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On the application of quasi-geostrophic equations to wind-driven barotropic currents in seas of moderate depth

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ABSTRACT

Quasi-geostrophic equations are derived, and the limitation of the approximations is discussed on the basis of a scale analysis. It is found that friction at the sea bottom is an important dynamical factor which determines an intrinsic time scale of importance to the accuracy of the equations. Analytical solutions are derived, giving the current resulting from specified external forces, and a comparison is made to results reported elsewhere. A special solution is shown to represent a trapped wave resembling a Kelvin wave, but with a reduction in phase speed caused by the implicit approximations. The implications of this solution are discussed in the concluding part of the paper, together with some remarks on a numerical model based on the quasi-geostrophic equations.

1. Introduction

The quasi-geostrophic formalism was first applied to ocean circulation by Charney (1955). In essence he used methods previously developed for the large-scale motion of the atmosphere. Making the distinction between "currents" and "waves", he argued that currents are most conveniently studied by means of the quasi-geostrophic equations, which may be described as prognostic equations where the inertia-gravity waves are excluded *a priori.*

Since their introduction into oceanography, these equations have been extensively used in studies of ocean dynamics, but not so extensively in similar problems concerning adjacent seas and coastal waters. A notable exception is a study of the Baltic Sea by Walin (1972). He found that the circulation induced by the wind is barotropic in the interior of the sea, provided that a certain condition on the internal static stability is fulfilled. Using slightly different notations Walin's condition may be written as

$$\Delta\rho/\rho_0 \ll (fL/gH)^2.$$

Here ρ_0 and $\Delta\rho$ are characteristic values of the mean density and departure from this density, while L and H are horizontal and vertical scales. The hydrographic characteristics of the Baltic were shown to agree with this condition, and we shall assume that the same is true for the water bodies we consider in this paper.

Although the circulation in the interior of the sea is barotropic, there may be narrow currents along the coast which are baroclinic in nature. This has been shown by Csanady (1971), Walin (1972) and others. The structure of such coastal currents will not be considered here, and boundary conditions for the barotropic interior are constructed on the basis of the restrictions imposed by the coast on the vertically integrated water transport.

Like Walin (1972) we shall derive the quasi-geostrophic equations on the basis of a scale analysis, but from a somewhat different viewpoint. As the bottom friction turns out to be important for the problems we deal with, and more so if the depth of the sea is small, we shall pay particular attention to this fact while deriving criteria for the validity of the approximations.

Later on a few analytical solutions of the

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equations are described. They may have quite general applications to water bodies of the prescribed dimension and structure, which is indicated by a comparison with results derived by a completely different method (Davies and Heaps, 1980).

2. Derivation of the quasi-geostrophic equations

2.1. *The vorticity equation*

Based on the assumption that the density gradients are irrelevant, the governing equations may be written

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + w\frac{\partial \mathbf{v}}{\partial z} + f\mathbf{k}\times\mathbf{v} = -\nabla(\phi + P) + \frac{\partial \tau}{\partial z}, \tag{1}$$

$$\frac{\partial w}{\partial z} = -\nabla\cdot\mathbf{v}, \tag{2}$$

$$\frac{1}{g}\frac{\partial \phi}{\partial t} + \frac{1}{g}\mathbf{v}_0\cdot\nabla\phi = w_0, \tag{3}$$

$$-\mathbf{v}_1\cdot\nabla h = w_1, \tag{4}$$

where the symbols have the following meaning:
ρ—density of sea water (a constant);
$\mathbf{v}$—horizontal vector velocity of the water;
w—vertical velocity component;
∇—horizontal Nabla operator;
$\mathbf{k}$—vertical unit vector;
x, y, z—Cartesian coordinates, the z-axis pointing upwards;
h—depth of water below mean sea level;
$w_0 = w(x, y, 0, t)$—vertical velocity at the sea surface;
$w_1 = w(x, y, -h, t)$—vertical velocity at the bottom;
ϕ/g—elevation of water level;
ρP—atmospheric pressure;
$\rho\tau$—horizontal stress vector.

The hydrostatic approximation has been used, i.e. we intend to study only motion structures in which the horizontal scale is large compared to the vertical scale.

The immediate effect of the wind stress is to set in motion the water masses close to the surface. This motion, in conjunction with the lateral boundaries and inhomogeneities in the wind stress, tends to establish horizontal pressure gradients which in turn accelerate the water masses at greater depths. The latter current, which is our primary focus of attention, will be referred to as the deep-sea current and will be designated by a bar over the relevant mathematical symbols. The Ekman currents close to the upper and lower boundaries will be called boundary layer currents, and will be designated by a suffix B when convenient.

The horizontal velocity may accordingly be divided into two parts

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}_B,$$

where $\bar{\mathbf{v}}$ is forced by the horizontal pressure gradient, i.e.

$$\frac{\partial \bar{\mathbf{v}}}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}} + f\mathbf{k}\times\bar{\mathbf{v}} = -\nabla(\phi + P). \tag{5}$$

Then

$$\frac{\partial \mathbf{v}_B}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\mathbf{v}_B + \mathbf{v}_B\cdot\nabla\bar{\mathbf{v}} + \mathbf{v}_B\cdot\nabla\mathbf{v}_B + w\frac{\partial \mathbf{v}_B}{\partial z} + f\mathbf{k}\times\mathbf{v}_B = \frac{\partial \boldsymbol{\tau}}{\partial z}. \tag{6}$$

Since $(\phi + P)$, according to the assumptions, is the same at all depths, the same is true for $\bar{\mathbf{v}}$. On the other hand, $\mathbf{v}_B$, which is forced mostly by the friction, will have significant values only near the surface and the bottom of the sea.

Assuming that $\phi/g \ll h$, we may define a volume flux per horizontal length unit

$$\mathbf{T}_B = \int_{-h}^{0} \mathbf{v}_B \, dz.$$

Then, from (6)

$$\frac{\partial \mathbf{T}_B}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\mathbf{T}_B + \mathbf{T}_B\cdot\nabla\bar{\mathbf{v}} + \int_{-h}^{0}\mathbf{v}_B\cdot\nabla\mathbf{v}_B\,dz + \int_{-h}^{0} w\frac{\partial \mathbf{v}_B}{\partial z}\,dz = -f\mathbf{k}\times\mathbf{T}_B + \boldsymbol{\tau}_0 - \boldsymbol{\tau}_1.$$

Here $\rho\boldsymbol{\tau}_0$ and $\rho\boldsymbol{\tau}_1$ are the horizontal stress vectors at the surface and bottom, respectively. In certain types of studies it has been customary to neglect the whole left-hand side, in which case we get

$$\mathbf{T}_B = -\frac{1}{f}\mathbf{k}\times(\boldsymbol{\tau}_0 - \boldsymbol{\tau}_1). \tag{7}$$

The justification for this simplification is extensively discussed in the Appendix.

We now attempt a scale analysis of eq. (5) in the light of a particular type of motion to be specified presently. As already mentioned we want to exclude the inertia-gravity waves. Such waves may be generated by external forces like atmospheric pressure and wind. However, when first generated, they can exist more or less independently of these forces. On the other hand they do not usually lead to long-term displacements of the water masses.

Generally speaking, we assume that a motion exists which depends directly on the external forces, or more precisely, on a quasi-balance between them. The more prominent of these forces in the case we study, are the wind stress on the ocean surface and the friction at the bottom. A balance between these two forces determines a velocity with characteristic size $U = S_0/c$, assuming that ρS_0 is a characteristic value of the wind stress and that the bottom stress is given by the approximate relation ρUc, where c is a constant. Furthermore, we may define a time scale, T, for adjustment towards the balance, as the ratio between the momentum of the water column, ρHU, and the force ρS_0. This gives $T = HU/S_0 = H/c$. Here H is the characteristic depth of the sea. Another time scale may be forced upon the water by variation in the wind stress and atmospheric pressure. We assume this latter time scale to be at least as large as the former one.

We also introduce a characteristic horizontal scale, L, determined basically by the size and shape of the sea and variations in the depth, and only to a small extent by the pressure and wind fields, which we assume to be of a much larger scale. Finally, on the basis of the geostrophic relation

$$\bar{\mathbf{v}} = f^{-1}\,\mathbf{k} \times \nabla(\phi + P), \tag{8}$$

we may determine a scale for ϕ and P equal to $LfU = fS_0L/c$. The arguments for using this assumption will be clarified later.

Now the scaled version of (5) becomes

$$\frac{c}{fH}\frac{\partial \bar{\mathbf{v}}}{\partial t} + \frac{S_0}{fcL}\,\bar{\mathbf{v}}\cdot\nabla\bar{\mathbf{v}} + \mathbf{k}\times\bar{\mathbf{v}} = -\nabla(\phi + P).$$

As characteristic numerical values we use $L = 10^5$ m, $S_0 = 10^{-4}$ m^2 s^{-2}, $H = 10^2$ m, $f = 10^{-4}$ s^{-1} and $c = 2.5 \times 10^{-3}$ m s^{-1}. Then $T = 4 \times 10^4$ s, $c/fH = 0.25$ while $S_0/fcL = 4 \times 10^{-3}$. Obviously the second term in (5) is of little importance. Also neglecting the first term gives the relation (8). However, a more accurate formula exists and will now be derived.

A time differentiation of the scaled equation (with the second term neglected), followed by a substitution for $\partial\bar{\mathbf{v}}/\partial t$ from the same equation, gives

$$\left(\frac{c}{fH}\right)^2 \frac{\partial^2 \bar{\mathbf{v}}}{\partial t^2} + \bar{\mathbf{v}} = \mathbf{k}\times\nabla(\phi + P) - \frac{c}{fH}\nabla\frac{\partial}{\partial t}(\phi + P).$$

If $(c/fH)^2 \ll 1$, the first term may be neglected. Returning to the unscaled equation we get the formula

$$\bar{\mathbf{v}} = f^{-1}\,\mathbf{k}\times\nabla(\phi + P) - f^{-2}\,\nabla\frac{\partial}{\partial t}(\phi + P). \tag{9}$$

The last term may be looked upon as a correction to the geostrophic formula (8). In the atmosphere it is known as the Brunt-Douglas isallobaric wind. In our case its characteristic size may be about 1/4 of the first term, as already shown.

Taking the vorticity and divergence of (5), without the non-linear term, we get

$$\frac{\partial\bar{\zeta}}{\partial t} + f\nabla\cdot\bar{\mathbf{v}} = 0,$$

$$\frac{\partial}{\partial t}(\nabla\cdot\bar{\mathbf{v}}) - f\bar{\zeta} = -\nabla^2(\phi + P),$$

where $\bar{\zeta} = \mathbf{k}\cdot\nabla\times\bar{\mathbf{v}}$. Elimination of $\bar{\zeta}$ from these two equations gives

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla\cdot\bar{\mathbf{v}} = -\frac{\partial}{\partial t}\nabla^2(\phi + P).$$

Under the same assumption as above, we may neglect the first term and get

$$\nabla^2\frac{\partial}{\partial t}(\phi + P) + f^2\,\nabla\cdot\bar{\mathbf{v}} = 0. \tag{10}$$

This is the quasi-geostrophic vorticity equation (Charney, 1955) in a form relevant to our assumptions. It is valid for motions where $(c/Hf)^2 \ll 1$, as was also the case with (9). It may easily be shown that it has no wave solution of the inertia-gravity type. Because of the importance of the non-dimensional number c/Hf we shall use the symbol R for this quantity in the rest of this paper. It is formally similar to the so-called Rossby boundary layer number u_*/Hf, where u_* is the friction velocity.

To summarize, we first note that $R = f^{-1}/T$ where T is the intrinsic time scale introduced above. The other time scale involved is imposed by the changing atmospheric disturbances. Both scales must be large compared to f^{-1} in order to use the quasi-geostrophic equation with sufficient accuracy. This condition is generally fulfilled for the atmospheric time scale, but not necessarily for T, unless the sea is sufficiently deep. With the values of c mostly recommended by the authors, the depth must be 50–100 m in middle and high latitudes. It appears that this fact has not been generally recognized.

From (2)–(4) it follows that

$$-h\nabla\cdot\bar{\mathbf{v}} - \int_{-h}^{0}\nabla\cdot\mathbf{v}_{\mathrm{B}}\,\mathrm{d}z - (\bar{\mathbf{v}} + \mathbf{v}_{\mathrm{B1}})\cdot\nabla h = \frac{\partial}{\partial t}\frac{\phi}{g} + (\bar{\mathbf{v}} + \mathbf{v}_{\mathrm{B0}})\cdot\nabla\frac{\phi}{g}.$$

Assuming that $\bar{\mathbf{v}}$ and $\bar{\mathbf{v}}_{\mathrm{B0}}$ are of the same order of magnitude, it follows from the scale analysis that the last term on the right-hand side is negligible compared to the first. Then the equation may be written

$$\frac{1}{g}\frac{\partial\phi}{\partial t} = -\nabla\cdot(\mathbf{T}_{\mathrm{B}} + h\bar{\mathbf{v}}). \tag{11}$$

Now $\nabla\cdot\bar{\mathbf{v}}$ may be eliminated between (10) and (11), giving

$$\left(\nabla^2 - \frac{f^2}{gh}\right)\frac{\partial\phi}{\partial t} = \frac{f^2}{h}\bar{\mathbf{v}}\cdot\nabla h + \frac{f^2}{h}\nabla\cdot\mathbf{T}_{\mathrm{B}} - \nabla^2\frac{\partial P}{\partial t}. \tag{12}$$

This equation may serve as a prognostic equation for ϕ provided that $\bar{\mathbf{v}}$ and τ_1 in eq. (7) can be derived from ϕ. For $\bar{\mathbf{v}}$ we use formula (9). The stress at the bottom depends on the velocity $\bar{\mathbf{v}}$, the roughness of the sea bed etc. In the following analysis we use the linear relation

$$\tau_1 = c\bar{\mathbf{v}}. \tag{13}$$

Then, when this relation together with (7) and (9) are substituted into (12), we get after some straightforward manipulation

$$\left(\nabla\cdot h\nabla - \frac{f^2}{g}\right)\frac{\partial\phi}{\partial t} = -c\nabla^2(\phi + P) - f\mathbf{k}\cdot\nabla h\times\nabla(\phi + P) + f\mathbf{k}\cdot\nabla\times\tau_0 - \nabla\cdot\left(h\nabla\frac{\partial P}{\partial t}\right). \tag{14}$$

2.2. *Lateral boundary conditions*

If the lateral boundary is a coastline, we base the kinematic condition on the assumption of no net flux of water across the boundary, i.e.

$$T_{\mathrm{BN}} + h\bar{v}_{\mathrm{N}} = 0. \tag{15}$$

Here, and in the following, the suffices N and S mean normal and tangential components, while $\partial/\partial n$ and $\partial/\partial s$ indicate differentiation along the normal and tangent, respectively.

For the solution of eq. (14), boundary conditions on ϕ are needed. These may be derived from the normal and tangential component of (9), in conjunction with (15). From (9) we get

$$\bar{v}_{\mathrm{N}} = -f^{-1}\frac{\partial}{\partial s}(\phi + P) - f^{-2}\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial t} + \frac{\partial P}{\partial t}\right),$$

$$\bar{v}_{\mathrm{S}} = f^{-1}\frac{\partial}{\partial n}(\phi + P) - f^{-2}\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial t} + \frac{\partial P}{\partial t}\right). \tag{16}$$

These equations, together with (15), (7) and (13) give the boundary condition for (14):

$$h\frac{\partial}{\partial n}\left(\frac{\partial\phi}{\partial t}\right) - \frac{c}{f}\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial t}\right) = -c\frac{\partial}{\partial n}(\phi + P) - fh\frac{\partial}{\partial s}(\phi + P) + f\tau_{0\mathrm{S}} - h\frac{\partial}{\partial n}\frac{\partial P}{\partial t} + \frac{c}{f}\frac{\partial}{\partial s}\frac{\partial P}{\partial t} \tag{17}$$

2.3. *Scaled equations*

For later reference we also give the scaled version of (14):

$$(\nabla\cdot h\nabla - \kappa^2)\frac{\partial\psi}{\partial t} = -\nabla^2\psi - R^{-1}\mathbf{k}\cdot\nabla h\times\nabla\psi + \mathbf{k}\cdot\nabla\times\tau_0 - \kappa^2\frac{\partial P}{\partial t}, \tag{18}$$

where $\kappa^2 = f^2L^2/gH$ has the characteristic size 10^{-1}, using the same numerical values as above. We have introduced the new variable $\psi = \phi + P$ for convenience. Since we consider cases with R small compared to unity, it would seem that the second term on the right dominates. This is certainly the case if $|\nabla h|$ is of order one or larger, in which case the current is approximately parallel to the isobaths.

Similarly the scaled version of the boundary condition (17) reads

$$\left(h\frac{\partial}{\partial n} - R\frac{\partial}{\partial s}\right)\frac{\partial\psi}{\partial t} = -\frac{\partial\psi}{\partial n} - R^{-1}h\frac{\partial\psi}{\partial s} + \tau_{0\mathrm{S}}. \tag{19}$$

Again, the second term on the right dominates if h is of order one and R is small. Then, the current is nearly parallel to the coast. We also note the scaled version of (9), which is

$$\bar{\mathbf{v}} = \mathbf{k} \times \nabla\psi - R\nabla\frac{\partial\psi}{\partial t}. \tag{20}$$

3. Analytical solutions

In the following a few solutions of the above equations are derived using analytical methods. They are selected so as to give insight into the type of problem we are aiming at in this paper, at the same time avoiding too complicated mathematics.

3.1. *Canal with constant depth*

We first consider an infinitely long canal of constant depth 1 between the shorelines $x = 0$ and $x = 1$. We search for solutions which are functions of x and t, but independent of y. Then (18) takes the form

$$\left(\frac{\partial^2}{\partial x^2} - \kappa^2\right)\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} = \frac{\partial\tau_{0y}}{\partial x} - \kappa^2\frac{\partial P}{\partial t}, \tag{21a}$$

while the boundary condition (19) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial t}\right) + \frac{\partial\psi}{\partial x} = \tau_{0y} \qquad \text{for } x = 0, 1 \tag{21b}$$

assuming that the atmospheric pressure and wind are uniform along the canal.

Initially we note that the general solution of the homogeneous problem may be written as

$$\psi_0 = \sum_{n=1}^{\infty} a_n e^{-\sigma_n t}\cos n\pi x, \tag{22}$$

where

$$\sigma_n = \left(1 + \frac{\kappa^2}{n^2\pi^2}\right)^{-1}, \tag{23}$$

and the a_n are constants to be determined from the value of ψ_0 at $t = 0$. Solutions of the non-homogeneous problem will now be worked out for the case where τ_{0y} = constant while $P(x,t)$ satisfies the condition $\partial P(0,t)/\partial x = \partial P(1,t)/\partial x = 0$. Then $\tau_{0y} = 0$ gives a non-homogeneous differential equation and a homogeneous boundary condition, and the other way around for $P = 0$.

3.1.1. *Wind stress zero.* Since $\partial P/\partial x = 0$ for $x = 0$, 1 we may write

$$P(x,t) = \sum_{n=1}^{\infty} b_n(t)\cos n\pi x, \tag{24}$$

where

$$b_n(t) = 2\int_0^1 P(\xi,t)\cos n\xi\pi\,\mathrm{d}\xi. \tag{25}$$

Then

$$\psi = \sum_{n=1}^{\infty} a_n(t)\cos n\pi x \tag{26}$$

is a solution, provided

$$(n^2\pi^2 + \kappa^2)a_n' + n^2\pi^2 a_n = \kappa^2 b_n'.$$

The last equation is an ordinary differential equation from which $a_n(t)$ may be determined. If $P(x,0) = \psi(x,0) = 0$ is chosen as the initial condition, the solution may be written as

$$a_n(t) = \frac{\kappa^2}{n^2\pi^2 + \kappa^2}\int_0^t e^{-\sigma_n(t-\tau)}b_n'(\tau)\,\mathrm{d}\tau, \tag{27}$$

where $b_n(t)$ is assumed to be known and σ_n is given by (23). Clearly, the rate of change of the horizontal pressure gradient is the important factor, not the pressure itself. From the exponential factor in (27), we see that the pressure change at a certain moment loses its influence with increasing time. The characteristic time for the influence is measured by σ_n^{-1}. It has its largest value for $n = 1$, but decreases asymptotically to 1 with increasing wave number. Using values appropriate to the North Sea, that is $L = 4 \times 10^5$ m, $H = 65$ m, $f = 1.22 \times 10^{-4}$ s^{-1} and $c = 2 \times 10^{-3}$ m s^{-1} (Davies and Heaps 1980), we get $\kappa^2 = 3.7$ and $\sigma_1^{-1} = 1.4$. Thus it is seen that σ_n^{-1}, which measures a characteristic adjustment time for the model, is in good agreement with the scaling assumptions above.

From ψ one may derive the current by means of (20). We shall not go into details, but only point out that currents develop both in the x and the y directions in response to the pressure change prescribed above. As a numerical example we choose $\rho P(x,t) = Kt\cos(\pi x/L)$ with $K = 100$ Pa/hour, and use the above values for the other constants. Then the maximum values of $\bar{u}$ and $\bar{v}$ are 0.5 and 1.0 cm s^{-1}, respectively. As we shall see presently, this is much less than the current which develops in response to a typical wind stress.

3.1.2. *Pressure change zero.* Since $P = 0$ in this case, we also have $\psi = \phi$. The non-homogeneous boundary condition can be satisfied by a particular solution

$$\phi_0 = \tau_{0y}(x - \tfrac{1}{2}). \tag{28}$$

The general solution is obtained by adding (22). To satisfy the initial condition $\phi(x,0) = 0$ we must solve the equation

$$\tau_{0y}(x - \tfrac{1}{2}) + \sum_{n=1}^{\infty} a_n \cos n\pi x = 0,$$

giving

$$a_n = \begin{cases} 4\tau_{0y}/n^2\pi^2 & n = 1, 3, 5 \ldots, \\ 0 & n = 2, 4, 6 \ldots. \end{cases} \tag{29}$$

The transient part of the solution decays with increasing time, and we are left with the stationary solution ϕ_0. The characteristic time for the transient part to become insignificant is again measured by σ_n^{-1}.

As an example we choose $\tau_{0y} = 2 \times 10^{-4}\ \mathrm{m^2\,s^{-2}}$, corresponding to a wind force of about 10 m $\mathrm{s^{-1}}$. Then the stationary current along the canal becomes 10 cm $\mathrm{s^{-1}}$. As already noted, this value is much larger than the typical value associated with pressure changes. Also note that wind force of the size mentioned may last for days with comparatively small changes. On the other hand, pressure changes of a significant size cannot last very long. For instance, a pressure tendency of the analytical form and amplitude used above would, when acting on a sea 400 km broad, create a surface wind of 10 m $\mathrm{s^{-1}}$ in about 4 h. We conclude, on the basis of all these examples, that the long-term water transport depends much more on the wind stress than on the changing atmospheric pressure.

3.2. *A more general stationary solution*

The stationary solution (28) has a straightforward physical interpretation, which becomes clear when we change to dimensional variables. Then (28) becomes

$$\phi_0 = \frac{f\tau_{0y}}{c}(x - \tfrac{1}{2}L),$$

which, according to (8), is associated with the current velocity $\bar{v}_0 = \tau_{0y}/c$ along the canal. For the current to be stationary, it is necessary that the volume transport across the canal, given by (7), vanish for any value of x. That this really is the case, is easily verified by means of (13). In other words, the deep sea current along the canal is exactly strong enough for the transport in the bottom boundary layer to compensate the drift in the boundary layer at the top.

We shall now derive a more general version of this solution, involving a variable depth, $h(x)$, and a current of constant strength, $\bar{u}$, across the canal. A stationary ϕ field, generated by a wind stress, constant in space and time, is governed by the equation

$$\nabla^2\phi + R^{-1}h'(x)\frac{\partial\phi}{\partial y} = 0,$$

while the boundary conditions are

$$\frac{\partial\phi}{\partial x} + R^{-1}h(0)\frac{\partial\phi}{\partial y} - \tau_{0y} = 0 \qquad \text{for } x = 0,$$

$$\frac{\partial\phi}{\partial x} + R^{-1}h(1)\frac{\partial\phi}{\partial y} - \tau_{0y} = 0 \qquad \text{for } x = 1.$$

An analytical solution subject to the constraint that $-\partial\phi/\partial y = \bar{u}$ (a constant) is easily derived:

$$\phi = \bar{u}R^{-1}\int_0^x h(\xi)\,\mathrm{d}\xi + \tau_{0y}x - \bar{u}y + \text{const.} \tag{30}$$

Here $\bar{u}$ is undetermined.

Hitherto the canal has been supposed to be open and unrestricted in the y direction. We now assume that it is closed somewhere by a boundary which we need not specify. The influence of this new boundary will be large in its neighbourhood, but not necessarily at greater distance. However, in the stationary case it imposes on the current the constraint that there can be no net volume transport across any line parallel to the x axis, that is

$$\int_0^1 \left(\bar{u} + R^{-1}h\frac{\partial\phi}{\partial x} - \tau_{0x}\right)\mathrm{d}x = 0. \tag{31}$$

This relation, together with (30), may be used to determine $\bar{u}$:

$$\bar{u} = -\frac{\int_0^1 (R^{-1}\tau_{0y}h - \tau_{0x})\,\mathrm{d}x}{\int_0^1 (1 + R^{-2}h^2)\,\mathrm{d}x}. \tag{32}$$

We assume that the depth scale is determined in such a way that

$$\int_0^1 h\,\mathrm{d}x = 1.$$

Then, with $h = 1 + \Delta h$

$$\bar{u} = -R\frac{\tau_{0y} - R\tau_{0x}}{1 + R^2 + \int_0^1 (\Delta h)^2\,dx}, \tag{33}$$

where the integral in most cases is small.

It is instructive to study the y-component of the geostrophic current

$$\bar{v} = \frac{\partial \phi}{\partial x} = \tau_{0y} + \bar{u}R^{-1}\,h(x). \tag{34}$$

It is seen to consist of two parts. The one is equal to the scaled stress component, τ_{0y}, while the other is positive if $\bar{u}$ is positive, and in strength proportional to the depth. The sign of $\bar{u}$ is determined by the direction of the wind stress. In fact, if α is the angle between the positive x-axis and the wind stress (see Fig. 1), $\bar{u}$ is negative if

$\alpha_1 < \alpha < \alpha_1 + \pi,$

where

$\mathrm{tg}\alpha_1 = R.$

As a typical value we choose $R = 0.25$, which gives $\alpha_1 = 14°$.

The physical content of this solution is similar to that of the solution with constant depth, previously described. Since we have assumed that there shall be no variation of the current in the direction along the canal, the surface elevation must vary linearly so that $\bar{u} = -\partial\phi/\partial y$ is a constant. However, the volume flux, $h\bar{u}$, varies. This variation, in its turn, must be compensated for by a variation in the boundary layer flux at the bottom, which in this model can only be produced by a variation in $\bar{v}$.

Davies and Heaps (1980) describe a numerical simulation of the wind-driven current in a model sea having the characteristic size and depth of the North Sea. A spectacular result is the surprisingly strong current in the deep eastern part ("The Norwegian Trench"). Obviously, this is in qualitative agreement with the results we have just derived. Below we also make a quantitative comparison. Figs. 3 and 7 in the paper by Davies and Heaps show the volume flux per grid-length through a cross section of the sea. In the formulation used in the present paper, this flux is the sum of the boundary layer flux, T_{By}, and the transport by the deep sea current, $\bar{v}h$. These quantities may easily be computed from (7), (13) and (34), using the same numerical values as Davies and Heaps, conveniently used as scaling constants. The result is shown as continuous lines in Figs. 2 and 3 for the non-dimensional wind stresses $\tau_0 = (0, -1)$ and $\tau_0 = (1, 0)$, respectively. The values computed by Davies and Heaps are entered as crosses. Obviously, the absolute values are not at all identical. This is not surprising since the models are very different, and so are also the methods by which the results are derived from the models. However, the relative variation across the

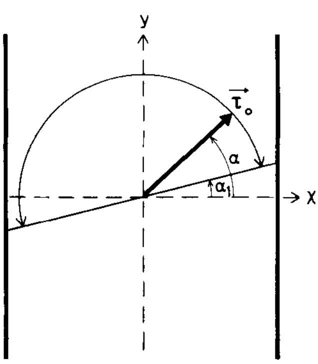

Fig. 1. Relation between the stationary current, $\bar{v}$, in a canal, and the constant wind stress, τ_0. The current may formally be assumed to consist of two parts: the first is proportional to τ_{0y} and independent of depth; the second is in strength proportional to the depth, and of negative sign if the direction of the stress is within the indicated sector.

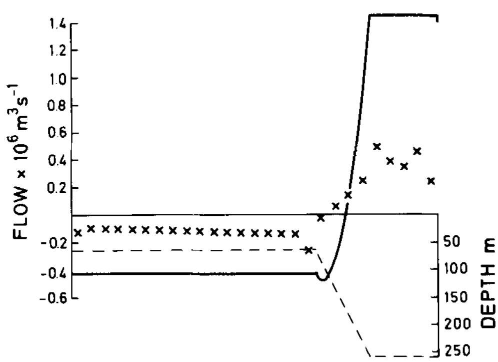

Fig. 2. Horizontal transport through a section of the model integrated by Davies and Heaps (1980), for northerly wind stress. The continuous line is computed from the analytical solution, while the crosses are taken from Fig. 3 in Davies and Heaps (1980). The numbers represent flow per grid length $\Delta x = 4 \times 10^5/27$ m. The broken line is the depth.

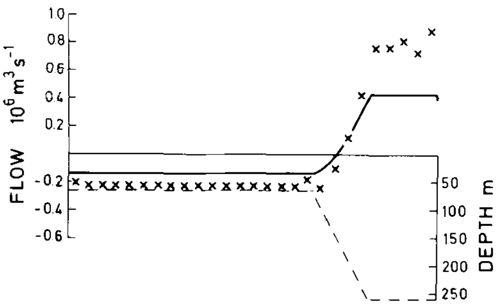

Fig. 3. Notation as in Fig. 2, but for westerly wind stress. The crosses are taken from Fig. 7 in Davies and Heaps (1980).

model sea is surprisingly similar. We take this as an indication that our analytical model and its physical interpretation may be quite relevant for actual cases.

4. A wave solution

Although inertia-gravity waves are excluded *a priori*, the quasi-geostrophic equations admit wave solutions of other kinds. As an example we shall study free waves trapped on a discontinuity in depth. Later on we present some arguments in favour of the hypothesis that such trapped waves are of particular importance to the response of the sea to changing atmospheric disturbances. For convenience the wind stress and atmospheric pressure are put equal to zero and the bottom friction is neglected. Then, since $c = 0$, we shall not use the scaled equations. We consider an idealized case where there is a discontinuity in the depth along the x-axis, so that

$$h = h_1 \qquad \text{when } y > 0,$$
$$h = h_2 \qquad \text{when } y < 0.$$

We also assume that $h_1 > h_2$. Then for $y \neq 0$ the following equations are valid

$$\left(\nabla^2 - \frac{f^2}{gh_1}\right)\frac{\partial \phi_1}{\partial t} = 0 \qquad y > 0, \tag{35a}$$

$$\left(\nabla^2 - \frac{f^2}{gh_2}\right)\frac{\partial \phi_2}{\partial t} = 0 \qquad y < 0. \tag{35b}$$

At the interface we assume continuity in the pressure force, i.e.

$$\phi_1 = \phi_2 \qquad y = 0, \tag{36}$$

and in the volume flux,

$$h_1 \bar{v}_1 = h_2 \bar{v}_2 \qquad y = 0. \tag{37}$$

The last of these conditions is actually a consequence of (11) and the first condition.

We search for a solution of the form

$$\phi_1 = \hat{\phi}_1 \, e^{i(kx - \nu t)} \, e^{-l_1 y} \qquad y > 0,$$
$$\phi_2 = \hat{\phi}_2 \, e^{i(kx - \nu t)} \, e^{l_2 y} \qquad y < 0,$$

where l_1 and l_2 are both positive, so that the solution dies away with increasing distance from the discontinuity. The differential equation is satisfied provided

$$l_1 = k\left(1 + \frac{f^2}{gh_1 k^2}\right)^{1/2},$$

$$l_2 = k\left(1 + \frac{f^2}{gh_2 k^2}\right)^{1/2}.$$

The first condition at the interface dictates the relation

$$\hat{\phi}_1 = \hat{\phi}_2 = \hat{\phi}_0.$$

Another relation comes from the second condition if $\bar{v}$ is substituted for, using (9):

$$\left(f\frac{\partial \phi_1}{\partial x} - \frac{\partial^2 \phi_1}{\partial y \partial t}\right)h_1 = \left(f\frac{\partial \phi_2}{\partial x} - \frac{\partial^2 \phi_2}{\partial y \partial t}\right)h_2.$$

Inserting here for ϕ_1, ϕ_2, l_1 and l_2 one derives the dispersion relation

$$\nu = \frac{f(h_1 - h_2)}{h_1(1 + f^2/gh_1 k^2)^{1/2} + h_2(1 + f^2/gh_2 k^2)^{1/2}}. \tag{38}$$

Kinematically this solution is similar to the so-called "double Kelvin wave". The difference shows up in the dispersion relation, which for the double Kelvin wave may be conveniently written as

$$\nu_0 = \frac{f(h_1 - h_2)}{h_1\left[1 + \dfrac{f^2 - \nu_0^2}{gh_1 k^2}\right]^{1/2} + h_2\left[1 + \dfrac{f^2 - \nu_0^2}{gh_2 k^2}\right]^{1/2}}. \tag{39}$$

As shown by Longuet-Higgins (1968) there is only one real value of ν_0 satisfying this equation and, furthermore, this solution is subject to the restriction $\nu_0 < f^2$. By inspection of (38) and (39) it becomes clear that ν converges to ν_0 as $\nu_0 \to 0$, so that formula (38) is more correct for small than for large values of ν.

If $h_2 = 0$, one gets

$$c = \nu/k = \sqrt{gh_1}\,(1 + k^2 gh_1/f^2)^{-1/2}. \qquad (40)$$

This formula may also be derived from the boundary condition (17). When k decreases, c approaches $\sqrt{gh_1}$, which is the true speed of the Kelvin wave. The shorter waves, however, have a much lower phase velocity. Accordingly, in contrast to the true Kelvin waves, these waves are dispersive with a group velocity equal to

$$c_{gr} = \frac{\partial \nu}{\partial k} = c(1 + k^2 gh/f^2)^{-1}.$$

Naturally the reduction of the phase speed is in accordance with the suppression of the second time derivative in the process of deriving the quasi-geostrophic equations.

The coastal wave is closely associated with the boundary condition (16), constructed on the basis of the generalized geostrophic relation (9) which allows a cross-isobaric flow. Since (9) is consistant with the vorticity equation (14), it is not surprising that similar wave solutions exist on a continuously sloping bottom, which may easily be shown.

5. Conclusions

We have studied some disturbances in the sea produced by changing atmospheric pressure and wind fields, with emphasis on the barotropic circulation in the interior. The quasi-geostrophic approach, which implies a simplification of the theoretical treatment, is shown to be sufficiently accurate provided the sea is not too shallow. On the other hand, if the sea is deep, a barotropic scheme may not be adequate. In that case a baroclinic model based on the same formal method may, in fact, be very attractive. However, this subject has not been considered in the present paper.

The analytical solutions, although seemingly restrictive, may nevertheless have quite general aspects. This we have tried to demonstrate in one specific case concerning the wind-driven current in the North Sea. The surprisingly strong flow in the Norwegian Trench is suggested by the general quasi-geostrophic formalism, although the actual analysis is carried out by a completely independent method.

Several types of trapped wave solutions may be derived from the quasi-geostrophic equations, and it is appropriate to attach some general comments to this situation. To fix our ideas, let us for the moment leave the quasi-geostrophic approximation and consider the Kelvin wave in contrast to the inertia-gravity wave. As is well known, the Kelvin wave is non-dispersive and, accordingly, the velocity of energy transfer is the same as the phase velocity. The inertia-gravity waves, on the other hand, have a smaller group velocity and more so for the longer waves. Furthermore, the Kelvin waves, being trapped at the coastline, conserve their amplitude, while the inertia-gravity waves may propagate in all directions from the source, and accordingly are attenuated. Therefore, the result of a changing external force generally propagates more effectively through the sea by the Kelvin waves. For this reason it is fortunate that similar trapped waves are present in the quasi-geostrophic system. It is true though, that the phase velocity, and in particular the group velocity, is reduced by the approximations, and this is especially true for the shorter waves. For this reason the quasi-geostrophic equations may not be accurate enough to simulate rapidly propagating disturbances like those discussed by Gjevik and Røed (1976).

This study has partly been motivated by a desire to investigate the possibility of basing a numerical model on the quasi-geostrophic equations. Simulations of wind-driven currents in seas of the hydrographic characteristics we have focused on, have been reported by many authors, beginning with Hansen (1956). A more recent example is Heaps (1972). As a rule, the models have been based on eqs. (1)–(4), i.e. inertia-gravity waves are included. This seems to be a complication compared to the quasi-geostrophic approach, as long as the emphasis is on the long-term displacement of the water masses, and not on tides and similar phenomena.

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7. Appendix

Using partial integration and the continuity equation (2), we get

$$\frac{\partial \mathbf{T}_B}{\partial t} + \mathbf{T}_B \cdot \nabla \bar{\mathbf{v}} + \nabla \cdot (\bar{\mathbf{v}} \mathbf{T}_B) + \int_{-h}^{0} \nabla \cdot (\mathbf{v}_B \mathbf{v}_B) \, dz$$
$$+ w_0 \mathbf{v}_{B0} - w_1 \mathbf{v}_{B1} = -f\mathbf{k} \times \mathbf{T}_B + \boldsymbol{\tau}_0 - \boldsymbol{\tau}_1. \quad \text{(A1)}$$

In a stationary case, $\mathbf{T}_B$ must balance the water transport by the geostrophic current. Therefore, it seems natural to use $UH = S_0 H/c$ as a scale for $\mathbf{T}_B$. The velocity $\mathbf{v}_B$ is confined to the boundary layers, and a characteristic size of the boundary layer thickness can be shown by similarity arguments to be $S_0^{1/2}/f$. Dividing the scale for $\mathbf{T}_B$ by this scale height, we find the characteristic size of $\mathbf{v}_B$ to be $S_0^{1/2} Hf/c$. The elevation of the sea surface is small compared to the variations in the depth of the sea, while $\mathbf{v}_{B0}$ is of the same order of magnitude as $\mathbf{v}_{B1}$. Therefore, $w_0 \mathbf{v}_{B0}$ may be neglected in relation to $w_1 \mathbf{v}_{B1}$. We shall need the characteristic size of w_1, which, according to (4), may be taken to be $UH/L = S_0 H/(cL)$.

We are now able to scale the equation and get

$$\frac{c}{fH}\frac{\partial \mathbf{T}_B}{\partial t} + \frac{S_0}{cfL}\{\mathbf{T}_B \cdot \nabla \bar{\mathbf{v}} + \nabla \cdot (\bar{\mathbf{v}} \mathbf{T}_B)\}$$
$$+ \frac{HS_0^{1/2}}{cL}\{\int_{-h}^{0} \nabla \cdot (\mathbf{v}_B \mathbf{v}_B) \, dz - w_1 \mathbf{v}_{B1}\}$$
$$= -\mathbf{k} \times \mathbf{T}_B + \frac{c}{fH}(\boldsymbol{\tau}_0 - \boldsymbol{\tau}_1), \quad \text{(A2)}$$

where we have taken advantage of the fact that the integrand has a significant size only in the boundary layers. Using the same numerical values as before, we find that the equation may be simplified to

$$\frac{c}{fH}\frac{\partial \mathbf{T}_B}{\partial t} = -\mathbf{k} \times \mathbf{T}_B + \frac{c}{fH}(\boldsymbol{\tau}_0 - \boldsymbol{\tau}_1). \quad \text{(A3)}$$

Again, we may simplify further if $c^2/(fH)^2 = R^2 \ll 1$, and, instead of (7), we then get the more accurate formula

$$\mathbf{T}_B = -f^{-1}\mathbf{k} \times (\boldsymbol{\tau}_0 - \boldsymbol{\tau}_1) + f^{-2}\frac{\partial}{\partial t}(\boldsymbol{\tau}_0 - \boldsymbol{\tau}_1). \quad \text{(A4)}$$

Introducing this expression, as before, in (12), we get in (14) the additional term $cf^{-2}\nabla^2 \partial^2 \phi/\partial t^2$ (which can be neglected compared to the first term on the right-hand side), and another term, $\nabla \cdot \partial \boldsymbol{\tau}_0/\partial t$, which generally is small, since $\boldsymbol{\tau}_0$ is approximately proportional to the geostrophic wind, which is non-divergent.

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