selection is that the same example has been used in the paper by Källén and Wiin-Nielsen (1980) for the deterministic problem. For these values of x_E and y_E one finds 3 steady states for the system (2.1) of which one is unstable and two are stable. One of the stable steady states is surrounded by an unstable limit cycle which was found numerically by a backward integration. The same example will be used later in this paper to explore the stochastic-dynamic problem without closure. The steady states corresponding to $x_E = 3$ and $y_E = 20$ are given in Table 2.

The first 3 cases correspond to the steady states in the deterministic system. In these cases we may evaluate (3.1) directly. It turns out that the determinant may be written in the form:

$$(\nu - (\bar{y}_s - 2))\cdot(\nu^2 - (\bar{y}_s - 2)\,\nu + 2\bar{x}_s^2 - (\bar{y}_s - 1))$$
$$(\nu^2 - 2(\bar{y}_s - 2)\,\nu + 4(2\bar{x}_s^2 - (\bar{y}_s - 1))) = 0. \tag{3.2}$$

The eigenvalues are therefore

$$\nu = \bar{y}_s - 2, \qquad \nu = \tfrac{1}{2}(\bar{y}_s - 2) \pm \tfrac{1}{2}\sqrt{\bar{y}_s^2 - 8\bar{x}_s^2},$$
$$\nu = (\bar{y}_s - 2) \pm \sqrt{\bar{y}_s^2 - 8\bar{x}_s^2}. \tag{3.3}$$

It is thus seen that the eigenvalues in our examples are:

steady state no. 1: -0.30; $-0.15 \pm i5.99$; $-0.30 \pm i11.98$
steady state no. 2: -1.68; $-0.84 \pm i6.28$; $-1.68 \pm i12.56$
steady state no. 3: 17.98; 18.98; -1.00; 37.95; -1.99

indicating that steady states nos. 1 and 2 are stable, while no. 3 is unstable. These results agree of course with those obtained earlier for the deterministic case.

Table 2. *The steady states for $x_E = 3$, $y_E = 20$*

Steady state no.	$\bar{x}_s$	$\bar{y}_s$	$\sigma_s(x^2)$	$\sigma_s(y^2)$	$\sigma_s(xy)$
1	−4.28	1.70	0	0	0
2	4.44	0.32	0	0	0
3	−0.16	19.98	0	0	0
4	2.34	2	12.51	25.03	−5.34
5	−3.84	2	3.24	6.47	0.84
6	2.14	10.15	5.28	96.54	−22.58
7	−2.22	10.84	4.24	83.45	18.81
8	0.08	1.01	18.97	0.47	−2.99

We next consider steady states nos. 4 and 5. It is seen that the fifth degree polynomial $F(\nu)$ resulting from the evaluation of (3.1) will start with the term $-\nu^5$. $F(0)$ is equal to the constant term in the polynomial. Since $F(\nu)$ will approach ∞ for $\nu \to \infty$, there will be a positive real root and consequently an unstable steady state if the constant term is positive. For steady states nos. 4 and 5 we find by evaluating the determinant that the constant term is:

$$4(x_s + x_E)(4x_s + x_E)(2x_s^2 - 1). \tag{3.4}$$

For steady state no. 4 we find that the value of (3.4) is 2627.2 while the value for no. 5 is 1183.2. It therefore follows that both of these steady states are unstable in our example.

The procedure just described may also be used to show that steady states no. 6 and 7 are unstable. The calculation of the constant term in the frequency equation is rather cumbersome, and it is not reproduced here.

The only remaining steady state is no. 8. In this case it was necessary to find all the roots in the frequency equation which for $x_E = 3$ and $y_E = 20$ becomes

$$\nu^5 + a_1\nu^4 + a_2\nu^3 + a_3\nu^2 + a_4\nu + a_5 = 0, \tag{3.5}$$

with $a_1 = 3.95$, $a_2 = 80.23$, $a_3 = 224.49$, $a_4 = 1582.50$ and $a_5 = 2837.77$. The equation has a negative real root which is -1.975. By division by $\nu + 1.975$ we get

$$\nu^4 + b_1\nu^3 + b_2\nu^2 + b_3\nu + b_4 = 0, \tag{3.6}$$

with $b_1 = 1.975$, $b_2 = 76.33$, $b_3 = 73.74$ and $b_4 = 1436.85$. Eq. (3.6) may be solved by the classical method for fourth degree equations and we get the following eigenvalues:

$$-1.975, \qquad -0.84 \pm i6.09, \qquad -0.15 \pm i6.17,$$

showing that steady state no. 8 is stable.

For our example ($x_E = 3, y_E = 20$) we have thus shown that in addition to the 2 stable steady states which belong to the deterministic equations, 1 stable steady state (no. 8) is introduced as a result of the closure approximation. The other 4 steady states introduced by the closure are unstable.

4. The system without closure

As described by Wiin-Nielsen (1979) it is possible to calculate the stochastic-dynamic variables from the formula

$$E[f(x, y, t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, t) \times \phi(x_0, y_0)\,dx_0\,dy_0 \tag{4.1}$$

in which E is the expected value, f an arbitrary function, ϕ the probability density function and (x_0, y_0) the initial phase space. We shall assume that

$$\phi = \frac{1}{2d_1 d_2 \pi}\exp\left[-\left(\frac{x_0 - \bar{x}_0}{d_1\sqrt{2}}\right)^2 - \left(\frac{y_0 - \bar{y}_0}{d_2\sqrt{2}}\right)^2\right], \tag{4.2}$$

where $(\bar{x}_0, \bar{y}_0)$ is the initial mean value and d_1, d_2 are the standard deviations. To evaluate (4.1) we introduce the new coordinates

$$\xi = \frac{x_0 - \bar{x}_0}{d_1\sqrt{2}}, \qquad \eta = \frac{y_0 - \bar{y}_0}{d_2\sqrt{2}}, \tag{4.3}$$

and we find

$$E[f(x,y,t)] = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,t)\exp[-\xi^2-\eta^2]\,d\xi\,d\eta. \tag{4.4}$$

In the following we shall be interested in the asymptotic values. We note that the deterministic problem has 3 steady states. Using the same example as in the previous sections, i.e., $x_E = 3$ and $y_E = 20$, we know from the results of Källén and Wiin-Nielsen (1980) that 1 of these steady states is unstable, while the other 2 are stable. Among the latter, 1 steady state, say (x_s', y_s'), is surrounded by an unstable limit cycle determined experimentally by a backward time integration. This limit cycle is reproduced in Fig. 2. All initial states inside the limit cycle will asymptotically approach (x_s', y_s') while all points outside the limit

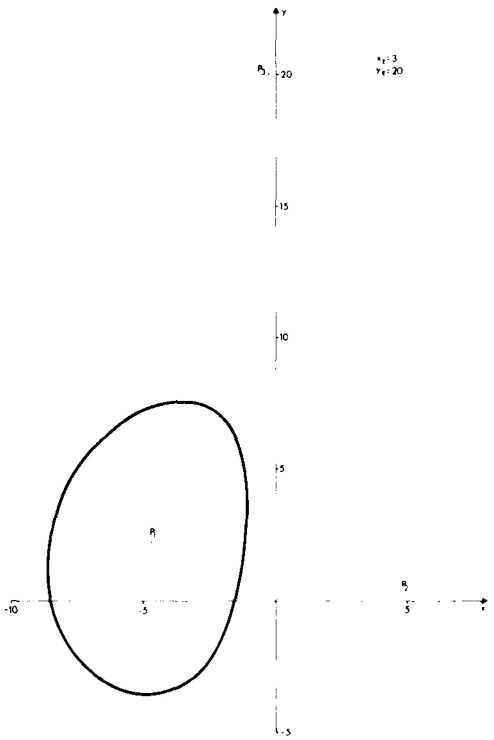

Fig. 2. The closed curve in the (x,y) plane is the unstable limit cycle for the deterministic problem. P_1, P_2 and P_3 are the steady states of which P_1 and P_2 are the stable states for the given values of x_E and y_E.

cycle will arrive in the other steady state (x_s'', y_s'') at large times. We may use these considerations to calculate (4.4) for $t \to \infty$. The formulae for the mean values are

$$\bar{x}_\infty = x_s'' + \frac{A}{\pi}(x_s' - x_s''),$$

$$\bar{y}_\infty = y_s'' + \frac{A}{\pi}(y_s' - y_s''), \tag{4.5}$$

where A is the area integral

$$A = \int_S \exp[-\xi^2 - \eta^2]\,d\xi\,d\eta. \tag{4.6}$$

S in (4.6) denotes the area inside the unstable limit cycle. We may write (4.6) in the form

$$A = \frac{\sqrt{\pi}}{2}\int_{\xi_1}^{\xi_2}\{\Phi(\eta_2) - \Phi(\eta_1)\}\,e^{-\xi^2}\,d\xi, \tag{4.7}$$

in which Φ is the error function, i.e.,

$$\Phi(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-z^2}\,dz. \tag{4.8}$$

Eq. (4.7) may be used to calculate A in a given case. Since the coordinates of the limit cycle are known in the (x,y) plane it is necessary to convert these coordinates to the (ξ, η) plane using (4.3) and knowing $\bar{x}_0$, $\bar{y}_0$, d_1 and d_2 in each case. Thereafter (4.7) is calculated by a numerical integration. The formulae for the second moments are

$$\sigma_\infty(x^2) = x_s''^2 + \frac{A}{\pi}(x_s'^2 - x_s''^2) - \bar{x}_\infty^2,$$

$$\sigma_\infty(y^2) = y_s''^2 + \frac{A}{\pi}(y_s'^2 - y_s''^2) - \bar{y}_\infty^2, \tag{4.9}$$

$$\sigma_\infty(xy) = x_s'' y_s'' + \frac{A}{\pi}(x_s' y_s' - x_s'' y_s'') - \bar{x}_\infty \bar{y}_\infty.$$

We may use the results in this section to compare the exact asymptotic values given in (4.5) and (4.9) with those obtained experimentally from numerical integrations of the stochastic-dynamic equations (2.4) with closure. For the initial state $\bar{x}_0 = 0$, $\bar{y}_0 = 5$, $\sigma_0(x^2) = \sigma_0(y^2) = 2$, $\sigma_0(xy) = 0$, we find that the integral A has the value $A = 1.1016$. The asymptotic values are therefore:

$$\bar{x}_\infty = 1.38,$$
$$\bar{y}_\infty = 0.80,$$
$$\sigma_\infty(x^2) = 17.32,$$
$$\sigma_\infty(y^2) = 0.44,$$
$$\sigma_\infty(xy) = -2.73.$$

The numerical integration of (2.4) from the initial state given above shows that the trajectory eventually approaches the stable steady state, no. 8, characterized by the values

$$\bar{x}_\infty = 0.08,$$
$$\bar{y}_\infty = 1.01,$$
$$\sigma_\infty(x^2) = 18.97,$$
$$\sigma_\infty(y^2) = 0.47,$$
$$\sigma_\infty(xy) = -2.99.$$

A considerable difference exists between the 2 sets of values, showing that the closure scheme employed here can be quite inaccurate for long integrations. The integral A defined by (4.6) or (4.7) may have values between 0 and π. If $A = 0$ we see from (4.5) and (4.9) that

$$\bar{x}_\infty = x_{s_2}, \quad \bar{y}_\infty = y_{s_2},$$
$$\sigma_\infty(x^2) = \sigma_\infty(y^2) = \sigma_\infty(xy) = 0,$$

showing that the asymptotic state is identical to steady state no. 2. On the other hand if $A = \pi$ we notice that

$$\bar{x}_\infty = x_{s_1}, \quad \bar{y}_\infty = y_{s_1},$$
$$\sigma_\infty(x^2) = \sigma_\infty(y^2) = \sigma_\infty(xy) = 0,$$

which is steady state no. 1.

It is also possible to show that $A = \frac{1}{2}\pi$ leads to values identical to those of steady state no. 8. From the first formula in (4.5) we find for $A = \frac{1}{2}\pi$

$$\bar{x}_\infty = \tfrac{1}{2}(x_{s_1} + x_{s_2}) = -\tfrac{1}{2}x_{s_3},$$

because $x_{s_1} + x_{s_2} + x_{s_3} = 0$ due to the properties of the roots in the cubic eq. (2.3).

From (4.5) we also find for $A = \frac{1}{2}\pi$ that

$$\bar{y}_\infty = \tfrac{1}{2}(y_{s_1} + y_{s_2}).$$

To show that this value agrees with the corresponding value in steady state no. 8, we derive first the equation for y_s from the basic deterministic equations by elimination of x_s. We find

$$(y_s - 1)^3 - (y_E - 1)(y_E - 1)^2 + x_E^2 = 0. \tag{4.10}$$

It therefore follows that

$$y_{s_1} + y_{s_2} + y_{s_3} = y_E + 2, \tag{4.11}$$

and thus

$$\begin{aligned}\tfrac{1}{2}(y_{s_1} + y_{s_2}) &= \tfrac{1}{2}(y_E + 2 - y_{s_3}) \\ &= 1 + \tfrac{1}{2}(y_E - y_{s_3}) = 1 + \tfrac{1}{2}x_{s_3}^2.\end{aligned} \tag{4.12}$$

Similar procedures, repeatedly using the properties of the roots in a cubic equation, may be used to show that the 3 values obtained from (4.9) for $A = \frac{1}{2}\pi$ coincide with the corresponding values for steady state no. 8.

We have thus shown that the 3 stable steady states which exist in the system with closure, i.e., nos. 1, 2 and 8, are identical to the steady state in the system without closure for the values of A of π, 0 and $\frac{1}{2}\pi$, respectively. The closure scheme is thus exact for these and only these values. The remaining question is therefore under which conditions A will take these values. A general, but incomplete analysis of this question can be obtained from (4.6) and (4.7) which define A. We note first of all that $\Phi(x)$, the error function, for practical purposes is equal to unity when $x \geqslant 2$. It therefore follows that if all η values are >2, we have $A \approx 0$. Since $\Phi(-x) = -\Phi(x)$ we will also have $A \approx 0$ when all η values are <-2. Eq. (4.7) may also be written in the form

$$A = \frac{\sqrt{\pi}}{2}\int_{\eta_1}^{\eta_2} (\Phi(\xi_2) - \Phi(\xi_1))\, e^{-\eta^2}\, d\eta, \tag{4.13}$$

and A will therefore be close to zero when all ξ values are either larger than 2 or less than -2. In terms of the initial values we may thus state that A is practically zero when

$$\bar{x}_0 < x_{0\,\mathrm{min}} - 2\sqrt{2d_1};$$
or $$\bar{x}_0 > x_{0\,\mathrm{max}} + 2\sqrt{2d_1},$$
or $$\bar{y}_0 < y_{0\,\mathrm{min}} - 2\sqrt{2d_2},$$
or $$\bar{y}_0 > y_{0\,\mathrm{max}} + 2\sqrt{2d_2},$$

where the subscripts "min" and "max" denote the extreme values of the coordinates of the closed curve defining A.

Considering the possibility that $A \approx \pi$, it is obvious that the region S in the (ξ, η) plane must reach into all 4 quadrants, and that the absolute values of ξ and η should be larger than 2 in most points of the boundary of S. As necessary, but not sufficient conditions we may thus state that

$$\begin{aligned} &\xi_{\min} < -2, \quad \eta_{\min} < -2, \\ &\xi_{\max} > 2, \quad \eta_{\max} > 2. \end{aligned} \tag{4.14}$$

These conditions lead to the inequalities

$$\begin{aligned} &x_{0\,\min} + 2\sqrt{2d_1} < \bar{x}_0 < \bar{x}_{0\,\max} - 2\sqrt{2d_1}, \\ &y_{0\,min} + 2\sqrt{2d_2} < \bar{y}_0 < y_{0\,\max} - 2\sqrt{2d_2}. \end{aligned} \tag{4.15}$$

Eq. (4.15) cannot be satisfied if d_1 and d_2 are too large. It is required that

$$d_1 < \frac{x_{0\,\max} - x_{0\,\min}}{4\sqrt{2}} = 1.32,$$

$$d_2 < \frac{y_{0\,\max} - y_{0\,\min}}{4\sqrt{2}} = 1.98.$$

Fig. 3 shows for $d_1 = d_2 = 1$ as an example the regions in the (x, y) plane in which A is approximately zero (the region outside the large square) and the smaller region inside which A may be close to π. These regions will change in location and size for other values of d_1 and d_2.

These considerations can be illustrated by numerical examples. Table 3 contains 3 cases. I.S. denotes the initial state, A.S. the asymptotic state, and A.S.C. the asymptotic state with closure. In each case the integral A was calculated using (4.6) and a step of 0.1.

The first example in Table 3 is one in which A is close to π. As expected, we find excellent agreement between A.S. and A.S.C. which turns out to be steady state no. 1. The second example has a value

Table 3

		I.S.	A.S.	A.S.C.	
	$\bar{x}$	−4.0	−4.28	−4.28	
	$\bar{y}$	4.0	1.70	1.70	Steady
$A = 3.1414$	$\sigma(x^2)$	0.5	9×10^{-5}	0.0	state 1
	$\sigma(y^2)$	0.5	2×10^{-4}	0.0	
	$\sigma(xy)$	0.0	5×10^{-4}	0.0	
	$\bar{x}$	0.0	3.76	4.44	
	$\bar{y}$	2.0	0.43	0.32	Steady
$A = 0.2433$	$\sigma(x^2)$	$2^{-1/2}$	5.47	0.00	state 2
	$\sigma(y^2)$	$2^{-1/2}$	0.13	0.00	
	$\sigma(xy)$	0.0	−0.84	0.00	
	$\bar{x}$	−1.1	−0.10	0.08	
	$\bar{y}$	4.0	1.04	1.01	Steady
$A = 1.6370$	$\sigma(x^2)$	0.5	18.98	18.97	state 8
	$\sigma(y^2)$	0.5	0.47	0.47	
	$\sigma(xy)$	0.0	−3.01	−2.99	

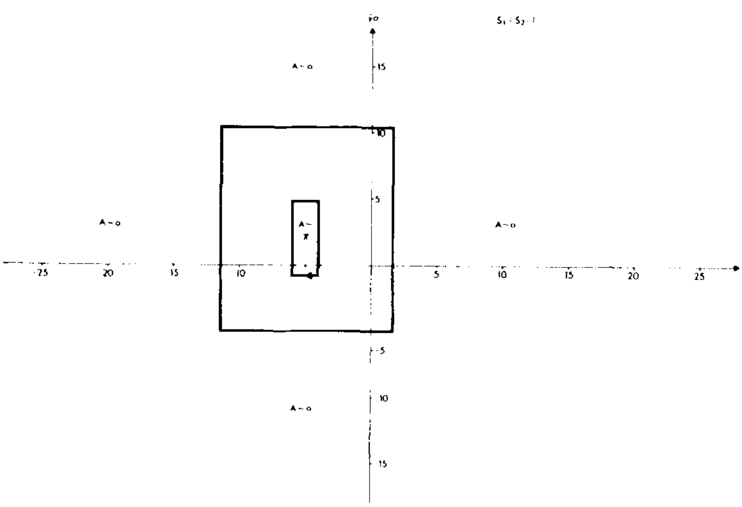

Fig. 3. Approximate regions in the $(\bar{x}_0, \bar{y}_0)$ plane in which the integral A has specific values. Outside the large rectangle A is close to zero, while $A \approx \pi$ inside the small rectangle.

of A which is different from the special values 0, $\frac{1}{2}\pi$ and π. Consequently there is considerable difference between the 2 asymptotic states for which A.S.C. is identical to steady state no. 2. Finally, the third example is one in which A is close to $\frac{1}{2}\pi$. The two asymptotic states agree closely. A.S.C. is steady state no. 8.

The analysis above has shown that A may take any value between 0 and π. The asymptotic solutions given by (4.5) and (4.9) can therefore vary continuously between the steady states (x_{s_1}, y_{s_1}) and (x_{s_2}, y_{s_2}). On the other hand, in the system with closure we can have only 3 solutions, steady states nos. 1, 2 and 8. It follows that the closure scheme can have substantial errors in the asymptotic states.

The 3 asymptotic steady states which exist in the system with closure are characterized by a vanishing uncertainty in nos. 1 and 2 while the uncertainty ellipse in no. 8 is a straight line. We shall now show these properties also hold for all the asymptotic states obtained in the system without closure.

Inserting from (4.5) into (4.9) we find that

$$\sigma_\infty(x^2) = = \frac{A}{\pi}\left(1 - \frac{A}{\pi}\right)(x_s' - x_s'')^2,$$

$$\sigma_\infty(y^2) = \frac{A}{\pi}\left(1 - \frac{A}{\pi}\right)(y_s' - y_s'')^2,$$

$$\sigma_\infty(xy) = \frac{A}{\pi}\left(1 - \frac{A}{\pi}\right)(x_s' - x_s'')(y_s' - y_s'').$$

It therefore follows that

$$\sigma_\infty(xy)^2 = \sigma_\infty(x^2)\,\sigma_\infty(y^2),$$

and the uncertainty ellipse is a straight line. The slope of the line is determined from

$$\cos 2\theta = \frac{(x_s' - x_s'')^2 - (y_s' - y_s'')^2}{(x_s' - x_s'')^2 + (y_s' - y_s'')^2},$$

$$\sin 2\theta = \frac{2(x_s' - x_s'')(y_s' - y_s'')}{(x_s' - x_s'')^2 + (y_s' - y_s'')^2},$$

giving the result that the slope is always $-9°$. On the other hand, the limiting value of the major half-axis is

$$a_* = \sqrt{2\mathscr{L}\frac{A}{\pi}\left(1 - \frac{A}{\pi}\right)[(x_s' - x_s'')^2 + (y_s' - y_s'')^2]}$$

or

$$a_* = 10.4\sqrt{\frac{A}{\pi}\left(1 - \frac{A}{\pi}\right)}.$$

The maximum value of a_* is thus 5.2 obtained for $A = \frac{1}{2}\pi$.

5. Some additional considerations

The examples in Section 4 are limited to initial states in which the co-variance $\sigma(xy)$ vanishes. On the other hand, several of the steady states for the system with closure do not satisfy this condition. For this reason it is desirable to consider a slightly generalized probability density function. For Gaussian distributions one may write the following expression

$$\Phi(x,y) = \frac{1}{2\pi\Delta^{1/2}}\exp\left[-\frac{1}{2\Delta}(\sigma(y^2)(x-\bar{x})^2 + \sigma(x^2)(y-\bar{y})^2 - 2\sigma(xy)(x-\bar{x})(y-\bar{y}))\right], \tag{5.1}$$

where

$$\Delta = \sigma(x^2)\,\sigma(y^2) - \sigma(xy)^2. \tag{5.2}$$

Eq. (5.1) satisfies the conditions that it has the variances $\sigma(x^2)$ and $\sigma(y^2)$, while the co-variance is $\sigma(xy)$. Verification may be obtained by direct calculation of these quantities.

The uncertainty ellipse may be defined as the curve inside which $\Phi(x,y)$ exceeds a given value. We note that

$$\Phi(\bar{x},\bar{y}) = \frac{1}{2\pi}\cdot\frac{1}{\Delta^{1/2}} \tag{5.3}$$

giving the maximum value of Φ. We may now define the uncertainty ellipse as the curve on which Φ is equal to $\mathscr{L}^{-1}\,\sigma(\bar{x},\bar{y})$, or

$$\Phi(x,y) = \frac{1}{2\pi\Delta^{1/2}}\frac{1}{\mathscr{L}} = \frac{1}{2\pi\Delta^{1/2}}\,e^{-\ln\mathscr{L}}. \tag{5.4}$$

Using (5.1) and (5.2) we find that the uncertainty ellipse is described by

$$\frac{x_1^2}{\sigma(x^2)} - 2\frac{\sigma(xy)}{\sigma(x^2)\,\sigma(y^2)}x_1y_1 + \frac{y_1^2}{\sigma(y^2)} = 2\ln\mathscr{L}\left(1 - \frac{\sigma(xy)^2}{\sigma(x^2)\,\sigma(y^2)}\right), \tag{5.5}$$

where $x_1 = x - \bar{x}$, $y_1 = y - \bar{y}$.

The formulae (4.5) and (4.9) for the asymptotic states derived in Section 4 are valid in the present case when A is given by

$$A = \pi \int_s \Phi(x_0, y_0)\,dx_0\,dy_0, \tag{5.6}$$

where Φ is defined by (5.1). The evaluation of A by numerical integration may be performed by using (5.1) directly, but it is also possible to reduce (5.6) to a simpler form by a reduction of the quadratic form in (5.1). Introducing first

$$x_2 = \left(\frac{\sigma(y^2)}{2\Delta}\right)^{1/2} x_1, \qquad y_2 = \left(\frac{\sigma(x^2)}{2\Delta}\right)^{1/2} y_1, \tag{5.7}$$

we find

$$A = \int_s \sqrt{1-r^2}\exp[-x_2^2 - y_2^2 + 2rx_2y_2]\,dx_2\,dy_2, \tag{5.8}$$

where the correlation coefficient r is

$$r = \frac{\sigma(xy)}{\sqrt{\sigma(x^2)\,\sigma(y^2)}}. \tag{5.9}$$

We define next

$$x_2 = \sqrt{\tfrac{1}{2}}\,(x_3 - y_3), \qquad y_2 = \sqrt{\tfrac{1}{2}}(x_3 + y_3), \tag{5.10}$$

giving

$$A = \int_s \sqrt{1-r^2}\exp[-(1-r)x_3^2 - (1+r)y_3^2]\,dx_3\,dy_3. \tag{5.11}$$

Finally for $\xi = \sqrt{1-r}\,x_3$, $\eta = \sqrt{1+r}\,y_3$ we get

$$A = \int_s \exp[-\xi^2 - \eta^2]\,d\xi\,d\eta. \tag{5.12}$$

Eq. (5.12) is in the same form as used in Section 4, but of course with a more complicated definition of the variables ξ and η.

The formalism developed in this section has been used to compare a number of integrations of the system with closure, with exact results. The initial states have been selected close to the unstable steady states of the system with closure. It is in these situations that one may expect the largest difference between the 2 systems, because one of them is close to an instability while the other is stable. The results are given in Tables 4–7. In these tables we have used to notation: I.S. for the initial state, A.S. for the asymptotic state without closure, and A.S.C. for the asymptotic state with closure.

The results listed in Tables 4–7 show very clearly that the system with closure is very inaccurate. This is of course due to the fact that this system has 3 asymptotic states while the system

Table 4

	I.S. ~4	A.S. $\pi^{-1}A = 0.0960$	A.S.C. Steady state no. 8
$\bar{x}$	2.34	3.60	0.08
$\bar{y}$	2.00	0.45	1.03
$\sigma(x^2)$	12.51	6.62	18.96
$\sigma(y^2)$	25.03	0.17	0.51
$\sigma(xy)$	−5.34	−1.03	−3.09

Table 5

	I.S. ~5	A.S. $\pi^{-1}A = 0.8642$	A.S.C. Steady state no. 8
$\bar{x}$	−3.74	−3.10	0.08
$\bar{y}$	2.10	1.51	1.03
$\sigma(x^2)$	3.24	8.90	18.96
$\sigma(y^2)$	6.47	0.23	0.51
$\sigma(xy)$	0.84	−1.41	−3.09

Table 6

	I.S. ~5	A.S. $\pi^{-1}A = 0.8423$	A.S.C. Steady state no. 1
$\bar{x}$	−3.8423	−3.18	−4.28
$\bar{y}$	2.0000	1.53	1.70
$\sigma(x^2)$	3.2361	8.38	0.0
$\sigma(y^2)$	6.4727	0.20	0.0
$\sigma(xy)$	0.8423	−1.31	0.0

Table 7

	I.S. ~7	A.S. $\pi^{-1}A = 0.3005$	A.S.C. Steady state no. 8
$\bar{x}$	−2.3	1.82	0.08
$\bar{y}$	10.9	0.73	1.03
$\sigma(x^2)$	4.2	15.98	18.96
$\sigma(y^2)$	84.0	0.41	0.51
$\sigma(xy)$	18.5	−2.52	−3.09

without closure has a continuum of such states. A comparison between Tables 5 and 6 shows furthermore that a slight change in the initial state may result in a different asymptotic state in the system with closure, while such a change is impossible in the exact calculation.

A calculation of the integral A for an initial state close to steady state no. 6 shows that due to the large value of the variances and co-variances in this case, we find a value of A which is exceedingly small. The exact calculation will therefore result in an asymptotic state very close to steady state no. 2. Numerical integrations of the system with closure also result in an asymptotic state equal to steady state no. 2.

In the present analysis we have repeatedly made use of the fact that the deterministic system possesses an unstable limit cycle around steady state no. 1. This property of the system was crucial in determining the asymptotic states in the system without closure, because the unstable limit cycle divides the phase space in 2 regions in such a way that initial states inside the limit cycle will approach steady state no. 1 asymptotically while initial states outside the limit cycle will approach steady state no. 2.

One may ask whether the stochastic-dynamic system with closure as described by the 5 equations (2.4) has an unstable limit cycle. To explore this question and to answer it partially, a numerical integration was performed. This integration started from an initial state very close to steady state no. 1, and therefore with very small values of $\sigma(x^2)$, $\sigma(y^2)$ and $\sigma(xy)$. Since this state is stable for time progressing in the positive direction, it is unstable for time going in the negative direction. Such a "backward" integration was therefore carried out using Heun's time differencing scheme and a time-step of 0.001. It turns out that such an integration after a long time leads to a periodic stable oscillation in the 5-dimensional phase space of $\bar{x}$, $\bar{y}$, $\sigma(x^2)$, $\sigma(y^2)$ and $\sigma(xy)$. The stable oscillations in the backward time integrations are shown in Figs. 4–8. It is seen that long integrations are necessary to arrive at the limit cycle. Since the limit cycle is stable in the backward integration, for our problem it is an unstable limit cycle. It is of course impossible to show it in the 5-dimensional phase space, but the closed curve $(\bar{x}(t), \bar{y}(t))$ is shown in Fig. 9. It is seen that the curve contains one loop which actually surrounds steady state 5 which is an unstable steady state. Such behaviour is naturally impossible in a 2-dimensional phase space, but the curve in Fig. 9 displays the behaviour of 2 variables only.

During the integration of the 5 simultaneous equations describing the stochastic-dynamic system with closure, the programme tested in each time-step whether the inequality

$$\sigma(xy) \leqslant \sigma(x^2)\,\sigma(y^2)$$

was satisfied. Due to possible inaccuracies in the

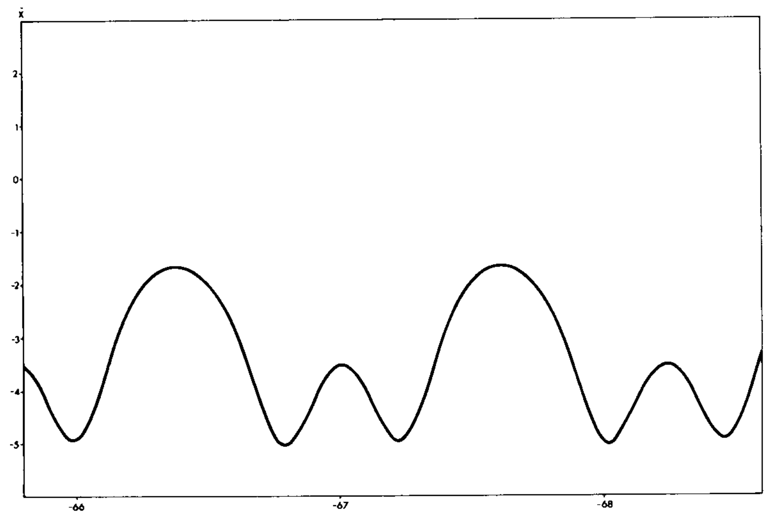

Fig. 4. $\bar{x}$ as a function of time in a backward calculation starting very close to steady state no. 1. It is seen that $\bar{x}(t)$ is a periodic function at this time.

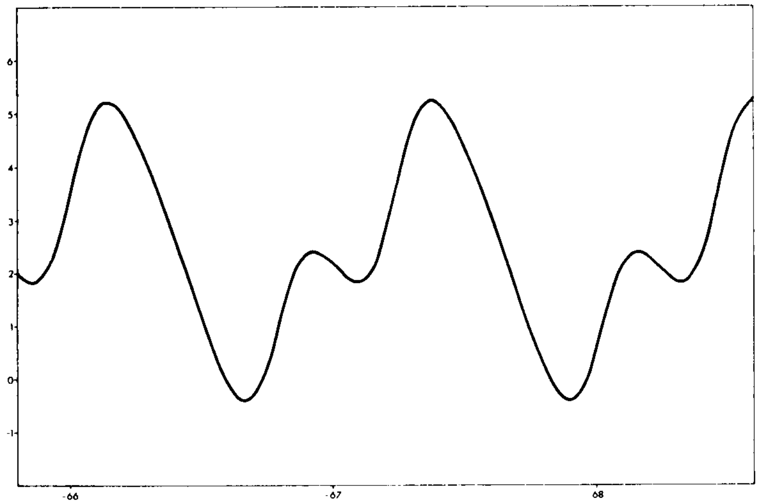

Fig. 5. $\bar{y} = \bar{y}(t)$. Arrangement as in Fig. 4.

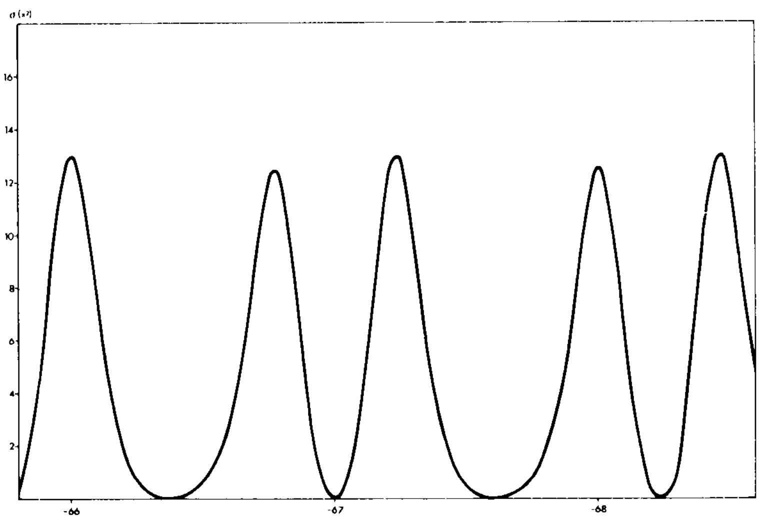

Fig. 6. $\sigma(x^2)$ as a function of time. Arrangement as in Fig. 4.

time-differencing scheme there is no guarantee that the inequality will hold at all times even if it is satisfied initially. In this way it was discovered that the limit cycle is of a very special nature, because it has a degenerate uncertainty ellipse at every point. Fig. 10, which contains a part of the curve $(\bar{x}(t), \bar{y}(t))$ and the degenerate uncertainty ellipses, illustrates this point.

6. Summary and conclusions

The investigations described in this paper deal with the solutions of the advective equation with forcing and dissipation reduced to a low-order system of 2 components. The main emphasis is on a stochastic-dynamic treatment of these equations considering the initial uncertainties. 2 different

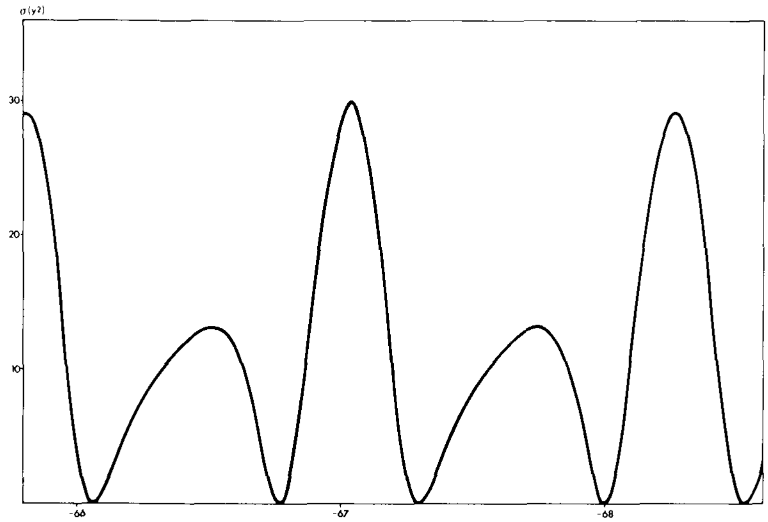

Fig. 7. $\sigma(y^2)$ as a function of time. Arrangement as in Fig. 4.

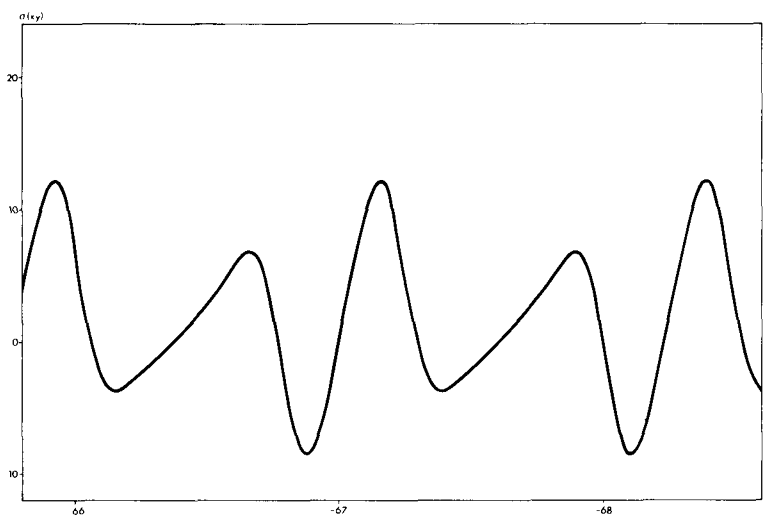

Fig. 8. $\sigma(xy)$ as a function of time. Arrangement as in Fig. 4.

stochastic-dynamic treatments are investigated. The first is the stochastic-dynamic equations after closure using as the closure mechanism the neglect of third and higher-order moments. The second is an exact evaluation of the solutions without any closure assumption.

For the system with closure, we find that one of the effects of the closure assumption is the introduction of 5 steady states in addition to the 3 steady states which exist for the deterministic problem. These steady states are given in Table 1. The stability of all steady states is investigated using an example in which the deterministic problem has 1 unstable and 2 stable steady states of which 1 stable state is surrounded by an unstable limit cycle. It is found that the additional steady states are unstable except for 1 state which happens to be located in the middle of the line connecting the other 2 stable steady states.

The system without closure is considered mainly in the asymptotic case where the solutions may be given immediately. The asymptotic values of the mean values ($\bar{x}$ and $\bar{y}$), the variances $\sigma(x^2)$ and $\sigma(y^2)$ and the co-variance $\sigma(xy)$ depend on the position of the 2 stable steady states and on the integral of the probability density function over the area enclosed by the unstable limit cycle for the deterministic problem. This integral can be evaluated with excellent accuracy by numerical methods. It is shown that all asymptotic states are situated on the line segment connecting the 2 deterministic stable steady states. In addition, it is demonstrated that all asymptotic states have a line segment as the uncertainty ellipse. The continuum of asymptotic states in the system without closure contains the 3 asymptotic states in the system with closure, because these 3 states are the end points and the middle point of the line segment corresponding to the values of 0, π and $\frac{1}{2}\pi$, respectively, of the integral over the region of the limit cycle.

The paper contains several examples comparing the asymptotic solutions of the 2 systems and a discussion of those cases in which the closure scheme may be accurate.

In Section 5 it is shown by numerical backward integrations in time that the systems with closure also contain an unstable limit cycle.

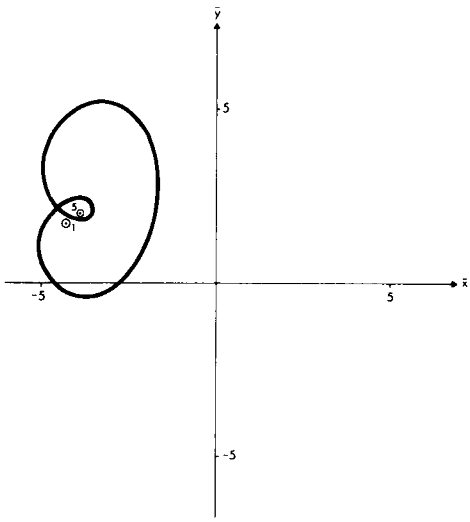

Fig. 9. The unstable limit cycle for the variables $\bar{x}(t)$ and $\bar{y}(t)$. The points marked 1 and 5 are 2 of the steady states.

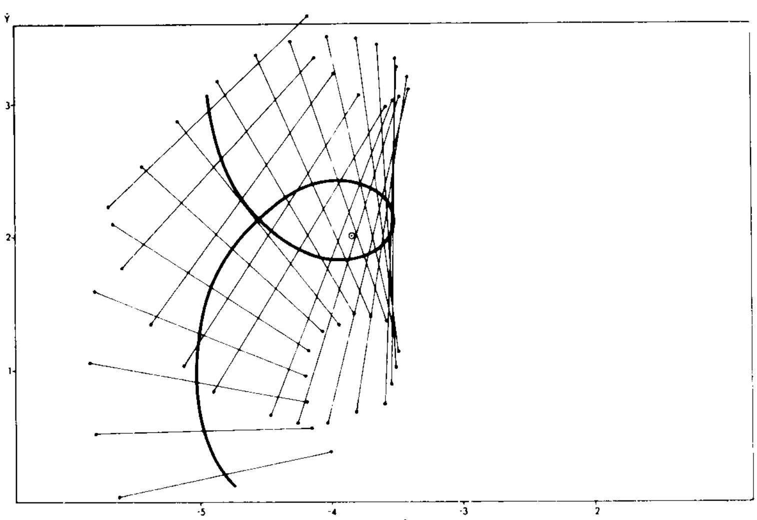

Fig. 10. A part of the curve displayed in Fig. 9. The degenerate uncertainty ellipses consisting of a line segment are shown for selected points.

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